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Closure properties for integral problems driven by regulated functions via convergence results

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A B S T R A C T

In this paper we give necessary and sufficient conditions for the convergence of Kurzweil–Stieltjes integrals with respect to regulated functions, using the notion of asymptotical equiintegrability. One thus generalizes several well-known convergence theorems. As applications, we provide existence and closure results for integral problems driven by regulated functions, both in single- and set-valued cases. In the particular setting of bounded variation functions driving the equations, we get features of the solution set of measure integrals problems.

1. Introduction

The role of convergence results for integrals in the theory of differential and integral equations is well-known. On the other hand, when studying a large number of problems one can notice the appearance of discontinuities in the behaviour of the functions, so we are led to the idea of working with measure driven problems, i.e.

$$x(t) = x_0 + \int_0^t f(s, x(s))dg(s) \tag{1}$$

or its multivalued counterpart,

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$$x(t) \in x_0 + \int_0^t F(s, x(s))dg(s), \tag{2}$$

where X is a Banach space, $\mathcal{P}_{cc}(X)$ is the family of all nonempty, closed and convex subsets of X, g is a real bounded variation function, $x_0 \in X$ and $f : [0,1] \times X \to X$, $F : [0,1] \times X \to \mathcal{P}_{cc}(X)$ are functions, resp. multifunctions.

There is a wide literature treating this subject (we refer to [1], [9], [13], [14] in the single-valued case and to [8], [11], [31], [37] in the set-valued setting). The motivation comes from the fact that one can thus cover the framework of usual differential problems (when g is absolutely continuous), of discrete problems (when g is a sum of step functions), of impulsive equations (for g being the sum between an absolutely continuous function and a sum of step functions), as well as retarded problems (see [1]). As proven in [13], dynamic equations on time scales and generalized differential equations can also be seen as measure differential equations.

On the other hand, it is of interest to develop an existence theory for this kind of problems in the more permissive case where the function g is only regulated (i.e. it has one-sided limits at every point) but it is not an easy task since the properties of primitives with respect to such functions are very weak (see e.g. [20] or [38]).

It is also important to have closure results for the studied problem, namely to check if when considering a sequence $(g_n)_n$ of functions converging to a function g the solutions of the equation governed by g_n is "close" (in some sense to be specified) to solutions of the equation governed by g.

To this purpose, it is necessary to have a convergence result for Stieltjes integrals of the following form:

$$\lim_{n\to\infty} \int_0^1 f_n(s)dg_n(s) = \int_0^1 f(s)dg(s)$$

and since when working with regulated functions the most appropriate integration theory is the Kurzweil–Stieltjes one, we focus in the first section of our paper (after the Preliminaries) on the matter of proving such a convergence theorem for the Kurzweil–Stieltjes integral.

Thus, we prove a necessary and sufficient assertion: the convergence holds if and only if f_n is asymptotically equiintegrable w.r.t. g_n on the unit interval. This is a concept (introduced in [2]) which encompasses that of equiintegrability, often implied when looking for the convergence of integrals. Our result generalizes [2, Theorem 8.12] where the functions are real-valued and $g_n = g$ for every $n \in \mathbb{N}$ and it is more general when compared to other results of convergence type (see Section 3).

Next, we apply the main theorem to get the existence of regulated solutions for integral equations and inclusions driven by regulated functions in general Banach spaces. In the single-valued case we apply a version of Schauder's fixed point theorem, while in the multivalued case we make use of a nonlinear alternative of Leray–Schauder type. In both situations, one of the main tools is the notion of equiregulatedness of a set of regulated functions (see [15]).

Afterwards, we focus on the closure properties of the solutions set for such problems; namely, to study if, when taking a sequence of regulated functions $(g_n)_n$ converging to a regulated function g, the solution set of the problem governed by g_n is close (in a specified sense) to the solution set of the problem governed by g. Such results are obtained via our main convergence theorem and are very important (in numerical analysis, for instance) since they allow one to study a general integral problem governed by a rough function by analysing similar problems governed by functions with much better properties.

We relate them to well-known results in literature in the case of problems governed by functions of bounded variation ([9], [18], [13], [25] in the single-valued case or [37], [31] in the set-valued setting).

2. Preliminary results

Let $(X, ||\cdot||)$ be a Banach space. For a function $u : [0,1] \to X$ the total variation will be denoted by $var(u)$ and u is said of bounded variation (or a BV function) if the total variation is finite. $BV([0,1], X)$ denotes the Banach space of functions $u : [0,1] \to X$ of bounded variation on $[0,1]$, endowed with the norm $||u||_{BV} = ||u(0)|| + var(u)$.

A function $u : [0,1] \to X$ is said to be regulated if there exist the limits $u(t+)$ and $u(s-)$ for all points $t \in [0,1)$ and $s \in (0,1]$. It is well-known that the set of discontinuities of a regulated function is at most countable, any bounded variation function is regulated, regulated functions are bounded and, if $G([0,1], X)$ is the set of regulated functions $u : [0,1] \to X$, then it is a Banach space when endowed with the norm $||u||_C = \sup_{t\in[0,1]} ||u(t)||$. If $x_0 \in X$, $B_R(x_0)$ is the open ball of radius R in $G([0,1], X)$ centred at the constant function $x(t) \equiv x_0$ and $\overline{B_R(x_0)}$ its closure. In particular, when x_0 is the origin of the space, i.e. the null function, denote by B_R the open ball of radius R in $G([0,1], X)$ centered at the origin of the space and by $\overline{B_R}$ its closure.

Let us recall some basic facts from the theory of Kurzweil–Stieltjes integration in Banach spaces, which is a particular case of Kurzweil integration ([22]).

