Abstract. Let $V$ be a variety of associative algebras with involution $*$ over a field $F$ of characteristic zero. Giambruno and Mishchenko proved in [6] that the $*$-codimension sequence of $V$ is polynomially bounded if and only if $V$ does not contain the commutative algebra $D = F \oplus F$, endowed with the exchange involution, and $M$, a suitable 4-dimensional subalgebra of the algebra of $4 \times 4$ upper triangular matrices, endowed with the reflection involution. As a consequence the algebras $D$ and $M$ generate the only varieties of almost polynomial growth. In [20] the authors completely classify all subvarieties and all minimal subvarieties of the varieties $\var(D)$ and $\var(M)$. In this paper we exhibit the decompositions of all minimal subvarieties of $\var(D)$ and $\var(M)$ and compute their $*$-colengths. Finally we relate the polynomial growth of a variety to the $*$-colengths and classify the varieties such that their sequence of $*$-colengths is bounded by three.

1. Introduction

Let $A$ be an associative algebra with involution ($*$-algebra) over a field $F$ of characteristic zero and let $c_n^*(A), n = 1, 2, \ldots,$ be its sequence of $*$-codimensions. In case $A$ satisfies a nontrivial identity, it was proved in [8] that $c_n^*(A)$ is exponentially bounded. In order to capture the exponential rate of growth of the sequence of $*$-codimensions, recently, in [7] the authors proved that for any associative $*$-algebra $A$, satisfying an ordinary identity,

$$\exp^*(A) = \lim_{n \to \infty} \sqrt[n]{c_n^*(A)}$$

exists and is an integer called the $*$-exponent of $A$.

Given a variety of $*$-algebras $V$, the growth of $V$ is the growth of the sequence of $*$-codimensions of any algebra $A$ generating $V$, i.e., $V = \var^*(A)$. In this paper we are interested in varieties of polynomial growth, i.e., varieties of $*$-algebras such that $c_n^*(V) = c_n^*(A)$ is polynomially bounded.

In such a case, if $A$ is an algebra with 1, in [19] it was proved that $c_n^*(A) = qn^k + O(n^{k-1})$ is a polynomial with rational coefficients whose leading term satisfies the inequalities $\frac{1}{4} \leq q \leq \sum_{1 \leq i \leq k} 2k^2(-1)^{k+1}$.

In case of polynomial growth Giambruno and Mishchenko proved in [6] that a variety $V$ has polynomial growth if and only if $V$ does not contain the commutative algebra $D = F \oplus F$, endowed with the exchange involution, and $M$, a suitable 4-dimensional subalgebra of the algebra of $4 \times 4$ upper triangular matrices, endowed with the reflection involution. As a consequence the $*$-algebras $D$ and $M$ generate the only varieties of almost polynomial growth, i.e., they grow exponentially but any proper subvariety is polynomially bounded.

In [20] the authors completely classify all subvarieties of the varieties $\var(D)$ and $\var(M)$. They also classify all their minimal subvarieties of polynomial growth. We recall that $V$ is a minimal variety of polynomial growth $n^k$ if asymptotically $c_n^*(V) \approx an^k$, for some $a \neq 0$, and $c_n^*(U) \approx bn^t$, with $t < k$, for any proper subvariety $U$ of $V$.
The relevance of the minimal varieties of polynomial growth relies in the fact that these were the building blocks that allowed the authors to give a complete classification of the subvarieties of the varieties of almost polynomial growth (see also [5, 11, 13, 14, 16, 17]).

An equivalent formulation of Giambruno-Mishchenko’s result can be given as follows. Let \( P_n^* \) be the vector space of multilinear polynomials of degree \( n \) and \( \text{Id}^*(A) \) the ideal of identities satisfied by a \( \ast \)-algebra \( A \). The space \( \frac{P_n^*}{P_n^* \cap \text{Id}^*(A)} \) has a structure of \( \mathbb{Z}_2 \wr S_n \)-module and its character \( \chi_n^*(A) \), by complete reducibility, decomposes as

\[
\chi_n^*(A) = \sum_{|\lambda| + |\mu| = n} m_{\lambda,\mu} \chi_{\lambda,\mu},
\]

where \( \chi_{\lambda,\mu} \) is the irreducible \( \mathbb{Z}_2 \wr S_n \)-character associated to the pair of partitions \( (\lambda, \mu) \) and \( m_{\lambda,\mu} \geq 0 \) is the corresponding multiplicity. Then

\[
l_n^*(A) = \sum_{|\lambda| + |\mu| = n} m_{\lambda,\mu},
\]

is called the \( n \)-th \( \ast \)-colength of \( A \). If \( A \) satisfies a non-trivial identity then \( l_n^*(A) \), \( n = 1, 2, \ldots \), is polynomially bounded [1].

In this paper we state the Giambruno-Mishchenko’s result as follows: if \( A \) is any \( \ast \)-algebra, \( c_n^*(A) \) is polynomially bounded if and only if the sequence of \( \ast \)-colengths is bounded by a constant, i.e., \( l_n^*(A) \leq k \), for some \( k \geq 0 \) and for all \( n \geq 1 \). Such result was proved for finite dimensional \( \ast \)-algebras in [24].

Moreover we exhibit the decompositions of the \( \ast \)-cocharacters of all minimal subvarieties of \( \text{var}^*(D) \) and \( \text{var}^*(M) \), compute their \( \ast \)-colengths and complete their \( \ast \)-codimensions. Finally we classify the varieties such that their sequence of \( \ast \)-colengths is bounded by three, for \( n \) large enough. Furthermore we show that if \( l_n^*(A) \leq 3 \), then for \( n \) large enough, \( l_n^*(A) \) is always constant.

2. Generalities and basic tools

Throughout this paper we shall denote by \( F \) a field of characteristic zero and by \( A \) an associative algebra, not necessarily with 1, endowed with an involution \( \ast \) over \( F \). Let us write \( A = A^+ \oplus A^- \), where \( A^+ = \{a \in A \mid a^* = a\} \) and \( A^- = \{a \in A \mid a^* = -a\} \) denote the sets of symmetric and skew elements of \( A \), respectively.

Let \( F(X, \ast) \) be the free associative algebra with involution on a countable set \( X = \{x_1, x_1^*, x_2, x_2^*, \ldots\} \) of noncommutative and skew variables over \( F \) (see [10]). It is useful to consider \( F(X, \ast) \) as generated by symmetric and skew variables: if we let \( y_i = x_i + x_i^* \) and \( z_i = x_i - x_i^* \) for \( i = 1, 2, \ldots \), then \( F(X, \ast) = F(y_1, z_1, y_2, z_2, \ldots) \).

We say that a polynomial \( f(y_1, y_2, \ldots, z_1, z_2, \ldots) \in F(X, \ast) \) is a \( \ast \)-identity of \( A \), and we write \( f \equiv 0 \), if \( f(a_1, a_2, \ldots, b_1, b_2, \ldots) = 0 \) for all \( a_1, a_2, \ldots, b_1, b_2, \ldots \in A^+ \) and \( b_1, b_2, \ldots \in A^- \).

The set \( \text{Id}^*(A) \) of all \( \ast \)-identities of \( A \) is a \( T^* \)-ideal of \( F(X, \ast) \), i.e., an ideal invariant under all endomorphisms of the free algebra commuting with the involution and is completely determined by its multilinear polynomials. We denote by \( P_n^* \) the space of all multilinear polynomials of degree \( n \) in the variables \( y_1, z_1, \ldots, y_n, z_n \), i.e.,

\[
P_n^* = \text{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n, \ w_i = y_i \text{ or } w_i = z_i, i = 1, \ldots, n\}.
\]

The dimension of the space \( P_n^*(A) = \frac{P_n^*}{P_n^* \cap \text{Id}^*(A)} \) is called the \( n \)-th \( \ast \)-codimension of \( A \) and is denoted by \( c_n^*(A) \).

For \( 0 \leq r \leq n \), let \( P_{r,n-r}^* \) denote the space of multilinear polynomials in the variables \( y_1, \ldots, y_r, z_{r+1}, \ldots, z_n \). In order to study the space \( P_n^* \cap \text{Id}^*(A) \) it is enough to study \( P_{r,n-r}^* \cap \text{Id}^*(A) \), for all \( r \geq 0 \).

Setting \( P_{r,n-r}^*(A) = \frac{P_{r,n-r}^*}{P_{r,n-r}^* \cap \text{Id}^*(A)} \) and \( c_{r,n-r}^*(A) = \dim P_{r,n-r}^*(A) \) we have that

\[
c_n^*(A) = \sum_{r=0}^{n} \binom{n}{r} c_{r,n-r}^*(A).
\]
Remark 2.1. If $A$ and $B$ are $\ast$-algebras, it is well known that $A \oplus B$ is a $\ast$-algebra and $\text{Id}^\ast(A \oplus B) = \text{Id}^\ast(A) \cap \text{Id}^\ast(B)$. Furthermore, $c^\ast_n(A \oplus B) \leq c^\ast_n(A) + c^\ast_n(B)$ and the equality holds if and only if
\[
\dim \frac{P^\ast_n}{\text{Id}^\ast(A) \cap \text{Id}^\ast(B)} = \dim \frac{P^\ast_n}{\text{Id}^\ast(A)} + \dim \frac{P^\ast_n}{\text{Id}^\ast(B)}.
\]
This is equivalent to saying that $\dim P^\ast_n = \dim(P^\ast_n \cap \text{Id}^\ast(A)) + \dim(P^\ast_n \cap \text{Id}^\ast(B))$, and, so, any polynomial in $P^\ast_n$ can be written as a sum of multilinear polynomials in $\text{Id}^\ast(A)$ and in $\text{Id}^\ast(B)$.

Similarly $c^\ast_{r,n-r}(A \oplus B) = c^\ast_{r,n-r}(A) + c^\ast_{r,n-r}(B)$ if and only if any polynomial in $P^\ast_{r,n-r}$ can be written as a sum of multilinear polynomials in $\text{Id}^\ast(A)$ and in $\text{Id}^\ast(B)$ with $r$ symmetric and $n - r$ skew variables.

Let $H_n$ be the hyperoctahedral group of degree $n$, i.e., $H_n = \mathbb{Z}_2 \wr S_n$, the wreath product of the multiplicative group of order two with $S_n$. The space $P^\ast_n$ has a natural left $H_n$-module structure induced by defining for $h = (a_1, \ldots, a_n; \sigma) \in H_n$, $hy_i = y_{\sigma(i)}$, $hz_i = z_{\sigma(i)} = \pm z_{\sigma(i)}$.