A partition of $[0,1]$ is a finite collection of pairs $\{(c_i, [t_{i-1}, t_i]), i = 1, \dots, l\}$, where $[t_{i-1}, t_i], i = 1, ..., l$ are non-overlapping intervals of $[0,1]$, $c_i \in [t_{i-1}, t_i]$, $i = 1, \dots, l$ and $\cup_{i=1}^{l} [t_{i-1}, t_i] = [0,1]$. A gauge δ is a positive function on $[0,1]$. For a given δ we say that a partition is δ-fine if $[t_{i-1}, t_i] \subset (c_i - \delta(c_i), c_i + \delta(c_i))$, $i = 1, \dots, l$.

Definition 1. A function $f : [0,1] \to X$ is said to be Kurzweil–Stieltjes integrable (briefly KS-integrable) w.r.t. $g : [0,1] \to \mathbf{R}$ on $[0,1]$ if there exists a vector $w \in X$, such that for every $\varepsilon > 0$ there exists a gauge δ_ε s.t.

$$\left\| \sum_{i=1}^{l} f(c_i)(g(t_i) - g(t_{i-1})) - w \right\| < \varepsilon$$

for any δ_ε-fine partition $\{(c_i, [t_{i-1}, t_i]), i = 1, \dots, l\}$ of $[0,1]$.

We set $w := (KS)\int_0^1 f dg$, or simply, since it is the only integral we consider and no confusion can arise, $w := \int_0^1 f dg$. This definition generalizes the usual definition of the Kurzweil–Henstock integral in which the function $g(t) = t$. The KS-integral has the usual properties of linearity, additivity with respect to adjacent intervals and the KS-integrability is preserved on all sub-intervals of $[0,1]$; the function $t \longleftrightarrow \int_0^t f dg$ is called the KS-primitive of f w.r.t. g on $[0,1]$. We recall that, for $X = \mathbb{R}^n$, if g is a left-continuous function of bounded variation, the corresponding Kurzweil–Stieltjes integral is equivalent to the Ward–Perron–Stieltjes integral (see [22, Theorem 1.2.1]), and that the Lebesgue–Stieltjes integrability, when defined over subset of $[0,1]$, implies the Ward–Perron–Stieltjes integrability ([28]); but the converse is not true.

The Kurzweil-type integrals have been extensively used in many papers on differential or integral equations (such as, in [32] or [38], see also [10], [13], [14], [19] or [29]).

Definition 2. A sequence $(f_n)_{n\in\mathbb{N}}$ is said to be KS-equiintegrable w.r.t. $(g_n)_n$ on $[0,1]$ if the integral $\int_0^1 f_n dg_n$ exists for all $n \in \mathbb{N}$ and for every $\varepsilon > 0$ there exists a gauge δ_ε s.t.

$$\left\| \sum_{i=1}^{l} f_n(c_i)(g_n(t_i) - g_n(t_{i-1})) - \int_0^1 f_n dg_n \right\| < \varepsilon$$

for any δ_ε-fine partition $\{(c_i, [t_{i-1}, t_i]), i = 1, \dots, l\}$ of $[0,1]$ and any $n \in \mathbb{N}$.

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The following property of the indefinite Kurzweil–Stieltjes integral implies that we shall obtain regulated solutions.

Proposition 3. *([38, Proposition 2.3.16] and [33]) Let $g : [0,1] \to \mathbb{R}$ and $f : [0,1] \to X$ be KS-integrable w.r.t. g.*
i) If g is regulated, then so is the primitive $h : [0,1] \to X$, $h(t) = \int_0^t f dg$ and for every $t \in [0,1]$,

$$h(t^+) - h(t) = f(t)\left[g(t^+) - g(t)\right] \quad \text{and} \quad h(t) - h(t^-) = f(t)\left[g(t) - g(t^-)\right].$$

ii) If g is of bounded variation and f is bounded, then h is of bounded variation.

Definition 4. ([16]) A set $\mathcal{A} \subset G([0,1], X)$ is said to be equiregulated if for every $\varepsilon > 0$ and every $t_0 \in [0,1]$ there exists $\delta > 0$ such that for all $x \in \mathcal{A}$:
i) for any $t_0 - \delta < t' < t_0$: $\|x(t') - x(t_0-)\| < \varepsilon$;
ii) for any $t_0 < t'' < t_0 + \delta$: $\|x(t'') - x(t_0+)\| < \varepsilon$.

In the sequel we will use the following

Theorem 5. *([16, Theorem 5.1]) If an equiregulated sequence converges pointwise, then it converges uniformly towards its limit.*

For applications, the following auxiliary result will be important.

Lemma 6. *Let $(h_\alpha)_{\alpha \in A}$ be a pointwise bounded family of X-valued functions on $[0,1]$, KS-equiintegrable w.r.t. the equiregulated family of real functions $(g_\alpha)_{\alpha \in A}$. Then the family $\left(\int_0^\cdot h_\alpha(s) dg_\alpha(s)\right)_{\alpha \in A}$ is equiregulated.*

Proof. We shall prove the condition in the definition of equiregularity only for the left limit (for the right limit the reasoning is similar).

Let $\varepsilon > 0$ be fixed. There is a gauge δ_ε such that for each δ_ε-fine partition $\{(c_i, [t_{i-1}, t_i], i = 1, ..., l\}$ of $[0,1]$,

$$\left\| \sum_{i=1}^{l} h_\alpha(c_i)(g_\alpha(t_i) - g_\alpha(t_{i-1})) - \int_0^1 h_\alpha dg_\alpha \right\| < \frac{\varepsilon}{2}, \ \forall \alpha \in A.$$

Fix $t_0 \in (0,1]$. There exists $M > 0$ such that $\|h_\alpha(t_0)\| \le M$ for every $\alpha \in A$.

On the other hand, as $(g_\alpha)_{\alpha \in A}$ is equiregulated, there exist $\bar{\delta}_\varepsilon > 0$ such that

$$\left|g_\alpha(t') - g_\alpha(t_0^-)\right| \le \frac{\varepsilon}{2M}, \ \forall \alpha \in A,$$

whenever $t_0 - \bar{\delta}_\varepsilon < t' < t_0$.

We shall prove that $\delta'_\varepsilon = \min(\delta_\varepsilon(t_0), \bar{\delta}_\varepsilon)$ is such that for every $t_0 - \delta'_\varepsilon < t' < t_0$:

$$\left\| \int_0^{t'} h_\alpha dg_\alpha - \int_0^{t_0^-} h_\alpha dg_\alpha \right\| < \varepsilon, \ \forall\, \alpha \in A.$$

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Indeed, as in the proof of [32, Theorem 1.16], write:

$$\int_0^{t'} h_\alpha dg_\alpha - \int_0^{t_0} h_\alpha dg_\alpha \tag{3}$$
$$= h_\alpha(t_0)(g_\alpha(t') - g_\alpha(t_0))$$
$$+ \left(\int_0^{t'} h_\alpha dg_\alpha - \int_0^{t_0} h_\alpha dg_\alpha - h_\alpha(t_0)(g_\alpha(t') - g_\alpha(t_0)) \right).$$

By Proposition 3:

$$\int_0^{t_0^-} h_\alpha dg_\alpha - \int_0^{t_0} h_\alpha dg_\alpha = h_\alpha(t_0)(g_\alpha(t_0^-) - g_\alpha(t_0)). \tag{4}$$

We subtract (3) and (4). One gets

$$\int_0^{t'} h_\alpha dg_\alpha - \int_0^{t_0^-} h_\alpha dg_\alpha$$
$$= h_\alpha(t_0)(g_\alpha(t') - g_\alpha(t_0^-))$$
$$+ \left(\int_0^{t'} h_\alpha dg_\alpha - \int_0^{t_0} h_\alpha dg_\alpha - h_\alpha(t_0)(g_\alpha(t') - g_\alpha(t_0)) \right).$$

Taking into account that $\{t_0, [t', t_0]\}$ is a δ_ε-fine (partial) partition of $[0, 1]$ and applying the analogue of Saks–Henstock Lemma for equiintegrable families ([33, Lemma 16] that can be straight away adapted for KS integral), we can make the last term, in norm, less than $\frac{\varepsilon}{2}$ for all $\alpha \in A$ and so,

$$\left\| \int_0^{t'} h_\alpha dg_\alpha - \int_0^{t_0^-} h_\alpha dg_\alpha \right\| < M \frac{\varepsilon}{2M} + \frac{\varepsilon}{2} = \varepsilon$$

for any $\alpha \in A$ and t' with $t_0 - \delta'_\varepsilon < t' < t_0$. □

In particular, when $g_\alpha = g$ for all $\alpha \in A$, we get the following result.

Corollary 7. *Let $(h_\alpha)_{\alpha \in A}$ be a pointwise bounded family of functions, KS-equiintegrable w.r.t. a regulated function g. Then the family $\left(\int_0^{\cdot} h_\alpha dg\right)_{\alpha \in A}$ is equiregulated.*

Remark 8. The preceding corollary generalizes [30, Proposition 3.4] where the stronger notion of variational Henstock-integrability ([23], [24]) was used instead.

A family $\mathcal{A}$ of X-valued functions defined on the unit interval is said to be pointwise relatively compact if for each $t \in [0, 1]$, $\mathcal{A}(t) \subset X$ is relatively compact.

We refer the reader to [7,21] for notions of set-valued analysis. We denote by $\mathcal{P}_{kc}(X)$ the subset of $\mathcal{P}_{cc}(X)$ consisting in all non-empty compact convex subsets of X. We endow $\mathcal{P}_{cc}(X)$ and $\mathcal{P}_{kc}(X)$ with the Hausdorff–Pompeiu distance; it is well-known that they become complete metric spaces. Any map $\Gamma : X \to \mathcal{P}_{cc}(X)$ is called a multifunction. A function $f : X \to X$ is called a selection of Γ if $f(x) \in \Gamma(x)$, for all $x \in X$.

A multifunction $\Gamma : X \to \mathcal{P}_{kc}(X)$ is upper semicontinuous at a point x_0 if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that the excess of $\Gamma(x)$ over $\Gamma(x_0)$ (in the sense of Hausdorff) is less than ε whenever $\|x - x_0\| < \delta_\varepsilon$: $\Gamma(x) \subset \Gamma(x_0) + \varepsilon B$, where B is the unit ball in X.

3. Convergence results

Let us introduce the following notion.

Definition 9. A sequence $(f_n)_{n\in\mathbb{N}}$ is said to be asymptotically KS-equiintegrable w.r.t. $(g_n)_{n\in\mathbb{N}}$ on $[0,1]$ if:

i) f_n is KS integrable w.r.t. g_n on $[0,1]$ for every $n \in \mathbb{N}$;

ii) for every $\varepsilon > 0$ there exists a gauge δ_ε s.t. for any δ_ε-fine partition $\mathcal{P} = \{(c_i, [t_{i-1}, t_i]), i = 1, \dots, l\}$ of $[0,1]$ there exists $N_{\mathcal{P}} \in \mathbb{N}$ s.t.

$$\left\| \sum_{i=1}^{l} f_n(c_i)(g_n(t_i) - g_n(t_{i-1})) - \int_0^1 f_n dg_n \right\| < \varepsilon,\ \forall n \geq N_{\mathcal{P}}. \tag{5}$$

Here is the main result of the paper.