Since $P^\ast_n \cap \text{Id}^\ast(A)$ is invariant under this action, the space $P^\ast_n / (P^\ast_n \cap \text{Id}^\ast(A))$ has the structure of a left $H_n$-module and its character $\chi^\ast_n(A)$, called the $n$th $\ast$-cocharacter of $A$, decomposes as
\[
\chi^\ast_n(A) = \sum_{|\lambda| + |\mu| = n} m_{\lambda,\mu} \chi_{\lambda,\mu},
\]
where $\lambda \vdash r$, $\mu \vdash n - r$, $r = 0, 1, \ldots, n$ and $m_{\lambda,\mu} \geq 0$ is the multiplicity of the irreducible $H_n$-character $\chi_{\lambda,\mu}$ associated to the pair $(\lambda, \mu)$.

Also
\[
l^\ast_n(A) = \sum_{|\lambda| + |\mu| = n} m_{\lambda,\mu}
\]
is called the $n$th $\ast$-colength of $A$.

Let $F^\ast_m(X, \ast) = \langle y_1, \ldots, y_m, z_1, \ldots, z_m \rangle$ denote the free associative algebra with involution in $m$ symmetric and skew variables and let $U = \text{span}_F \{y_1, \ldots, y_m\}$, $V = \text{span}_F \{z_1, \ldots, z_m\}$. There is a natural left action of the group $GL(U) \times GL(V) \cong GL_m \times GL_m$ on the space $U \oplus V$ and we can extend this action diagonally to get an action on $F^\ast_m(X, \ast)$. Note that for any algebra $A$ with involution, the space $F^\ast_m(X, \ast) \cap \text{Id}^\ast(A)$ is invariant under this action.

So by considering $F^\ast_m(X, \ast)$, the space of all homogeneous polynomials of degree $n$ in the variables $y_1, \ldots, y_m, z_1, \ldots, z_m$, we have that
\[
F^\ast_m(A) := F^\ast_m(X, \ast) / (F^\ast_m(X, \ast) \cap \text{Id}^\ast(A))
\]
is a $GL_m \times GL_m$-module and we denote its character by $\psi^\ast_n(A)$. It is well known (see [2, Theorem 12.4.4]) that there is a one-to-one correspondence between irreducible $GL_m \times GL_m$-characters and pairs of partitions $(\lambda, \mu)$, with $\lambda \vdash n - r$ and $\mu \vdash r$, $r = 0, \ldots, n$ where $\lambda$ and $\mu$ are partitions with at most $m$ parts.

If $\psi_{\lambda,\mu}$ denotes the irreducible $GL_m \times GL_m$-character corresponding to $(\lambda, \mu)$ then we can write
\[
\psi^\ast_n(A) = \sum_{|\lambda| + |\mu| = n} \tilde{m}_{\lambda,\mu} \psi_{\lambda,\mu}
\]
where $\tilde{m}_{\lambda,\mu}$ are the corresponding multiplicities and $h(\lambda)$ (respectively $h(\mu)$) denotes the height of the Young diagram corresponding to $\lambda$ (respectively $\mu$).

In order to calculate the multiplicity $m_{\lambda,\mu}$ of an irreducible character $\chi_{\lambda,\mu}$ in the decomposition (2.2), we use the following relationship proved by Giambruno in [3, Theorem 3]
\[
m_{\lambda,\mu} = \tilde{m}_{\lambda,\mu}, \quad \text{for all } \lambda \vdash n - r \text{ and } \mu \vdash r \text{ with } h(\lambda), h(\mu) \leq m.
\]

It is well known that an irreducible submodule of $F^\ast_m(A)$ corresponding to the pair $(\lambda, \mu)$ is generated by a non-zero polynomial $f_{\lambda,\mu}$, called highest weight vector, of the form (see for instance [2, Theorem 12.4.12])
\[
f_{\lambda,\mu}(y_1, \ldots, y_p, z_1, \ldots, z_q) = \prod_{i=1}^l St_{h(\lambda)}(y_1, \ldots, y_{h(\lambda)}) \prod_{i=1}^n St_{h(\mu)}(z_1, \ldots, z_{h(\mu)}) \sum_{\sigma \in S_n} \alpha_{\sigma} \sigma,
\]
where $\alpha_\sigma \in F$, $St_k(x_1, \ldots, x_k) = \sum_{\sigma \in S_k} (\text{sign } \sigma) x_{\sigma(1)} \cdots x_{\sigma(k)}$ is the standard polynomial of degree $k$ and $S_n$ acts from right by permuting places in which the variables occur.

Let $T_\lambda$ and $T_\mu$ be two Young tableaux. We denote by $f_{T_\lambda, T_\mu}$ the highest weight vector obtained from (2.5) by considering the only permutation $\sigma \in S_n$ such that the integers $\sigma(1), \ldots, \sigma(h_1(\lambda))$, in this order, fill in from top to bottom the first column of $T_\lambda$, $\sigma(h_1(\lambda) + 1), \ldots, \sigma(h_1(\lambda) + h_2(\lambda))$ the second column of $T_\lambda$, $\ldots, \sigma(h_1(\lambda) + \cdots + h_{\lambda-1}(\lambda) + 1), \ldots, \sigma(r)$ the last column of $T_\lambda$; also $\sigma(r+1), \ldots, \sigma(r + h_1(\mu))$ fill in the first column of $T_\mu$, $\ldots, \sigma(r + h_1(\mu) + \cdots + h_{\mu-1}(\mu) + 1), \ldots, \sigma(n)$ the last column of $T_\mu$.

**Remark 2.2.** (see [2]) In the decomposition (2.3) we have $\tilde{m}_{\lambda, \mu} \neq 0$ if and only if there exists a pair of tableaux $(T_\lambda, T_\mu)$ such that the corresponding highest weight vector $f_{T_\lambda, T_\mu}$ is not a $*$-identity of $A$. Moreover $\tilde{m}_{\lambda, \mu}$ is the maximal number of linearly independent highest weight vectors $f_{T_\lambda, T_\mu}$ in $F_m^\pi(A)$.

### 3. Varieties of almost polynomial growth and their subvarieties

The purpose of this section is to study the sequences of $*$-cocharacters, $*$-codimensions and $*$-colengths of the minimal subvarieties of polynomial growth of the varieties of almost polynomial growth, which are classified in [20].

We denote by $UT_s = UT_s(F)$ the algebra of the $s \times s$ upper triangular matrices over $F$ and by $I_s$ the $s \times s$ identity matrix. Recall that the varieties of almost polynomial growth are generated by the following two algebras (see [6])

1) $F \oplus F$, the two-dimensional commutative algebra with the exchange involution $(a, b)^* = (b, a)$;

2) $M = \begin{pmatrix} u & r & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & v \\ 0 & 0 & 0 & u \end{pmatrix}$, the subalgebra of $UT_4$ with the reflection involution, i.e.,

the involution obtained by reflecting a matrix along its secondary diagonal: if $a = \alpha(e_{11} + e_{44}) + \beta(e_{22} + e_{33}) + \gamma e_{12} + \delta e_{34}$ then $a^* = \alpha(e_{11} + e_{44}) + \beta(e_{22} + e_{33}) + \delta e_{12} + \gamma e_{34}$, where the $e_{ij}$'s denote the usual matrix units.

The above algebras characterize the varieties of $*$-algebras of polynomial growth.

**Theorem 3.1.** [6, Theorem 4.7] Let $A$ be a $*$-algebra. Then the sequence $c_n^*(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if $M, D \in \text{var}^*(A)$.

We start by presenting $*$-algebras belonging to the variety generated by $D$ and generating minimal varieties of polynomial growth (see [20]).

For $k \geq 2$, let

$$C_k = \{ \alpha I_k + \sum_{1 \leq i < k} \alpha_i E_i \mid \alpha, \alpha_i \in F \}$$

be the commutative subalgebra of $UT_k$ with involution given by

$$(\alpha I_k + \sum_{1 \leq i < k} \alpha_i E_i)^* = \alpha I_k + \sum_{1 \leq i < k} (-1)^i \alpha_i E_i^*.$$

Here $E_1 = \sum_{i=1}^{k-1} e_i e_{i+1}$.

Since $D$ is commutative, any antiisomorphism of $D$ is an automorphism and, so, $D$ can be viewed as a superalgebra with grading $(D^{(0)}, D^{(1)})$, where $D^{(0)} = D^+$ and $D^{(1)} = D^-$. Hence, the classification of the $*$-superalgebras, up to $T^*$-equivalence, inside $\text{var}^*(D)$ and the classification of the superalgebras inside $\text{var}^{\vartheta}(D)$ are equivalent. In the light of these considerations we have the following.

**Theorem 3.2.** [20, Lemma 9],[23, Theorem 8.3] Let $k \geq 2$. Then

1) $\text{Id}^*(C_k) = \langle [y_1, y_2], [y, z], [z_1, z_2], z_1 \cdots z_k \rangle_{T^*}$

2) $c_n^*(C_k) = \sum_{j=0}^{k-1} \binom{k}{j} \approx \frac{1}{(k-1)!} n^{k-1}, n \rightarrow \infty$.

3) $\chi_n^*(C_k) = \sum_{j=0}^{k-1} \chi(n-j), (j)$ and $\ell_n^*(C_k) = k$. 


Given two $*$-algebras $A$ and $B$, we say that $A$ is $T^*$-equivalent to $B$, and we write $A \sim_{T^*} B$, in case $\text{Id}^*(A) = \text{Id}^*(B)$.

The following theorem classifies the subvarieties and the minimal varieties of $\text{var}^*(D)$.

**Theorem 3.3.** [20, Theorem 7 and Corollary 3] Let $A$ be a $*$-algebra such that $\text{var}^*(A) \subseteq \text{var}^*(D)$. Then

1. either $A \sim_{T^*} N$ or $A \sim_{T^*} C \oplus N$ or $A \sim_{T^*} C_k \oplus N$, for some $k \geq 2$, where $N$ is a nilpotent $*$-algebra and $C$ is a non-nilpotent commutative $*$-algebra with trivial involution.
2. The algebra $A$ generates a minimal variety of polynomial growth if and only if $A \sim_{T^*} C_k$, for some $k \geq 2$.

Next we exhibit the decomposition of the $*$-cocharacter of all minimal subvarieties of $\text{var}^*(M)$. We start by recalling $*$-algebras inside $\text{var}^*(M)$ generating minimal varieties of polynomial growth. For any $k \geq 2$, consider the following subalgebras of $UT_{2k}$ endowed with the reflection involution:

$$N_k = \text{span}_F \{ I_{2k}, E, \ldots, E^{k-2}; e_{12} = e_{2k-1,2k}, e_{13}, \ldots, e_{1k}, e_{k+1,2k}, \ldots, e_{2k-2,2k} \}$$

$$U_k = \text{span}_F \{ I_{2k}, E, \ldots, E^{k-2}; e_{12} = e_{2k-1,2k}, e_{13}, \ldots, e_{1k}, e_{k+1,2k}, \ldots, e_{2k-2,2k} \}$$

$$A_k = \text{span}_F \{ e_{11} + e_{2k-2k}, E, \ldots, E^{k-2}; e_{12}, e_{13}, \ldots, e_{1k}, e_{k+1,2k}, \ldots, e_{2k-1,2k} \}$$

where $E = \sum_{i=2}^{k-1} e_i + e_{2k-i,2k-i}$.