Theorem 10. *Let $f_n : [0,1] \to X$ converge pointwise to $f : [0,1] \to X$ and $g_n : [0,1] \to \mathbb{R}$ converge pointwise to $g : [0,1] \to \mathbb{R}$.*

Then the following conditions are equivalent:
1) the sequence $(f_n)_n$ is asymptotically KS-equiintegrable w.r.t. $(g_n)_n$ on $[0,1]$;
2) f_n is integrable w.r.t g_n on $[0,1]$ for each $n \in \mathbb{N}$, f is KS-integrable w.r.t. g on $[0,1]$ and

$$\lim_{n\to\infty} \int_0^1 f_n dg_n = \int_0^1 f dg. \tag{6}$$

Proof. $1) \Rightarrow 2)$

Let us first show that the sequence $\left(\int_0^1 f_n dg_n\right)_n$ is Cauchy (therefore convergent).

Fix $\varepsilon > 0$. From the asymptotical KS-equiintegrability hypothesis, there exists a gauge δ_ε such that for any δ_ε-fine partition $\mathcal{P} = \{(c_i, [t_{i-1}, t_i]), i = 1, \dots, l\}$ of $[0,1]$ there exists $N_{\mathcal{P}} \in \mathbb{N}$ s.t. (5) is satisfied.

Fix now a δ_ε-fine partition $\mathcal{P}_\varepsilon = \{c_i, [t_{i-1}, t_i], i = 1, ..., l\}$.

As the partition is fixed and the sequence $(g_n)_n$ is pointwise bounded, there exists M_ε s.t. $\sum_{i=1}^{l} |g_n(t_i) - g_n(t_{i-1})| \leq M_\varepsilon$ for each n and one can choose $N_\varepsilon^1 \in \mathbb{N}$ s.t. for any $m, n \geq N_\varepsilon^1$,

$$\|f_n(c_i) - f_m(c_i)\| \leq \frac{\varepsilon}{M_\varepsilon},\ \forall i \in \{1, ..., l\} \tag{7}$$

whence

$$\left\| \sum_{i=1}^{l} (f_n(c_i) - f_m(c_i))(g_n(t_i) - g_n(t_{i-1})) \right\| \leq \varepsilon. \tag{8}$$

On the other hand,

$$\left\|\sum_{i=1}^{l} f_m(c_i)((g_n-g_m)(t_i)-(g_n-g_m)(t_{i-1}))\right\|$$
$$\le \sum_{i=1}^{l} \|f_m(c_i)\| \left(|(g_n-g_m)(t_i)|+|(g_n-g_m)(t_{i-1})|\right).$$

Now since $(f_n)_n$ is pointwise bounded and the partition is fixed, there exists $\overline{M}_\varepsilon$ s.t. $\sum_{i=1}^{l}\|f_m(c_i)\| \le \overline{M}_\varepsilon$ and one can find $N_\varepsilon^2 \in \mathbb{N}$ s.t. for any $m,n \ge N_\varepsilon^2$,

$$\max_{i=1}^{l}\left(|(g_n-g_m)(t_i)|\right) \le \frac{\varepsilon}{2\overline{M}_\varepsilon},$$

and so,

$$\left\|\sum_{i=1}^{l} f_m(c_i)((g_n-g_m)(t_i)-(g_n-g_m)(t_{i-1}))\right\| \le \varepsilon. \tag{9}$$

Then for every $m,n \ge N_\varepsilon = \max(N_\varepsilon^1, N_\varepsilon^2, N_{\mathcal{P}_\varepsilon})$, by (5), (7), (8) and (9),

$$\begin{aligned}\left\|\int_0^1 f_n dg_n - \int_0^1 f_m dg_m\right\| &\le \left\|\int_0^1 f_n dg_n - \sum_{i=1}^{l} f_n(c_i)(g_n(t_i)-g_n(t_{i-1}))\right\| \\ &+ \left\|\sum_{i=1}^{l} f_n(c_i)(g_n(t_i)-g_n(t_{i-1})) - \sum_{i=1}^{l} f_m(c_i)(g_m(t_i)-g_m(t_{i-1}))\right\| \\ &+ \left\|\int_0^1 f_m dg_m - \sum_{i=1}^{l} f_m(c_i)(g_m(t_i)-g_m(t_{i-1}))\right\| \\ &< 2\varepsilon + \left\|\sum_{i=1}^{l} (f_n(c_i)-f_m(c_i))(g_n(t_i)-g_n(t_{i-1}))\right\| \\ &+ \left\|\sum_{i=1}^{l} f_m(c_i)((g_n-g_m)(t_i)-(g_n-g_m)(t_{i-1}))\right\| \\ &\le \sum_{i=1}^{l} \|(f_n(c_i)-f_m(c_i))\|\, |g_n(t_i)-g_n(t_{i-1})| + 3\varepsilon \le 4\varepsilon,\end{aligned}$$

and therefore the sequence $\left(\int_0^1 f_n dg_n\right)_n$ is Cauchy, so convergent. Let us denote by I its limit.

Let us now prove that f is KS-integrable w.r.t. g and that $\int_0^1 f dg = I$.

Indeed, let $\varepsilon > 0$. Choose a gauge δ_ε from the asymptotical KS-integrability assumption, and let $\mathcal{P} = \{(c_i, [t_{i-1}, t_i]), i = 1, \dots, l\}$ be any δ_ε-fine partition of $[0,1]$.

Then there exists $N_{\mathcal{P}} \in \mathbb{N}$ s.t.

$$\left\|\sum_{i=1}^{l} f_n(c_i)(g_n(t_i)-g_n(t_{i-1})) - \int_0^1 f_n dg_n\right\| < \frac{\varepsilon}{3},\ \forall n \ge N_{\mathcal{P}}.$$

Therefore passing to the limit for $n \to \infty$ we get

$$\left\| \sum_{i=1}^{l} f(c_i)(g(t_i) - g(t_{i-1})) - I \right\| < \varepsilon$$

and the integrability of f w.r.t. g and the equality (6) are proved.

2) $\Rightarrow$ 1)

Let $\varepsilon > 0$. Since f is integrable w.r.t. g there exists a gauge δ_ε s.t.

$$\left\| \sum_{i=1}^{l} f(c_i)(g(t_i) - g(t_{i-1})) - \int_0^1 f dg \right\| < \frac{\varepsilon}{3} \tag{10}$$

for any δ_ε-fine partition $\{c_i, [t_{i-1}, t_i], i = 1, ..., l\}$.

Fix now a δ_ε-fine partition $\mathcal{P}_\varepsilon = \{c_i, [t_{i-1}, t_i], i = 1, ..., l\}$. We have

$$\begin{aligned}
&\left\| \sum_{i=1}^{l} f_n(c_i)(g_n(t_i) - g_n(t_{i-1})) - \int_0^1 f_n dg_n \right\| \\
&\leq \left\| \sum_{i=1}^{l} (f_n - f)(c_i)(g_n(t_i) - g_n(t_{i-1})) \right\| \\
&+ \left\| \sum_{i=1}^{l} f(c_i)((g_n - g)(t_i) - (g_n - g)(t_{i-1})) \right\| \\
&+ \left\| \sum_{i=1}^{l} f(c_i)(g(t_i) - g(t_{i-1})) - \int_0^1 f dg \right\| \\
&+ \left\| \int_0^1 f_n dg_n - \int_0^1 f dg \right\|.
\end{aligned}$$

As previously seen, there exists $N^1_{\mathcal{P}_\varepsilon} \in \mathbb{N}$ s.t. for any $n \geq N^1_{\mathcal{P}_\varepsilon}$,

$$\left\| \sum_{i=1}^{l} (f_n - f)(c_i)(g_n(t_i) - g_n(t_{i-1})) \right\| + \left\| \sum_{i=1}^{l} f(c_i)((g_n - g)(t_i) - (g_n - g)(t_{i-1})) \right\| \leq \frac{\varepsilon}{3}.$$

Besides, one can find $\overline{N}_\varepsilon \in \mathbb{N}$ s.t.

$$\left\| \int_0^1 f_n dg_n - \int_0^1 f dg \right\| < \frac{\varepsilon}{3}, \ \forall n \geq \overline{N}_\varepsilon.$$

Using the relation (10) we get that for every $n \geq N_{\mathcal{P}_\varepsilon} = \max(N^1_{\mathcal{P}_\varepsilon}, \overline{N}_\varepsilon)$,

$$\left\| \sum_{i=1}^{l} f_n(c_i)(g_n(t_i) - g_n(t_{i-1})) - \int_0^1 f_n dg_n \right\| < \varepsilon$$

so the sequence is asymptotically KS-equiintegrable w.r.t. g_n. $\square$

Remark 11. Our result is "the best possible" from the point of view that it asserts sufficiency and necessity. It is the analogue for the Kurzweil integration theory of [36, Theorem 2.8] (available for Lebesgue integrals). In particular, for real-valued functions f_n and for identical functions $g_n(s) = g(s) = s$ for all $n \in \mathbb{N}$, this result for Kurzweil integral can be found in [2, Theorem 8.12].

Remark 12. The notion of asymptotical KS-equiintegrability is more general than that of KS-equiintegrability. In [5, p. 295] there is an example (in the particular case $g_n(t) = t$ for any $n \in \mathbb{N}$) of a sequence of functions KS-integrable, pointwise convergent to the null function such that the sequence of its primitives converges to the primitive of the null function (therefore, as a consequence of Theorem 10, it is asymptotically KS-integrable); though, the sequence of its primitives is not uniformly-ACG^*, therefore the sequence is not KS-equiintegrable (see [17, Chapter 13]).

Corollary 13. *Let $f_n : [0,1] \to X$ converge pointwise to $f : [0,1] \to X$ and $g_n : [0,1] \to \mathbb{R}$ converge pointwise to $g : [0,1] \to \mathbb{R}$.*

If $(f_n)_n$ is KS-equiintegrable w.r.t. $(g_n)_n$ on $[0,1]$, then f is KS-integrable w.r.t. g on $[0,1]$ and

$$\lim_{n\to\infty} \int_0^t f_n dg_n = \int_0^t f dg, \ \forall t \in [0,1]. \tag{11}$$

Proof. Since each KS-equiintegrable sequence is also asymptotically KS-equiintegrable, by Theorem 10 we get that f is KS-integrable w.r.t. g on $[0,1]$ and that equality (11) holds for $t = 1$.

Repeating the proof as in case of the Henstock–Kurzweil integral with g(t)=t (see [34, Theorem 3.5.5]), we obtain that $(f_n)_n$ is KS-equiintegrable w.r.t. $(g_n)_n$ on $[0,t]$, for all $t \in [0,1]$. Therefore equality (11) holds. □

Remark 14. Corollary 13 holds true in particular if $g_n = g$ for all $n \in \mathbb{N}$ and generalizes also [4, Theorem 6.1] (where $g_n = g$ for any $n \in \mathbb{N}$ and g is an ACG^*-function).

We are checking next that Theorems I 4.17, I 4.18 in [35] are generalized as well.

Proposition 15. *Let $f_n : [0,1] \to X$ converge uniformly to a bounded function $f : [0,1] \to X$ and $g_n : [0,1] \to \mathbb{R}$ converge in variation to a BV function g.*

Assume that the integrals $\int_0^1 f_n(s)dg_n(s)$ exist for all $n \in \mathbb{N}$. Then $(f_n)_n$ is KS-equiintegrable (and hence asymptotically KS-equiintegrable) with respect to $(g_n)_n$.