Notice that in case $k = 2$, we have that $U_2$ is $T^*$-equivalent to the commutative algebra with trivial involution, so $\text{Id}^*(U_2) = \langle [y_1, y_2], z_1 \rangle_{T^*}$ and $c_n^*(U_2) = 1$.

The following results describe the $T^*$-ideals of the above algebras and explicit the $*$-codimensions of $N_k$ and $U_k$.

**Lemma 3.4.** [20, Lemma 2] Let $k \geq 2$. Then

1. $\text{Id}^*(N_k) = \langle [y_1, \ldots, y_{k-1}], z_1 z_2 \rangle_{T^*}$, in case $k \geq 3$ and $\text{Id}^*(N_k) = \langle [y_1, y_2], [y, z], z_1 z_2 \rangle_{T^*}$, in case $k = 2$.
2. $c_n^*(N_k) = 1 + \sum_{j=1}^{k-2} \binom{n}{j} (2j - 1) + \binom{n}{k-1} (k-1) \approx q n^{k-1}$, for some $q > 0$.

**Lemma 3.5.** [20, Lemma 3] Let $k \geq 3$. Then

1. $\text{Id}^*(U_k) = \langle [z, y_1, \ldots, y_{k-2}], z_1 z_2 \rangle_{T^*}$.
2. $c_n^*(U_k) = 1 + \sum_{j=1}^{k-2} \binom{n}{j} (2j - 1) + \binom{n}{k-1} (k-2) \approx q n^{k-1}$, for some $q > 0$.

**Lemma 3.6.** [20, Lemma 3] Let $k \geq 2$. Then

$$\text{Id}^*(A_k) = \langle y_1 \cdots y_{k-2} S y_1, y_k, y_{k+1}, y_{k-1}, y_{k+2}, \cdots y_{2k-1}, y_1 \cdots y_{k-1} z y_k \cdots y_{2k-1}, z_1 z_2 \rangle_{T^*}$$

The relevance of the above $*$-algebras is shown in the following.

**Theorem 3.7.** [20, Theorem 6 and Corollary 1] Let $A$ be a $*$-algebra such that $\text{var}^*(A) \subseteq \text{var}^*(M)$. Then

1. $A$ is $T^*$-equivalent to one of the following $*$-algebras: $N$, $N_k \oplus N$, $U_k \oplus N$, $N_k \oplus U_k \oplus N$, $A_t \oplus N$, $N_k \oplus A_t \oplus N$, $U_k \oplus A_t \oplus N$, $N_k \oplus U_k \oplus A_t \oplus N$, for some $k, t \geq 2$, where $N$ is a nilpotent $*$-algebra.
2. $A$ generates a minimal variety of polynomial growth if and only if either $A \sim_{T^*} U_r$ or $A \sim_{T^*} N_k$ or $A \sim_{T^*} A_t$, for some $k \geq 2, r > 2$.

Next we determine the $*$-codimensions of the algebra $A_k$, for any $k \geq 2$. We start by considering the case $k = 2$.

**Lemma 3.8.** $c_n^*(A_2) = 4n - 1$, for $n \geq 3$. 

Proof. We have $\text{Id}^*(A_2) = (St_3(y_1, y_2, y_3), y_1z_2, z_1z_2)_{T^*}$. Since $z_1z_2 \in \text{Id}^*(A_2)$, by [21, Remark 8], we have $z_1w_2 \in \text{Id}^*(A_2)$ for any monomial $w$ of $F(X, \ast)$, and, so $c_{n-r, r}(A_k) = 0$ for all $r \geq 2$. Thus by (2.1)

\begin{equation}
(3.1) \quad c_n^r(A_2) = c_{n,0}(A_2) + nc_{n-1,1}(A_2).
\end{equation}

We start by considering $P_{n,0}^*(A_2)$. By the Poincaré-Birkhoff-Witt theorem (see [2]), every monomial in $y_1, \ldots, y_n$ can be written as a linear combination of products of the type

\begin{equation}
(3.2) \quad y_1 \cdots y_n w_1 \cdots w_m
\end{equation}

where $w_1, \ldots, w_m$ are left normed Lie commutators in $y_i$’s and $i_1 < \cdots < i_s$. Since $[y_1, y_2][y_3, y_4] \in \text{Id}^*(A_2)$, we get that, modulo $(\langle y_1, y_2, [y_3, y_4] \rangle_{T^*})$, at most one commutator can appear in (3.2) and the elements in (3.2) are polynomials of type

\begin{equation}
\text{or } y_1 \cdots y_n \text{ with } r > j_i < \cdots < j_t.
\end{equation}

Moreover, modulo $(y_1[y_2, y_3])_{T^*}$, we have that

\begin{equation}
[y_r, y_{j_1}, \ldots, y_{j_t}] = [y_r, y_{j_1}]y_{j_2} \cdots y_{j_t} = y_r \cdots y_{j_t}.
\end{equation}

Then modulo $\text{Id}^*(A_2)$, every polynomial in $P_{n,0}^*$ can be written as a linear combination of elements of the type

\begin{equation}
(3.3) \quad [y_r, y_{i_2}] \cdots \hat{y}_r \cdots [y_n, y_j] \quad \text{and} \quad y_1 \cdots y_n,
\end{equation}

$2 \leq r \leq n, 1 \leq i \leq j \leq n$, where the symbol $\hat{y}_r$ means that the variable $y_r$ is omitted. Notice that the variables of the first type only appear in case $s = 0$ in (3.2). Because of $[y_1, y_2][y_3, y_4] \in \text{Id}^*(A_2)$ the variables out of the commutator in the polynomials of the second type in (3.3) can be ordered. Moreover, since $St_3(y_1, y_2, y_3) \in \text{Id}^*(A_2)$, $y_1[y_2, y_3] = y_2[y_1, y_3] + y_3[y_2, y_1]$ can be applied and we obtain that the polynomials

\begin{equation}
(3.4) \quad [y_r, y_{i_2}] \cdots \hat{y}_r \cdots [y_n, y_j] \quad \text{and} \quad y_1 \cdots y_n, \quad 2 \leq r \leq n
\end{equation}

generate $P_{n,0}^* \cap \text{Id}^*(A_2)$.

We claim that these polynomials form a basis of $P_{n,0}^*(A_2)$. Suppose that $f \in P_{n,0}^* \cap \text{Id}^*(A_2)$ is a linear combination of the polynomials in (3.4) and write

\begin{equation}
f = \alpha y_1 \cdots y_n + \sum_{j=2}^n \alpha_j [y_j, y_1]y_2 \cdots \hat{y}_r \cdots y_n + \sum_{j=2}^n \beta_j y_2 \cdots \hat{y}_r \cdots y_n[y_j, y_1].
\end{equation}

By making the evaluation $y_i = e_{i_1} + e_{i_2}$, for all $i = 1, \ldots, n$, we get $\alpha(e_{i_1} + e_{i_2}) = 0$, and, so, $\alpha = 0$. Now for a fixed $j$, the evaluation $y_j = e_{j_2} + e_{j_3}$ and $y_1 = e_{i_1} + e_{i_2}$, for all $i \neq j$ gives $\alpha_j e_{j_3} - \beta_j e_{j_2} = 0$, and so, $\alpha_j = \beta_j = 0$ and the claim is proved. Thus $c_{n,0}^r(A_2) = 1 + 2(n - 1) = 2n - 1$.

We now consider $P_{n-1,1}^*(A_2)$. Since $y_1z_2 \in \text{Id}^*(A_2)$, then, modulo $P_{n-1,1}^* \cap \text{Id}^*(A_2)$, $P_{n-1,1}^*$ can be generated by the monomials

\begin{equation}
z_n y_1 \cdots y_{n-1} \text{ and } y_1 \cdots y_{n-1} z_n.
\end{equation}

We claim that these polynomials form a basis of $P_{n-1,1}^* \cap \text{Id}^*(A_2)$. Let $f = \alpha z_n y_1 \cdots y_{n-1} + \beta y_1 \cdots y_{n-1} z_n \in P_{n-1,1}^* \cap \text{Id}^*(A_2)$. By making the evaluation $z_n = e_{i_2} - e_{i_3}$ and $y_i = e_{i_1} + e_{i_2}$, for all $i \neq n$, we get $-\alpha e_{i_3} + \beta e_{i_2} = 0$ and so $\alpha = \beta = 0$. Thus $c_{n-1,1}^r(A_2) = 2$.

Hence, from (3.1) it follows that $c_n^r(A_2) = 2n - 1 + 2n = 4n - 1$. \hfill \square

Remark 3.9. For $k \geq 3$, let

\begin{equation}
I_1 = \langle [y_1, y_2] [y_3, y_4], [y_1, y_2] [y_3, y_4] [y_5, \ldots, y_{k+1}] \rangle_{T^*} \quad \text{and} \quad I_2 = \langle [y_1, y_2] [y_3, y_4], y_3 \cdots y_{k+1} [y_1, y_2] \rangle_{T^*}.
\end{equation}

By [13, Lemma 3.1],

\begin{equation}
c_{n,0}^r(I_1) = c_{n,0}^r(I_2) = 1 + \sum_{j=0}^{k-2} \binom{n}{j} (n - j - 1).
\end{equation}

Moreover, if $I$ is the $T^*$-ideal $I_1 \cap I_2$ then, by [13, Lemma 3.4],

\begin{equation}
I = \langle [y_1, y_2] [y_3, y_4], y_1 \cdots y_{k-1} [y_k, y_{k+1}] y_{k+2} \cdots y_{2k} \rangle_{T^*}.
\end{equation}
From Remark 2.1, we have the strict inequality \( c_{n,0}^s(I) < c_{n,0}^s(I_1) + c_{n,0}^s(I_2) \) since \( y_1 \cdots y_n \) is a polynomial in \( P_{n,0}^s \) which is not in \((P_{n,0}^s \cap I_1) + (P_{n,0}^s \cap I_2) \). Moreover, since \( I \cap P_{n,0}^s \subset \text{Id}^s(A_k) \cap P_{n,0}^s \), we have

\[
(3.6) \quad c_{n,0}^s(A_k) \leq c_{n,0}^s(I) < c_{n,0}^s(I_1) + c_{n,0}^s(I_2) = 2 + 2 \sum_{j=0}^{k-2} \binom{n}{j} (n - j - 1).
\]

**Lemma 3.10.** Let \( k \geq 2 \). Then

\[
c_{n}^s(A_k) = 1 + 2 \sum_{j=0}^{k-2} \binom{n}{j} (n - j) + 2 \sum_{j=0}^{k-2} \binom{n}{j} (n - j - 1) \approx qn^{k-1}, \text{ for some } q > 0.
\]

**Proof.** The result has already been proved for \( k = 2 \) in Lemma 3.8 so we consider \( k \geq 3 \). Since \( z_1z_2 \in \text{Id}^s(A_k) \), by [21, Remark 8] we have that \( z_1wz_2 \in \text{Id}^s(A_k) \), for any monomial \( w \) of \( F(X,*) \), and, so \( P_{n-r,r}^s(A_k) = \{0\} \) for all \( r \geq 2 \) and

\[
(3.7) \quad c_{n}^s(A_k) = c_{n,0}^s(A_k) + nc_{n-1,1}^s(A_k).
\]

Let us study the dimensions of \( P_{n,0}^s(A_k) \) and \( P_{n-1,1}^s(A_k) \). We start by considering \( P_{n,0}^s(A_k) \). We claim that the following polynomials in \( P_{n,0}^s \)

\[
(3.8) \quad y_1 \cdots y_n, y_1 \cdots y_i \{y_r, y_m\} y_j_1 \cdots y_j_s, \ y_1 \cdots y_{p_n} y_{r_1} y_{r_2} \cdots y_{q_r}
\]

where \( t \leq k-1, \ i_1 < \cdots < i_t, \ r > m < j_1 < \cdots < j_s \) and \( v < k-1, a > b < p_1 < \cdots < p_n, q_1 < \cdots < q_v \) are linearly independent modulo \( \text{Id}^s(A_k) \). Suppose that \( f \in P_{n,0}^s \cap \text{Id}^s(A_k) \) is a linear combination of the above polynomials and write

\[
f = \alpha y_1 \cdots y_n + \sum_{t \leq k-1} \sum_{r,l} \alpha_{r,l} y_1 \cdots y_{i_t} \{y_r, y_{m_n}\} y_{j_1} \cdots y_{j_s},
\]

where \( t + s = n - 2 \) and for any fixed \( t \) and \( s \), \( I = \{i_1, \ldots, i_t\} \) and \( J = \{j_1, \ldots, j_s\} \). If \( t < k-1 \) then \( i_1 < \cdots < i_t \) and \( r > m < j_1 < \cdots < j_s \) and if \( s < k-1 \) then \( r > m < i_1 < \cdots < i_t \) and \( j_1 < \cdots < j_s \).

First suppose that \( \alpha \neq 0 \). Then by making the evaluation \( y_1 = \cdots = y_n = e_{11} + e_{2k,2k} \) we get \( \alpha(e_{11} + e_{2k,2k}) = 0 \) and so \( \alpha = 0 \), a contradiction.

Now suppose that \( \alpha_{r,l} \neq 0 \), for some \( t \leq k-1, r, I \), and \( J \). Then by making the evaluation \( y_i = \cdots = y_{i_t} = E, y_r = E_{11} + e_{2k-1,2k} \) and \( y_m = y_{j_1} = \cdots = y_{j_s} = e_{11} + e_{2k,2k}, \) we get \( \alpha_{r,l} e_{2k-1,2k} - \alpha_{r,l} e_{12} = 0 \), and, so, \( \alpha_{r,l} = \alpha_{r,l} = 0 \), a contradiction. Similarly, if \( \alpha_{r,l} \neq 0 \), for some \( s < k-1, r, I \), and \( J \), by making the evaluation \( y_m = y_{j_1} = \cdots = y_{j_s} = e_{11} + e_{2k,2k}, y_r = e_{12} + e_{2k-1,2k} \) and \( y_m = y_{j_1} = \cdots = y_{j_s} = E \) we get \( \alpha_{r,l} = \alpha_{r,l} = 0 \), a contradiction as above.

In (3.8) we have \( 1 + 2 \sum_{j=0}^{k-2} \binom{n}{j} (n - j - 1) \) polynomials which are linearly independent modulo \( P_{n,0}^s \cap \text{Id}^s(A_k) \) so we have

\[
1 + 2 \sum_{j=0}^{k-2} \binom{n}{j} (n - j - 1) \leq c_{n,0}^s(A_k).
\]

On the other hand, by (3.6) we get

\[
c_{n,0}^s(A_k) < 2 + 2 \sum_{j=0}^{k-2} \binom{n}{j} (n - j - 1).
\]

Thus we conclude that \( c_{n,0}^s(A_k) = 1 + 2 \sum_{j=0}^{k-2} \binom{n}{j} (n - j - 1) \).

Now we consider \( P_{n-1,1}^s(A_k) \). Since \( y_1 \cdots y_{n-1} = y_1 \cdots y_{2k-2} \in \text{Id}^s(A_k) \), then \( P_{n-1,1}^s \) can be generated modulo \( \text{Id}^s(A_k) \) by the monomials

\[
y_1 \cdots y_i, y_{j_1} \cdots y_{j_s},
\]

where \( i_1 < \cdots < i_t, j_1 < \cdots < j_s \) and we have \( t < k-1 \) or \( s < k-1 \).
We next show that these polynomials are linearly independent modulo $\text{Id}^*(A_k)$. Suppose that $f \in P^*_{n-1,1} \cap \text{Id}^*(A_k)$ is a linear combination of the polynomials above and write

$$ f = \sum_{t \leq k-1} \sum_{i,j} \alpha_{t,j} y_{i_1} \cdots y_{i_t} z y_{j_1} \cdots y_{j_s}, $$

where $t + s = n - 1$ and for any fixed $t$ and $s$, $i_1 < \cdots < i_t$, $j_1 < \cdots < j_s$, $I = \{i_1, \ldots, i_t\}$ and $J = \{j_1, \ldots, j_s\}$.

Suppose $\alpha_{t,j} \neq 0$, for some $t < k - 1$ and $J$. By making the evaluation $z_n = e_{12} - e_{2k - 1 - 2k}$, $y_{i_1} = \cdots = y_{i_t} = E$ and $y_{j_1} = \cdots = y_{j_s} = e_{11} + e_{2k, 2k}$ we get $-\alpha_{t,j} e_{2k - t - 1, 2k} + \alpha_{j,J} e_{1,2 + t} = 0$, thus $\alpha_{t,j} = \alpha_{j,J} = 0$, a contradiction.

Suppose now $\alpha_{t,j} \neq 0$, for some $s < k - 1$ and $I$. Then the evaluation $z_n = e_{12} - e_{2k - 1 - 2k}$, $y_{i_1} = \cdots = y_{i_s} = e_{11} + e_{2k, 2k}$ and $y_{j_1} = \cdots = y_{j_s} = E$ gives $\alpha_{t,j} = 0$, a contradiction. Thus the polynomials in (3.9) form a basis of $P^*_{n-1,1}(A_k)$ and by counting we get $c^*_n(A_k) = 2 \sum_{j=0}^{k-2} \binom{n-j}{j}$.

Finally, by (3.7), we have

$$ c^*_n(A_k) = 1 + 2 \sum_{j=0}^{k-2} \binom{n-j}{j} (n-j-1) + 2 \sum_{j=0}^{k-2} \binom{n-j}{j} (n-j). $$

□

Next we explicitly determine the sequences of $*$-cocharacters and $*$-colengths of the minimal varieties $\text{var}^*(A) \subseteq \text{var}^*(M)$. If $\chi^*_n(A) = \sum_{|\lambda| + |\mu| = n} m_{\lambda, \mu} \chi_{\lambda, \mu}$ is the decomposition of the $n$th $*$-cocharacter of $A$, we denote by $d_{\lambda, \mu}$ the degree of the $H_\lambda$-character $\chi_{\lambda, \mu}$.

We shall prove all theorems by using induction on $k$, so for each class of algebras $N_k, U_k$ and $A_k$ we start with a lemma about the sequence of $*$-cocharacters in a particular case. We start with the study of $*$-cocharacters and $*$-colengths of the minimal varieties $\text{var}^*(A_k)$.

**Lemma 3.11.** $\chi^*_n(A_2) = \chi(n,0) + 2\chi(n-1,1) + 2\chi(n-1,0)$ and $l_n^*(A_2) = 5$

**Proof.** Let $\chi^*_n(A_2) = \sum_{|\lambda| + |\mu| = n} m_{\lambda, \mu} \chi_{\lambda, \mu}$ be the decomposition of the $n$th $*$-cocharacter of $A_2$. Notice that

$$ d_{(n),0} + 2d_{(n-1),(1)} + 2d_{(n-1),0} = 1 + 2n + 2(n-1) = c^*_n(A_2). $$

Then, since that $m_{(n),0} = 1$, if we find two linearly independent highest weight vectors for each pair of partitions $((n-1),(1))$ and $((n-1),\emptyset)$ which are not $*$-identities of $A_2$ we may conclude that $\chi^*_n(A_2)$ has the desired decomposition.

In fact, let

$$ f_1 = y^{n-1}z \quad \text{and} \quad f_2 = zy^{n-1} $$

be highest weight vectors associated to the pair of partitions $((n-1),(1))$ and corresponding to the pairs of tableaux:

(3.10) \begin{align*}
\begin{array}{lllllllll}
1 & 2 & \cdots & n-1 & \hline
n & \end{array} & \begin{array}{lllllllll}
1 & 2 & \cdots & n \hline
n-1 & \end{array},
\end{align*}

respectively. It is clear that by making the evaluation $y = e_{11} + e_{44}$ and $z = e_{12} - e_{24}$, we get that $f_1 = e_{12} \neq 0$ and $f_2 = -e_{34} \neq 0$. This says that $f_1$ and $f_2$ are not $*$-identities of $A_2$. Moreover by making the same evaluation we have that $\alpha f_1 + \beta f_2 = 0$ implies $\alpha = \beta = 0$, so these polynomials are linearly independent modulo $\text{Id}^*(A_2)$.

On the other hand,

$$ g_1 = [y_1, y_2] y_1^{n-2} \quad \text{and} \quad g_2 = y_1^{n-2}[y_1, y_2] $$

are the highest weight vector associated to the pair of partitions $((n-1),\emptyset)$ and corresponding to the pairs of tableaux:

(3.11) \begin{align*}
\begin{array}{llllllllll}
1 & 2 & \cdots & n-1 & \hline
n-1 & \end{array} & \begin{array}{llllllllll}
1 & \cdots & n-2 & \hline
\emptyset & \end{array},
\end{align*}

respectively.
By making the evaluation $y_1 = e_{11} + e_{44}$ and $y_2 = e_{12} + e_{34}$, we get that $g_1 = -e_{34} \neq 0$ and $g_2 = e_{12} \neq 0$. It shows that $g_1$ and $g_2$ are not $\ast$-identities of $A_2$ and by making the same evaluation we have that $\alpha g_1 + \beta g_2 = 0$ implies $\alpha = \beta = 0$, so these polynomials are linearly independent modulo $\text{Id}^\ast(A_2)$.