Proof. We want to show that for $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that for every partition and for every $n, m \geq N_\varepsilon$

$$\left\| \sum_{i=1}^{l} f_n(c_i)(g_n(t_i) - g_n(t_{i-1})) - \sum_{i=1}^{l} f_m(c_i)(g_m(t_i) - g_m(t_{i-1})) \right\| < \varepsilon. \tag{12}$$

Since $(f_n)_n$ is uniformly convergent to a bounded function, it is uniformly bounded by a constant M, also as $g_n : [0,1] \to \mathbb{R}$ converge in variation to a BV function g, we can assume that $var(g_n) < M$. Fix $\varepsilon > 0$, we can find $N_\varepsilon \in \mathbb{N}$ such that for every $n, m \geq N_\varepsilon$

$$\|f_n(t) - f_m(t)\| < \frac{\varepsilon}{2M}, \ \forall t \in [0,1] \quad \text{and} \quad var(g_n - g_m) < \frac{\varepsilon}{2M}.$$

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Let $\mathcal{P} = \{(c_i, [t_{i-1}, t_i]), i = 1, \dots, l\}$ be any partition of $[0,1]$, then

$$\begin{aligned}
&\left\| \sum_{i=1}^{l} f_n(c_i)(g_n(t_i) - g_n(t_{i-1})) - \sum_{i=1}^{l} f_m(c_i)(g_m(t_i) - g_m(t_{i-1})) \right\| \\
&\le \left\| \sum_{i=1}^{l} f_n(c_i)(g_n(t_i) - g_n(t_{i-1})) - \sum_{i=1}^{l} f_n(c_i)(g_m(t_i) - g_m(t_{i-1})) \right\| \\
&+ \left\| \sum_{i=1}^{l} f_n(c_i)(g_m(t_i) - g_m(t_{i-1})) - \sum_{i=1}^{l} f_m(c_i)(g_m(t_i) - g_m(t_{i-1})) \right\| \\
&= \left\| \sum_{i=1}^{l} f_n(c_i)[(g_n(t_i) - g_m(t_i)) - (g_n(t_{i-1}) - g_m(t_{i-1}))] \right\| \\
&+ \left\| \sum_{i=1}^{l} (f_n(c_i) - f_m(c_i))(g_m(t_i) - g_m(t_{i-1})) \right\| \\
&< M \sum_{i=1}^{l} |(g_n(t_i) - g_m(t_i)) - (g_n(t_{i-1}) - g_m(t_{i-1}))| \\
&+ \frac{\varepsilon}{2M} \sum_{i=1}^{l} |g_m(t_i) - g_m(t_{i-1})| \\
&\le M var(g_n - g_m) + \frac{\varepsilon}{2M} var(g_m) < \varepsilon.
\end{aligned}$$

Therefore condition (12) is satisfied. Now doing the same calculation as in [17, exercise 13.10], from the assumption that each f_n is KS-integrable w.r.t. g_n and from (12) we get that the sequence $(f_n)_n$ is KS-equiintegrable w.r.t. $(g_n)_n$ and then $(f_n)_n$ is asymptotically KS-integrable with respect to $(g_n)_n$. □

Following the same steps as in the previous Proposition 15 and using the idea of the proof of [18, Lemma 2.2], we get to the same conclusion by weakening the convergence assumptions on g_n in the case f_n are regulated.

Proposition 16. *Let $f_n : [0,1] \to X$ be a sequence of regulated functions which converges uniformly to $f : [0,1] \to X$ and let $g_n : [0,1] \to \mathbb{R}$ converge uniformly to a BV function g. Assume that $var(g_n) \le M$ for every $n \in \mathbb{N}$.*

Then the integrals $\int_0^1 f_n dg_n$ exist and $(f_n)_n$ is KS-equiintegrable (hence asymptotically KS-equiintegrable) with respect to $(g_n)_n$.

Proof. Again we shall show that for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that for every partition and for every $n, m \ge N_\varepsilon$

$$\left\| \sum_{i=1}^{l} f_n(c_i)(g_n(t_i) - g_n(t_{i-1})) - \sum_{i=1}^{l} f_m(c_i)(g_m(t_i) - g_m(t_{i-1})) \right\| < \varepsilon.$$

Fix $\varepsilon > 0$. Since $(f_n)_n$ is uniformly convergent to f which is regulated, there exists a step function $u : [0,1] \to X$ and there exists $N_\varepsilon > 0$ such that for any $n > N_\varepsilon$,

$$\|f - u\|_C < \varepsilon \text{ and } \|f_n - u\|_C < \varepsilon.$$

With the remark that u has bounded variation (see [6], page 237), N_ε can be chosen such that

$$\|g_n - g\|_C < \frac{\varepsilon}{\|u\|_{BV}},\ \forall n > N_\varepsilon.$$

Let now $\mathcal{P} = \{(c_i, [t_{i-1}, t_i]), i = 1, \ldots, l\}$ be any partition of $[0,1]$. Then for any $n, m > N_\varepsilon$,

$$\begin{aligned}
&\left\| \sum_{i=1}^{l} f_n(c_i)(g_n(t_i) - g_n(t_{i-1})) - \sum_{i=1}^{l} f_m(c_i)(g_m(t_i) - g_m(t_{i-1})) \right\| \\
&\le \left\| \sum_{i=1}^{l} (f_n(c_i) - u(c_i))(g_n(t_i) - g_n(t_{i-1})) \right\| + \left\| \sum_{i=1}^{l} (f_m(c_i) - u(c_i))(g_m(t_i) - g_m(t_{i-1})) \right\| \\
&+ \left\| \sum_{i=1}^{l} u(c_i) \left[(g_n - g_m)(t_i)) - (g_n - g_m)(t_{i-1})) \right] \right\| \\
&\le \left\| \sum_{i=1}^{l} (f_n(c_i) - u(c_i))(g_n(t_i) - g_n(t_{i-1})) \right\| + \left\| \sum_{i=1}^{l} (f_m(c_i) - u(c_i))(g_m(t_i) - g_m(t_{i-1})) \right\| \\
&+ \left\| \sum_{i=1}^{l} u(c_i) \left[(g_n - g)(t_i)) - (g_n - g)(t_{i-1})) \right] \right\| + \left\| \sum_{i=1}^{l} u(c_i) \left[(g_m - g)(t_i)) - (g_m - g)(t_{i-1})) \right] \right\|.
\end{aligned}$$

Then as in the proof of [38, Lemma 2.3.6],

$$\begin{aligned}
&\left\| \sum_{i=1}^{l} f_n(c_i)(g_n(t_i) - g_n(t_{i-1})) - \sum_{i=1}^{l} f_m(c_i)(g_m(t_i) - g_m(t_{i-1})) \right\| \\
&\le \|f_n - u\|_C \cdot var(g_n) + \|f_m - u\|_C \cdot var(g_m) + 2\|g_n - g\|_C \cdot \|u\|_{BV} + 2\|g_m - g\|_C \cdot \|u\|_{BV} \\
&< (2M + 4)\varepsilon
\end{aligned}$$

and from here the proof goes as in Proposition 15. □

4. Existence and closure results in the single-valued setting

We shall start by giving an existence result, using the following generalization of Schauder's fixed point theorem.

Theorem 17. *Let $\mathcal{K}$ be a closed convex set in a Banach space and assume that $T : \mathcal{K} \to \mathcal{K}$ is a continuous mapping such that $T(\mathcal{K})$ is a relatively compact subset of $\mathcal{K}$. Then T has a fixed point.*

Lemma 18. *Let $h : [0,1] \to \mathbb{R}$ be a regulated function, $x_0 \in X$ and $f : [0,1] \times X \to X$ satisfy the condition that $f(\cdot, x(\cdot))$ is KS-integrable w.r.t. h for any $x \subset G([0,1], X)$. Suppose that for some $R > 0$, the family*

$$\left\{ \int_0^{\cdot} f(s, x(s))dh(s),\ x \in \overline{B_R(x_0)} \right\}$$

is equiregulated.

Then one can find a constant $M_R > 0$ such that for any $x \in \overline{B_R(x_0)}$,

$$\left\| \int_0^{\cdot} f(s, x(s))dh(s) \right\|_C \leq M_R.$$

Proof. The collection is equiregulated and $\left\{ \int_0^{\cdot} f(s, x(s))dh(s),\ x \in \overline{B_R(x_0)} \right\}$ is bounded. Thus, an application of [16, Proposition 5.7] gives us the uniform boundedness. □

Theorem 19. *Let $h : [0, 1] \to \mathbb{R}$ be a regulated function and $f : [0, 1] \times X \to X$ satisfy the following assumptions:*
i) $f(s, \cdot)$ is continuous, for each $s \in [0, 1]$ and for any regulated function $x : [0, 1] \to X$, $f(\cdot, x(\cdot))$ is KS-integrable w.r.t. h;
ii1) for any pointwise convergent sequence $(x_n)_n \subset G([0, 1], X)$ bounded in the norm $\| \cdot \|_C$, the sequence $\{f(\cdot, x_n(\cdot)), n \in \mathbb{N}\}$ is asymptotically KS-equiintegrable w.r.t. h on $[0, t]$ for every $t \in [0, 1]$ and
ii2) for any $R > 0$, the subset of $G([0, 1], X)$

$$\left\{ \int_0^{\cdot} f(s, x(s))dh(s),\ x \in \overline{B_R(x_0)} \right\}$$

is equiregulated and pointwise relatively compact;
iii) There exists $R_0 > 0$ such that the constant M_{R_0} whose existence is stated in Lemma 18 satisfies $M_{R_0} \leq R_0$.

Then the integral measure equation

$$x(t) = x_0 + \int_0^t f(s, x(s))dh(s)$$

has regulated solutions with $\|x - x_0\|_C \leq R_0$.

Proof. Let $\mathcal{K} = \overline{B_{R_0}(x_0)}$. It is nonempty, closed and convex.

Define now the operator $T : \mathcal{K} \to \mathcal{K}$ by

$$(Tx)(t) = x_0 + \int_0^t f(s, x(s))dh(s)$$

and prove that it satisfies the hypothesis of Schauder's fixed point theorem.

Obviously, for any $x \in \mathcal{K}$, $Tx \in \mathcal{K}$ since it is regulated (Proposition 3) and, by Lemma 18,

$$\left\| \int_0^t f(s, x(s))dh(s) \right\| \leq M_{R_0} \leq R_0,\ \forall t \in [0, 1].$$

First, let us check that $T(\mathcal{K})$ is relatively compact. Since it is equiregulated and for each $t \in [0, 1]$, $T\mathcal{K}(t)$ is relatively compact in X by hypothesis ii2), the relative compactness of $T(\mathcal{K})$ as a subset of $G([0, 1], X)$ is a consequence of [16, Theorem 6.2].

We have to prove the continuity of T. Let $x_n \in \mathcal{K}$ uniformly converge to x. Then $f(s, x_n(s)) \to f(s, x(s))$ for each $s \in [0, 1]$ and it is asymptotically KS-equiintegrable w.r.t. h on $[0, t]$ for every $t \in [0, 1]$ by assumption ii1). By Theorem 10,

$$\int_0^t f(s, x_n(s))dh(s) \to \int_0^t f(s, x(s))dh(s).$$

At the same time, this sequence is equiregulated, therefore by [16, Theorem 5.1] it converges uniformly and so, T is continuous.