Thus $\chi_n^\ast(A_2) = \chi(n, \emptyset) + 2\chi(n-1, (1)) + 2\chi(n-1, \emptyset)$ and $l_n^\ast(A_2) = 5$.

Before giving the decomposition of the $\chi_n^\ast(A_k)$, for any $k \geq 2$, we prove the following.

**Remark 3.12.** Let $k \geq 2$. Then

$$c_n^\ast(A_k) = d_{(n, \emptyset)} + \sum_{j=1}^{k-1} 2(k-j)d_{(n-j, j, \emptyset)} + \sum_{j=1}^{k-2} 2(k-j-1)d_{(n-j-1, j, 1, \emptyset)} + \sum_{j=0}^{k-2} 2(k-j-1)d_{(n-j-1, j, 1, \emptyset)}.$$

**Proof.** We use induction on $k$. By Lemma 3.11, we have that $\chi_n^\ast(A_2) = \chi(n, \emptyset) + 2\chi(n-1, \emptyset) + 2\chi(n-1, (1))$. This says that $c_n^\ast(A_2) = d_{(n, \emptyset)} + 2d_{(n-1, \emptyset)} + 2d_{(n-1, (1))}$ and, so the result is true for $k = 2$.

Now we suppose the result is true for some $k \geq 2$. By Lemma 3.10, we have that

$$c_n^\ast(A_{k+1}) = c_n^\ast(A_k) + 2\left(\begin{array}{c} n \\ k-1 \end{array}\right)(n-k) + 2\left(\begin{array}{c} n \\ k-1 \end{array}\right)(n-k+1).$$

Hence, by using that

$$\sum_{j=1}^{k} d_{(n-j, j, \emptyset)} + \sum_{j=1}^{k-1} d_{(n-j-1, j, 1, \emptyset)} = \left(\begin{array}{c} n \\ k-1 \end{array}\right)(n-k) + \sum_{j=0}^{k-1} d_{(n-j-1, j, 1, \emptyset)},$$

we have

$$c_n^\ast(A_{k+1}) = c_n^\ast(A_k) + 2\left(\begin{array}{c} n \\ k-1 \end{array}\right)(n-k) + 2\left(\begin{array}{c} n \\ k-1 \end{array}\right)(n-k+1)
= c_n^\ast(A_k) + 2\sum_{j=1}^{k} d_{(n-j, j, \emptyset)} + 2\sum_{j=1}^{k-1} d_{(n-j-1, j, 1, \emptyset)} + 2\sum_{j=0}^{k-1} d_{(n-j-1, j, 1, \emptyset)}
= d_{(n, \emptyset)} + \sum_{j=1}^{k} 2(k+1-j)d_{(n-j, j, \emptyset)} + \sum_{j=1}^{k-1} 2(k-j)d_{(n-j-1, j, 1, \emptyset)} + \sum_{j=0}^{k-1} 2(k-j-1)d_{(n-j-1, j, 1, \emptyset)}.$$

Thus the result is true for any $k \geq 2$. \hfill $\square$

In the next lemmas, we shall adopt the convention that the symbols $\bar{\cdot}$, $\tilde{\cdot}$ and $\bar{\cdot}\tilde{\cdot}$ indicate alternation on a given set of variables. Thus, for instance, the notation $\bar{y}_1\tilde{y}_1\bar{y}_4\tilde{y}_2\bar{y}_2\tilde{y}_3$ indicates the polynomial

$$\sum_{\sigma \in S_2} (\text{sign } \sigma) (\text{sign } \tau_1) (\text{sign } \tau_1) y_{\sigma(1)} y_{\sigma(1)} y_{\tau(1)} y_{\tau(1)} y_{\sigma(2)} y_{\sigma(2)} y_{\tau(2)} y_{\tau(2)}.$$

Now we are in position to compute the $\ast$-cocharacter and the $\ast$-colength of $A_k$, for any $k \geq 2$.

**Theorem 3.13.** For $k \geq 2$, we have

1. $\chi_n^\ast(A_k) = \chi(n, \emptyset) + \sum_{j=1}^{k-1} 2(k-j)\chi(n-j, j, \emptyset) + \sum_{j=1}^{k-2} 2(k-j-1)\chi(n-j-1, j, 1, \emptyset) + \sum_{j=0}^{k-2} 2(k-j-1)\chi(n-j-1, j, 1, \emptyset)$

2. $l_n^\ast(A_k) = 3k^2 - 5k + 3$.

**Proof.** By the previous remark, we have that, for any $k \geq 2$,

$$c_n^\ast(A_k) = d_{(n, \emptyset)} + \sum_{j=1}^{k-1} 2(k-j)d_{(n-j, j, \emptyset)} + \sum_{j=1}^{k-2} 2(k-j-1)d_{(n-j-1, j, 1, \emptyset)} + \sum_{j=0}^{k-2} 2(k-j-1)d_{(n-j-1, j, 1, \emptyset)}.$$
It is clear that \( m_{(n),\emptyset} = 1 \). In order to prove the desired decomposition of \( \chi^*_n(A_k) \), we shall prove that the irreducible characters \( \chi_{(n-j),\emptyset} \), \( \chi_{(n-l-1,1,\emptyset)} \) and \( \chi_{(n-l-1,1,\emptyset)} \), for \( 1 \leq j \leq k-1, 1 \leq l \leq k-2 \) and \( 0 \leq t \leq k-2 \), appear in the decomposition of the \( * \)-cocharacter \( \chi^*_n(A_k) \) with multiplicity \( m_{(n-j),\emptyset} = 2(k-j), m_{(n-l-1,1,\emptyset)} = 2(k-l-1) \) and \( m_{(n-l-1,1,\emptyset)} = 2(k-t-1) \), respectively.

(i) For the pair of partitions \(((n-1,1),\emptyset)\), for any \( 0 \leq p \leq k-2 \) we consider the following pairs of tableaux:

\[
\begin{pmatrix}
  1 & \cdots & p & p+1 & \cdots & n \\
  \hline
  p+2 & \cdots & n-p \\
\end{pmatrix}, \emptyset
\]

and their corresponding highest weight vectors, respectively,

\[ f_p = y_1^p[y_1, y_2] y_1^{n-p-2} \]

By making the evaluation \( y_1 = e_1 + e_{2k, 2k} + E \) and \( y_2 = e_1 + e_{2k-1, 2k} \), we get that

\[ f_p(y_1, y_2) = c_{2k-p-2, 2k} - c_{2k-p-1, 2k} \] and \( g_p(y_1, y_2) = e_1 + e_{2k-1, 2k} \).

Then \( f_p \) and \( g_p \) are not \( * \)-identities of \( A_k \), for any \( 0 \leq p \leq k-2 \), and these \( 2(k-1) \) polynomials are linearly independent modulo \( \text{Id}^*(A_k) \). Hence \( m_{(n-1,1),\emptyset} \geq 2(k-1) \).

(ii) For fixed \( 2 \leq j \leq k-1 \), for the pair of partitions \(((n-j,j),\emptyset)\) and for \( 0 \leq p \leq k-j-1 \) and \( w = n-p \), we consider the following pairs of tableaux:

\[
\begin{pmatrix}
  1 & \cdots & p+j & p+j+1 & \cdots & n \\
  \hline
  p+1 & \cdots & n-p \\\n\end{pmatrix}, \emptyset
\]

and their corresponding highest weight vectors, respectively,

\[ f_p = y_1^p \bar{y}_1 \cdots \bar{y}_j \bar{y}_2 \cdots \bar{y}_j y_1^{n-j-p} \] and \( g_p = y_1^n \bar{y}_1 \cdots \bar{y}_j \bar{y}_2 \cdots \bar{y}_j y_1^p \).

We have, by making the evaluation \( y_1 = e_{11} + e_{2k, 2k} + E \) and \( y_2 = e_{11} + e_{2k, 2k} + e_{12} + e_{2k-1, 2k} \), that \( f_p(y_1, y_2) = c_{j, \alpha+1, 2k} + c_{j, \beta+1, 2k} \), with \( \alpha \neq 0 \) and \( \beta \neq 0 \). Then, for any \( 0 \leq p \leq k-j-1 \), \( f_p \) and \( g_p \) are not \( * \)-identities of \( A_k \). Moreover, the same evaluation shows that these \( 2(k-j) \) polynomials are linearly independent modulo \( \text{Id}^*(A_k) \). Thus \( m_{(n-j,j),\emptyset} \geq 2(k-j) \), for any \( 2 \leq j \leq k-1 \).

(iii) Now, for fixed \( 1 \leq l \leq k-2 \), for the pair of partitions \(((n-l-1,1,1),\emptyset)\) and for \( 0 \leq p \leq k-j-2 \) and \( w = n-p \), we consider the following pairs of tableaux:

\[
\begin{pmatrix}
  1 & \cdots & p+1 & p+2 & \cdots & n \\
  \hline
  p+1 & \cdots & n-p+1 \\\n\end{pmatrix}, \emptyset
\]

and their corresponding highest weight vectors, respectively,

\[ f_p = y_1^p \bar{y}_1 \cdots \bar{y}_l \bar{y}_1 \bar{y}_2 \cdots \bar{y}_l y_1^{n-l-2p-1} \] and \( g_p = y_1^n \bar{y}_1 \cdots \bar{y}_l \bar{y}_1 \bar{y}_2 \cdots \bar{y}_l y_1^p \).

Evaluating \( y_1 = e_{11} + e_{2k+2k} + E, y_2 = E \) and \( y_3 = e_{12} + e_{2k-1, 2k} \), we get that \( f_p(y_1, y_2, y_3) = c_{l, \alpha+1, 2k} + c_{l, \beta+1, 2k} \), with \( \alpha \neq 0 \) and \( \beta \neq 0 \). Thus \( f_p \) and \( g_p \), for any \( 0 \leq p \leq k-j-2 \), are not \( * \)-identities of \( A_k \) and these \( 2(k-l) \) polynomials are linearly independent modulo \( \text{Id}^*(A_k) \). Hence we have that \( m_{(n-l-1,1,1)} \geq 2(k-l-1) \), for any \( 1 \leq l \leq k-2 \).
(iv) Finally, for fixed $0 \leq t \leq k-2$, for the pair of partitions $((n-t-1,t), (1))$ and for $0 \leq p \leq k-j-2$ and $w = n-p$, we consider the following pairs of tableaux:

$$
\begin{pmatrix}
p + 1 & \cdots & p + t & 1 & p & p + 2t + 2 & \cdots & n, & p + t + 1 \\
p + t + 2 & \cdots & p + 2t & 1 & p + 2t + 2 & \cdots & n & \end{pmatrix}
$$

(3.14)

and their corresponding highest weight vectors, respectively,

$$
f_p = y_1^p \bar{y}_1 \cdots \bar{y}_i z \bar{y}_j \cdots \bar{y}_k \ y_1^{n-p-2t-1} \text{ and } g_p = y_1^{n-p-2t-1} \bar{y}_1 \cdots \bar{y}_i z \bar{y}_j \cdots \bar{y}_k y_1^p.
$$

By making the evaluation $y = e_1 + e_{2k} + E$ and $z = e_{12} - e_{2k-1,2k}$, in case $t = 0$, and $y_1 = e_1 + e_{2k} + E$, $y_2 = E$ and $z = e_{12} - e_{2k-1,2k}$ otherwise, we get that $f_p(y_1, y_2, z) = \alpha e_{2k-t-p-1,2k}$ and $g_p(y_1, y_2, z) = \beta e_{1,t+p+1}$, with $\alpha \neq 0$ and $\beta \neq 0$. Thus $m_{(n-t-1,t), (1)} \geq 2(k-t-1)$, for any $0 \leq t \leq k-2$, since $f_p$ and $g_p$ are not $*$-identities of $A_k$, for all $0 \leq p \leq k-t-2$, and these $2(k-t-1)$ polynomials are linearly independent modulo $\text{Id}^*(A_k)$.

Thus we have that

$$
c_n^*(A_k) \geq d_{(n),\emptyset} + \sum_{j=1}^{k-1} (2k-j)d_{(n-j,j),\emptyset} + \sum_{j=1}^{k-2} (2k-j-1)d_{(n-j-1,j,1),\emptyset} + \sum_{j=0}^{k-2} (2k-j-1)d_{(n-j-1,j,1), (1)} = c_n^*(A_k).
$$

Hence $\chi_n^*(A_k)$ has the desired decomposition. It is easy to show that $l_n^*(A_k) = 3k^2 - 5k + 3$, $\forall k \geq 2$, and the result is proved.

Now we study the $*$-cocharacters and the $*$-colengths of the minimal variety $\text{var}^*(N_k)$, for all $k \geq 2$.

**Lemma 3.14.** $\chi_n^*(N_2) = \chi_{(n),\emptyset} + \chi_{(n-1), (1)}$ and $l_n^*(N_2) = 2$.

**Proof.** Notice that we have

$$
d_{(n),\emptyset} + d_{(n-1), (1)} = 1 + n = c_n^*(N_2).
$$

Then, since $m_{(n),\emptyset} = 1$, if we find a highest weight vector for the pair of partitions $((n-1), (1))$ which is not a $*$-identity of $N_2$, we may conclude that $\chi_n^*(N_2)$ has the desired decomposition.

In fact, let $f_1 = y^n z$ be the highest weight vector associated to the pair of partitions $((n-1), (1))$ and corresponding to the pair of tableaux:

$$
\begin{pmatrix}
1 & \cdots & n-1 & n \\
\end{pmatrix}
$$

By making the evaluation $y = I$ and $z = e_{12} - e_{34}$, we get that $f = e_{12} - e_{34} \neq 0$. This says that $f$ is not a $*$-identity of $N_2$. Hence we have $\chi_n^*(N_2) = \chi_{(n),\emptyset} + \chi_{(n-1), (1)}$ and $l_n^*(N_2) = 2$.

**Remark 3.15.** Let $k \geq 2$. Then

$$
c_n^*(N_k) = d_{(n),\emptyset} + \sum_{j=1}^{k-1} (k-j-2)d_{(n-j,j),\emptyset} + \sum_{j=1}^{k-2} (k-j-1)d_{(n-j-1,j,1),\emptyset}.
$$

**Proof.** We shall use induction on $k$. From Lemma 3.14 it follows that the result is true for $k = 2$. Now we suppose the result is true for some $k \geq 2$. By Lemma 3.4, we have that

$$
c_n^*(N_{k+1}) = c_n^*(N_k) + \binom{n}{k-1} (k-2) + \binom{n}{k} k.
$$

Hence, by using that, for all $r \geq 1$,

$$
\sum_{j=0}^r d_{(n-j,j-1), (1)} = \binom{n}{r+1} (r+1) \text{ and } \sum_{j=1}^r [d_{(n-j,j),\emptyset} + d_{(n-j-1,j,1),\emptyset}] = \binom{n}{r+1} r,
$$

\]
we get the following:
\[
c^*_n(N_{k+1}) = c^*_n(N_k) + \left(\binom{n}{k-1}\right)(k-2) + \binom{n}{k}k
\]
\[
= c^*_n(A_k) + \sum_{j=1}^{k-2} [d_{(n-j,j),\emptyset} + d_{(n-j-1,j),\emptyset}] + \sum_{j=0}^{k-1} d_{(n-j-1,j),(1)}
\]
\[
= d_{(n),\emptyset} + \sum_{j=1}^{k-2} (k - j - 1)[d_{(n-j,j),\emptyset} + d_{(n-j-1,j),\emptyset}] + \sum_{j=0}^{k-1} (k - j)d_{(n-j-1,j),(1)}
\]
Thus the result is true for any \( k \geq 2 \).

**Theorem 3.16.** For \( k \geq 3 \), we have
\[
(1) \quad \chi^*_n(N_k) = \chi(n,\emptyset) + \sum_{j=1}^{k-3} (k - j - 2)[\chi(n-j,j),\emptyset + \chi(n-j-1,j),\emptyset] + \sum_{j=0}^{k-2} (k - j - 1)\chi(n-j-1,j),(1).
\]
\[
(2) \quad l^*_n(N_k) = \frac{3k^2 - 11k + 14}{2}.
\]

**Proof.** The proof is similar to the proof of Lemma 3.13. By the previous remark, we have that, for any \( k \geq 3 \),
\[
c^*_n(N_k) = d_{(n),\emptyset} + \sum_{j=1}^{k-3} (k - j - 2)[d_{(n-j,j),\emptyset} + d_{(n-j-1,j),\emptyset}] + \sum_{j=0}^{k-2} (k - j - 1)d_{(n-j-1,j),(1)}.
\]
It is clear that \( m_{(n),\emptyset} = 1 \). In order to prove the desired decomposition of \( \chi^*_n(N_k) \), we shall prove that the characters \( \chi(n-j,j),\emptyset, \chi(n-t-1,l),\emptyset \) and \( \chi(n-1-t,l),\emptyset \), for \( 1 \leq j, l \leq k - 3 \) and \( 0 \leq t \leq k - 2 \), appear in the decomposition of the \( * \)-cocharacter \( \chi^*_n(N_k) \) with multiplicity \( m_{(n-j,j),\emptyset} = k - j - 2 \), \( m_{(n-t-1,l),\emptyset} = k - l - 2 \) and \( m_{(n-1-t,l),\emptyset} = k - t - 1 \), respectively.

(i) For fixed \( 1 \leq j \leq k - 3 \), the pair of partitions \((n - j, j, \emptyset)\) and for \( 0 \leq p \leq k - j - 3 \), we consider the pair of tableaux (3.12) given in Lemma 3.13 whose corresponding highest weight vector is
\[
f_p = y_1^{n-2j-p} y_1^{(j)} y_2^{(j)} y_3^{(j)}.\]
By making the evaluation \( y_1 = I + E \) and \( y_2 = I + e_{13} + e_{2k-2,2k} \) we get
\[
f_p(y_1, y_2) = \alpha \sum_{i=0}^{k-2} \binom{n-2j-p}{i} e_{2k-j-i-2,2k} + \beta \sum_{i=0}^{p} \binom{p}{i} e_{1,3+j+i},
\]
with \( \alpha \) and \( \beta \) non-zero values. Then, for any \( 0 \leq p \leq k - j - 3 \), \( f_p \) is not a \( * \)-identity of \( N_k \). Moreover, the same evaluation shows that these \((k - j - 2)\) polynomials are linearly independent modulo Id\(^*\)(\( N_k \)). Thus \( m_{(n-j,j),\emptyset} \geq k - j - 2 \), for any \( 1 \leq j \leq k - 3 \).

(ii) Now, for fixed \( 1 \leq l \leq k - 3 \), the pair of partitions \((n - l - 1, l, \emptyset)\) and \( 0 \leq p \leq k - j - 3 \), we consider the pair of tableaux (3.13) with the following corresponding highest weight vector:
\[
g_p = y_1^{n-p-2l-1} y_1^{(l-1)} y_1^{(l-1)} y_2^{(l-1)} y_3^{(l-1)} y_3^{(l-1)} y_3^{(l-1)}.\]
Evaluating \( y_1 = I + E \), \( y_2 = E \) and \( y_3 = e_{13} + e_{2k-2,2k} \), we also get that
\[
g_p(y_1, y_2, y_3) = \alpha \sum_{i=0}^{k-2} \binom{n-2j-p}{i} e_{2k-j-i-2,2k} + \beta \sum_{i=0}^{p} \binom{p}{i} e_{1,3+j+i},
\]
with \( \alpha \) and \( \beta \) non-zero values. Thus \( g_p \), for any \( 0 \leq p \leq k - j - 3 \), is not a \( * \)-identity of \( N_k \) and these \((k-l-2)\) polynomials are linearly independent modulo Id\(^*\)(\( N_k \)). Hence we have that \( m_{(n-l-1,l),\emptyset} \geq (k - l - 2) \), for any \( 1 \leq l \leq k - 3 \).

(iii) Finally, for fixed \( 0 \leq t \leq k - 2 \), for the pair of partitions \((n - t - 1, l, \emptyset)\) and for \( 0 \leq p \leq k - j - 2 \), we consider the pair of tableaux (3.14) and its corresponding highest weight vector.
By making the evaluation $y_1 = I + E$ and $z = e_{12} - e_{2k-1,2k}$, in case $t = 0$, and $y_1 = I + E$, $y_2 = E$ and $z = e_{12} - e_{2k-1,2k}$ otherwise, we get that

$$h_p = y_1^{n-p-2r-1} \frac{y_1 \cdots y_z t}{t} y_2 \cdots y_t y_1^p.$$ 

with $\alpha$ and $\beta$ non-zero values. Thus $m_{(n-t,1,1),(1)} \geq (k-t-1)$, for any $0 \leq t < k-2$, since that $h_p$ is not a $*$-identity of $N_k$, for all $0 \leq p < k-t-2$, and these $(k-t-1)$ polynomials are linearly independent modulo $\text{Id}^*(N_k)$.

Thus we have that

$$e_{n}^*(N_k) \geq d_{(n),0} + \sum_{j=1}^{k-2} (k-j-2) [d_{(n-j,1),0} + d_{(n-j-1,1,1),0} + \sum_{j=0}^{k-3} (k-j-1) d_{(n-j-1,1),(1)}].$$

Hence, by the previous remark, $\chi_{n}^*(N_k)$ has the desired decomposition and $l_{n}^*(N_k) = \frac{3k^2 - 11k + 14}{2}$. \hfill $\Box$

We finish this section by calculating the $*$-cocharacters and $*$-colengths of $\text{var}^*(U_k)$, for all $k \geq 3$.

**Lemma 3.17.** $\chi^*_n(U_3) = \chi(n),0 + \chi(n-1,1),0 + \chi(n-2,1,1),0 + \chi(n-1),(1)$ and $l^*_n(U_3) = 4$.

**Proof.** Notice that

$$d_{(n),0} + d_{(n-1),(1)} + d_{(n-1,1),0} + d_{(n-2,1,1),0} = 1 + (n-1) + \frac{(n-1)(n-2)}{2} = e_{n}^*(U_3).$$

Then, since $m_{(n),0} = 1$, if we find a highest weight vector for each pair of partitions $((n-1),(1))$, $((n-1,1),0)$ and $(n-1,1,1,0)$ which is not a $*$-identity of $U_3$ we may conclude that $\chi_{n}^*(U_3)$ has the desired decomposition.

In fact, let $f = g^{n-1}z$ be the highest weight vector associated to the pair of partitions $((n-1),(1))$ and corresponding to the pair of tableaux:

$$\begin{pmatrix} 1 & 2 & \cdots & n-1 \\ \end{pmatrix}, \begin{pmatrix} n \\ \end{pmatrix}.$$ 

By making the evaluation $g = I$ and $z = e_{13} - e_{46}$, we get that $f = e_{13} - e_{46} \neq 0$ and, so, $f$ is not a $*$-identity of $U_3$.

Now we consider $g = [y_1,y_2] y_4^{n-2}$ the highest weight vector associated to the pair of partitions $((n-1,1),0)$ and corresponding to the pair of tableaux:

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 \\ \end{pmatrix}, \begin{pmatrix} 0 \\ \end{pmatrix}.$$ 

By making the evaluation $y_1 = I + e_{12} + e_{56}$ and $y_2 = e_{23} + e_{45}$, we get that $g = e_{13} - e_{46} \neq 0$. It shows that $g$ is not a $*$-identity of $U_3$.

Finally we consider $h = St_3(y_1,y_2,y_3) y_4^{n-3}$ the highest weight vector associated to the pair of partitions $((n-1,1^2),0)$ and corresponding to the pair of tableaux:

$$\begin{pmatrix} 1 & 2 \\ \vdots & 3 \\ \vdots & n \\ \end{pmatrix}, \begin{pmatrix} 0 \\ \end{pmatrix}.$$ 

By making the evaluation $y_1 = I$, $y_2 = e_{23} + e_{45}$ and $y_3 = e_{12} + e_{56}$, we get that $h = -e_{13} + e_{46} \neq 0$ and this says that $h$ is not a $*$-identity of $U_3$. Hence we have that $\chi_{n}^*(U_3) = \chi(n),0 + \chi(n-1,1),0 + \chi(n-2,1,1),0 + \chi(n-1),(1)$ and $l_{n}^*(U_3) = 4$. \hfill $\Box$

The proof of the next result is similar to the proof of Lemma 3.16.

**Theorem 3.18.** For $k \geq 3$, we have

1. $\chi^*_n(U_k) = \chi(n),0 + \sum_{j=1}^{k-2} (k-j-1) [\chi(n-j,j),0 + \chi(n-j-1,j,1),0] + \sum_{j=0}^{k-3} (k-j-2) \chi(n-j-1,j),(1)$.
2. $l_{n}^*(U_k) = \frac{3k^2 - 9k + 8}{2}$. 

13
4. Characterizing varieties of small *-colength

In this section we shall classify the varieties such that their sequence of *-colengths is bounded by three, for \( n \) large enough. We start by considering the algebra \( G_2^* \), the Grassmann algebra with 1 generated by the elements \( e_1, e_2 \) over \( F \) subject to the condition \( e_1e_2 + e_2e_1 = e_1^2 = e_2^2 = 0 \), and endowed with the involution * such that \( e_i^* = -e_i \), for \( i = 1, 2 \). We have the following.

**Lemma 4.1.** For the algebra \( G_2^* \) we have

1. \( \text{Id}^*(G_2^*) = \langle [y_1, y_2], [y, z], z_1z_2 + z_2z_1, z_1z_2z_3 \rangle_{T^*} \).
2. \( c_n^*(G_2^*) = 1 + n + \frac{n(n-1)}{2} \).
3. \( \chi_n^*(G_2^*) = 2 \) if \( n \) is even, \( \chi_n^*(G_2^*) = 3 \) if \( n \) is odd.

**Proof.** In [21, Lemma 10] the authors determined the \( T^* \)-ideal and computed the \( n \)-th \( * \)-dimension of the algebra \( G_2^* \). Here we shall prove that \( \chi_n^*(G_2^*) = 2 \) if \( n \) is even, \( \chi_n^*(G_2^*) = 3 \) if \( n \) is odd. We start by noticing that \( (G_2^*)^+ = \text{span}_F \{1\} \) and \( (G_2^*)^- = \text{span}_F \{e_1, e_2, e_1e_2\} \). Moreover, we have

\[
d_{\langle n \rangle, \emptyset} + d_{\langle n-1, (1) \rangle} + d_{\langle n-2, (1^2) \rangle} = 1 + n + \frac{n(n-1)}{2} = c_n^*(G_2^*).
\]

Then, since \( m_{\langle n \rangle, \emptyset} = 1 \), we just need to find a highest weight vector for each pair of partitions \( \langle (n-1), (1) \rangle \) and \( \langle (n-2), (1^2) \rangle \) which is not a \( * \)-identity of \( G_2^* \) to conclude that \( \chi_n^*(G_2^*) = 2 \) if \( n \) is even, \( \chi_n^*(G_2^*) = 3 \) if \( n \) is odd.

In fact, let \( f = y^{n-1}z_1 \) and \( g = y^{n-2}[z_1, z_2] \) be the highest weight vectors associated to the pairs of partitions \( \langle (n-1), (1) \rangle \) and \( \langle (n-2), (1^2) \rangle \) and corresponding to the pairs of tableaux, respectively:

\[
\begin{pmatrix}
1 & 2 & \ldots & n-1 & n\\
\end{pmatrix}
\]

(4.1)

By making the evaluation \( y = 1, z_1 = e_1 \) and \( z_2 = e_2 \), we get that \( f = e_1 \neq 0 \) and \( g = 2e_1e_2 \neq 0 \); then \( f \) and \( g \) are not \( * \)-identities of \( G_2^* \) and the proof is complete. \( \square \)

Next we consider the algebra \( G_2^* \oplus C_3 \) and the algebra \( G_2^* \oplus C_3 \), the Grassmann algebra with 1 generated by the elements \( e_1, e_2, e_3 \) over \( F \) subject to the condition \( e_ie_j + e_je_i = e_i^2 = e_j^2 = 0 \), for all \( i, j = 1, 2, 3 \), and endowed with the involution * such that \( e_i^* = -e_i \), for \( i = 1, 2, 3 \). The next lemma can be proved as the previous one.

**Lemma 4.2.** For the algebras \( G_2^* \) and \( G_2^* \oplus C_3 \) we have

1. \( \text{Id}^*(G_2^*) = \langle [y_1, y_2], [y, z], z_1z_2 + z_2z_1, z_1z_2z_3 \rangle_{T^*} \).
2. \( c_n^*(G_2^*) = 1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6} \).
3. \( \chi_n^*(G_2^*) = \chi_{\langle n \rangle, \emptyset} + \chi_{\langle n-1, (1) \rangle} + \chi_{\langle n-2, (1^2) \rangle} + \chi_{\langle n-3, (1^3) \rangle} \).
4. \( \text{Id}^*(G_2^* \oplus C_3)^+ = \langle [y_1, y_2], [y, z], z_1z_2z_3 \rangle_{T^*} \).
5. \( c_n^*(G_2^* \oplus C_3) = n^2 + 1 \).
6. \( \chi_n^*(G_2^* \oplus C_3) = \chi_{\langle n \rangle, \emptyset} + \chi_{\langle n-1, (1) \rangle} + \chi_{\langle n-2, (1^2) \rangle} + \chi_{\langle n-3, (1^3) \rangle} \).
7. \( l_n^*(G_2^* \oplus C_3) = 4 \).

Recall that if \( A = F + J \) is a finite dimensional algebra over \( F \) where \( J = J(A) \) is its Jacobson radical, then \( J \) can be decomposed into the direct sum of \( B \)-bimodules

\[
J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}
\]

(4.2)

where for \( i \in \{0, 1\} \), \( J_{ik} \) is a left faithful module or a 0-left module according as \( i = 1 \) or \( i = 0 \), respectively. In a similar way, \( J_{ik} \) is a right faithful module or a 0-right module according as \( k = 1 \) or \( k = 0 \), respectively. Moreover, for \( i, k, r, s \in \{0, 1\} \), \( J_{ir}J_{rs} \subseteq J_{is} \), \( J_{ik}J_{rs} = 0 \) for \( k \neq r \) and \( J_{11} = BN \) for some nilpotent subalgebra \( N \) of \( A \) commuting with \( B \) [9].

Notice that if the algebra \( A \) has an involution *, then \( J_{00} \) and \( J_{11} \) are stable under the involution whereas \( J_{01} = J_{10} \).

In what follows we use the following result.
Proposition 4.3. [20, Theorem 2] Let \( A \) be an algebra with involution over a field \( F \) of characteristic zero and suppose that \( c_n^*(A) \), \( n = 1, 2, \ldots \), is polynomially bounded. Then \( A \cong \bigoplus_i B_i \) where, for each \( i \in \{1, \ldots, m\} \), \( B_i \) is a finite-dimensional algebra with involution over \( F \) and \( \dim B_i/J(B_i) \leq 1 \), for all \( i = 1, \ldots, m \).

Now, by applying [24, Corollary 5.5] we get the following.

Theorem 4.4. Let \( A \) be an algebra with involution over a field \( F \) of characteristic zero. Then \( c_n^*(A) \), \( n = 1, 2, \ldots \), is polynomially bounded if and only if \( l_n^*(A) \leq k \), for some constant \( k \) and for all \( n \geq 1 \).

Proof. If \( c_n^*(A) \), \( n = 1, 2, \ldots \), is polynomially bounded then by Proposition 4.3, \( A \) satisfies the same \( \ast \)-identities as a finite dimensional algebra and the result follows by applying Corollary 5.5 in [24]. Conversely, suppose that \( l_n^*(A) \leq k \), for some constant \( k \) and for all \( n \geq 1 \). Then by [22] and [6], \( M \) and \( D \) do not belong to the variety generated by \( A \) since their \( \ast \)-colengths are not bounded by any constants. Then, by Theorem 3.1, \( c_n^*(A) \), \( n = 1, 2, \ldots \), is polynomially bounded. \( \square \)

Much effort has been put into the study of algebras with coengths bounded by a constant (see [4, 15, 12, 18] for the ordinary and graded cases). Here we deal with the case of algebras with involution.

Lemma 4.5. [21, Lemma 14] If \( A = F + J \) is a finite-dimensional algebra with involution where \( J = J_{00} \oplus J_{10} \oplus J_{11} \) and \( A_2 \notin \text{var}^*(B) \) then \( J_{10} = J_{01} = 0 \).

Next we study \( \ast \)-algebras of the type \( F + J_{11} \).

Lemma 4.6. Let \( B = F + J_{11} \). If \( C_i \notin \text{var}^*(B) \), for \( i \geq 2 \), then \( z^{i-1} \equiv 0 \) on \( B \).

Proof. We give a proof of the result by following closely the proof of [21, Lemma 27]. Suppose that there exists \( a \in J_{11} \) such that \( a^{-1} \neq 0 \) and consider the \( \ast \)-subalgebra \( R \) of \( B \) generated by \( 1 \) and \( a \) over \( F \). Then if \( I \) is the \( \ast \)-ideal generated by \( a \), we have that the algebra \( R = R/I \) has induced involution and \( R = \text{span}(\overline{1}, \overline{a}, \overline{a^2}, \ldots, \overline{a^{\ast-1}}) \). It is easily seen that \( \overline{R} = C_i \) through the isomorphism \( \varphi \) such that \( \varphi(\overline{1}) = e_{11} + \cdots + e_{ii} \), \( \varphi(\overline{a}) = e_{12} + \cdots + e_{i-1\, i} \). Hence \( C_i \in \text{var}^*(B) \) and we have reached a contradiction. \( \square \)

Lemma 4.7. Let \( B = F + J_{11} \).

(1) If \( U_3 \notin \text{var}^*(B) \) then \( [y_1, y_2] \equiv 0 \) on \( B \).

(2) If \( N_3 \notin \text{var}^*(B) \) then \( [y, z] \equiv 0 \) on \( B \).

Proof. Suppose, for a contradiction, that \( [y_1, y_2] \neq 0 \). Let \( a, b \in J_{11}^+ \) be such that \( [a, b] \neq 0 \) and consider the \( \ast \)-subalgebra \( R \) generated by \( 1, a, b \) over \( F \) and let \( I \) be the \( \ast \)-ideal generated by \( a^2, b^2, ab + ba \). So the \( \ast \)-algebra \( R = R/I \) is linearly generated by \( \{\overline{1}, \overline{a}, \overline{b}, \overline{ab}\} \) and we claim that \( \text{Id}^*(R) = \text{Id}^*(U_3) \). Clearly \( z_{12} \equiv 0 \) and \( [z, y] \equiv 0 \) are \( \ast \)-identities of \( R \), and so, \( \text{Id}^*(U_3) \subseteq \text{Id}^*(R) \).

Let \( f \in P_n^* \cap \text{Id}^*(R) \) a multilinear polynomial of degree \( n \). By [21, Lemma 19] we can write \( f \) (mod \( \text{Id}^*(U_3) \)) as:

\[
 f = \alpha y_1 \cdots y_n + \sum_{1 \leq i < j \leq n} \alpha_{ij} y_{i1} \cdots y_{in} - [y_i, y_j] + \sum_{i=1}^n \alpha_i y_{i1} \cdots y_{in-1},
\]

where \( i_1 < i_2 < \cdots < i_{n-2} \) and \( j_1 < j_2 < \cdots < j_{n-1} \). By making the evaluations \( y_1 = \cdots = y_n = \overline{1} \) and \( z_i = 0 \) for \( i = 1, \ldots, n \), we get \( \alpha = 0 \). Also, for a fixed \( i < j \) the evaluation \( y_i = \overline{a}, y_j = \overline{b}, y_k = \overline{1} \) for \( k \notin \{i, j\} \) and \( z_i = 0 \) for \( i = 1, \ldots, n \), gives \( \alpha_{ij} = 0 \). Finally the evaluation \( z_i = [\overline{a}, \overline{b}], y_j = \overline{1} \) for \( j \neq i \) gives \( \alpha_i = 0 \). Hence \( f \in \text{Id}^*(U_3) \) and, so, \( \text{Id}^*(R) \subseteq \text{Id}^*(U_3) \). Thus \( U_3 \in \text{var}^*(B) \) and the proof of the first part is complete.

The second part of the lemma is proved similarly. \( \square \)

Lemma 4.8. Suppose that \( B = F + J_{11} \) satisfies \( z_{12}z_2 + z_{21}z_1 \equiv 0 \). If \( z_{12}z_2z_3 \neq 0 \) then \( G_3^* \in \text{var}^*(B) \).

Proof. Consider \( a, b, c \in J_{11} \) such that \( abc \neq 0 \). Let \( R \) be the subalgebra of \( B \) generated by \( 1, a, b, c \). Since \( z_{12}z_2 + z_{21}z_1 \equiv 0 \) in \( R \) we have \( a^2 = b^2 = c^2 = 0 \) and so \( R = \text{span}\{1, a, b, c, ab, ac, bc, abc\} \). As a consequence, the correspondence

\[
 1 \mapsto 1, \ a \mapsto e_1, \ b \mapsto e_2, \ c \mapsto e_3
\]

defines an isomorphism between \( R \) and \( G_3^* \). \( \square \)
Lemma 4.9. If $B = F + J_{11}$ is such that $[z_1, z_2] \neq 0$ then $G_2^* \in \text{var}^*(B)$.

Proof. Consider $a, b \in J_{11}$ such that $[a, b] \neq 0$. Let $R$ be the subalgebra of $B$ generated by $1, a, b$ and let $I$ be the $*$-ideal generated by $a^2, b^2, ab + ba$. So the $*$-algebra $\mathcal{R} = R/I$ is linearly generated by $\{\bar{1}, \bar{a}, \bar{b}, \bar{ab}\}$. We have $\mathcal{R}$ is isomorphic to $G_2^*$ and so $G_2^* \in \text{var}^*(B)$.

Now we are in position to prove the main result of this section which allows us to classify the varieties with $*$-colengths bounded by $3$, for $n$ large enough.

Theorem 4.10. Let $A$ be an algebra with involution over a field $F$ of characteristic zero. The following conditions are equivalent.

1. $l_n^*(A) \leq 3$, for $n$ large enough.
2. $A_2, N_3, U_3, C_4, G_3^2 \oplus C_4 \notin \text{var}^*(A)$.
3. $A$ is $T^*$-equivalent to $N$ or $C \oplus N$ or $C_2 \oplus N$ or $C_3 \oplus N$, $G_3^2 \oplus N$, where $N$ is a nilpotent $*$-algebra and $C$ is a commutative non-nilpotent algebra with trivial involution.

Proof. First, notice that the condition (1) implies the condition (2) since by Lemmas 3.11, 3.17, 3.16, 4.2 and Theorem 3.2 we have that $l_n^*(A_2) = 5, l_n^*(N_3) = l_n^*(U_3) = l_n^*(G_2^2 \oplus C_4) = l_n^*(C_4) = 4$. Also, the condition (3) implies the condition (1), by Lemmas 3.14, 3.2 and 4.1.

Suppose now that $A_2, N_3, U_3, C_4, G_3^2 \oplus C_4 \notin \text{var}^*(A)$. Since $C_4 \notin \text{var}^*(D)$ and $A_2 \in \text{var}^*(M)$, it follows that $D, M \notin \text{var}^*(A)$. Hence, by Theorem 3.1, the $*$-codimensions of $A$ are polynomially bounded and by Proposition 4.3, we may assume that

$$A = B_1 \oplus \cdots \oplus B_m$$

is a direct sum of finite-dimensional $*$-algebras where either $B_i$ is nilpotent or $B_i = F + J(B_i)$.

If $B_i$ is nilpotent for all $i$, then $A$ is a nilpotent $*$-algebra and we are done in this case.

Therefore we may assume that there exists $i = 1, \ldots, m$ such that $B_i = F + J(B_i)$ and $J(B_i) = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$.

Since $A_2 \notin \text{var}^*(B_i)$, by Lemma 4.5, we have that $J_{01} = J_{10} = 0$ and so, $B_i = (F + J_{11}) \oplus J_{00}$ is a direct sum of $*$-algebras and we study $B = F + J_{11}$.

Since $N_3, U_3 \notin \text{var}^*(B)$, by Lemma 4.7, it follows that $[y_1, y_2] \equiv 0$ and $[y, z] \equiv 0$ are $*$-identities of $B$.

Now we have to consider two different cases:

1. $[z_1, z_2] \equiv 0$ on $B$
2. $[z_1, z_2] \neq 0$ on $B$.

In case (1), we have that $B \in \text{var}^*(D)$. Since $C_4 \notin \text{var}^*(B)$, by Theorem 3.3 we must have that $B$ is $T^*$-equivalent to either $C$ or $C_2$ or $C_3$.

Now assume that $[z_1, z_2] \neq 0$ on $B$. So, by Lemma 4.9, $G_2^* \in \text{var}^*(B)$. On the other hand, since $G_2^* \oplus C_3 \notin \text{var}^*(A)$ we must have that $C_3 \notin \text{var}^*(A)$. Hence, by Lemma 4.6, $z^2 \equiv 0$ on $B$ and after linearizing we get that $z_1 z_2 + z_2 z_1 \equiv 0$ on $B$. Finally, since $G_2^* \notin \text{var}^*(B)$, by Lemma 4.8, we have that $z_1 z_2 z_3 \equiv 0$.

Hence $\text{Id}^*(G_2^*) \subseteq \text{Id}^*(B)$ and it follows that $B$ is $T^*$-equivalent to $G_2^*$.

Recalling that $A = B_1 \oplus \cdots \oplus B_m$ and putting together all pieces, we get the desired conclusion.

Actually, notice that if $l_n^*(A) \leq 3$, then for $n$ large enough, $l_n^*(A)$ is always constant.

In conclusion we have the following classification: for any $*$-algebra $A$ and $n$ large enough,

1. $l_n(A) = 0$ if and only if $A \sim_T N$.
2. $l_n(A) = 1$ if and only if $A \sim_T C \oplus N$.
3. $l_n(A) = 2$ if and only if $A \sim_T C_2 \oplus N$.
4. $l_n(A) = 3$ if and only if either $A \sim_T C_3 \oplus N$ or $A \sim_T G_2^2 \oplus N$,

where $N$ is a nilpotent $*$-algebra and $C$ is a commutative non-nilpotent algebra with trivial involution.

Acknowledgment
The authors would like to thank the referee for useful suggestions.
References