In conclusion, the conditions of the fixed point result are checked and so, we get the existence of regulated solutions. □

Remark 20. Let us remark that the equiregulatedness hypothesis ii2) is a natural one, it was also used in some of the previous papers dealing with integral equations in the framework of Kurzweil integrals; e.g. hypothesis (2) of [14, Theorem 7.1] implies the equiregulatedness of $\left\{\int_0^{\cdot} f(s, x(s))dh(s),\ x \in \overline{B_R(x_0)}\right\}$, by [16, Proposition 5.9].

Let us present in the sequel two applications of our result, to very general problems where the assumptions of Theorem 19 can easily be checked.

Proposition 21. *Let $h : [0, 1] \to \mathbb{R}$ be left-continuous non-decreasing and $f : [0, 1] \times \mathbb{R}^k \to \mathbb{R}^k$ satisfy the following assumptions:*
1) f is measurable w.r.t. the first argument, continuous w.r.t. the second one and for any $R > 0$ there exists a KS-integrable (w.r.t. h) function $\overline{M}_R : [0, 1] \to \mathbb{R}$ such that

$$\|f(s, x(s))\| \le \overline{M}_R(s),\ \forall s \in [0, 1], x \in \overline{B_R(x_0)};$$

2) for any pointwise convergent sequence $(x_n)_n \subset G([0, 1], \mathbb{R}^k)$ bounded in the norm $\|\cdot\|_C$, the sequence $\{f(\cdot, x_n(\cdot)), n \in \mathbb{N}\}$ is asymptotically KS-equiintegrable w.r.t. h on $[0, t]$ for every $t \in [0, 1]$.

If one can find $R_0 > 0$ such that

$$\int_0^1 \overline{M}_{R_0}(s)dh(s) \le R_0,$$

then the problem has regulated solutions with $\|x - x_0\|_C \le R_0$.

Proof. It suffices to check the hypothesis of the previous theorem.

Thus, for any regulated function $x : [0, 1] \to \mathbb{R}^k$, denoting by $R = \|x - x_0\|_C$, the function $f(\cdot, x(\cdot))$ is measurable and majorized in norm by $\overline{M}_R(\cdot)$ which is positive and KS-integrable. By Proposition 4 in [12], its primitive $F(t) = \int_0^t \overline{M}_R(s)dh(s)$ is differentiable w.r.t. h, dh-a.e. and $F_h'(t) = \overline{M}_R(t)$, dh-a.e (see [27, Theorem 6.5]).

Besides, as F is non-decreasing, in the same way as in [17, Theorem 4.10] it can be proved that F_h' is Lebesgue–Stieltjes integrable w.r.t. h. It follows that $\overline{M}_R$ is Lebesgue–Stieltjes integrable w.r.t. h.

Therefore, $f(\cdot, x(\cdot))$ is Lebesgue–Stieltjes integrable w.r.t. h, and so KS-integrable as well (see [28]).

The equi-regulatedness of the family

$$\left\{\int_0^{\cdot} f(s, x(s))dh(s), x \in \overline{B_R(x_0)}\right\}$$

is, by [16, Proposition 5.9], a consequence of the inequality

$$\left\| \int_0^{t_1} f(s,x(s))dh(s) - \int_0^{t_2} f(s,x(s))dh(s) \right\| \leq \int_{t_1}^{t_2} \overline{M}_R(s)dh(s).$$

Moreover, for each $t \in [0,1]$ and any $x \in \overline{B_R(x_0)}$,

$$\left\| \int_0^{t} f(s,x(s))dh(s) \right\| \leq \int_0^{t} \overline{M}_R(s)dh(s),$$

therefore

$$\left\{ \int_0^{\cdot} f(s,x(s))dh(s),\ x \in \overline{B_R(x_0)} \right\}$$

is pointwise relatively compact.

The role of M_R is played by $\int_0^1 \overline{M}_R(s)dh(s)$, therefore hypothesis iii) in Theorem 19 is also verified. □

We shall see next that usual locally Lipschitz assumptions guarantee the hypothesis of the previous result.

Corollary 22. *Let $h : [0,1] \to \mathbb{R}$ be a left-continuous non-decreasing function and $f : [0,1] \times \mathbb{R}^k \to \mathbb{R}^k$ satisfy the following assumptions:*
i) f is measurable w.r.t. the first argument and for any $R > 0$, there exists $L_R > 0$ such that for every $x, y \in \overline{B_R(x_0)}$ and $s \in [0,1]$,

$$\|f(s,x) - f(s,y)\| \leq L_R \|x - y\|.$$

ii) $\|f(\cdot, x_0)\|$ is KS-integrable w.r.t. h;
iii) There exists $R_0 > 0$ such that $L_{R_0}(h(1) - h(0)) < 1$.

Then the integral measure equation

$$x(t) = x_0 + \int_0^t f(s,x(s))dh(s)$$

has regulated solutions with $\|x - x_0\|_C \leq R_0$.

Proof. Indeed, the continuity of $f(s,\cdot)$ comes from hypothesis i), while for any regulated function $x : [0,1] \to \mathbb{R}^k$, denoting by $R = \|x - x_0\|_C$,

$$\|f(s,x(s))\| \leq \|f(s,x_0)\| + L_R\|x(s) - x_0\| \leq \|f(s,x_0)\| + L_R R, \quad \forall s \in [0,1]$$

whence, as seen before, since it is measurable and majorized by a KS-integrable and positive function, $f(\cdot, x(\cdot))$ is Lebesgue–Stieltjes integrable and so, KS-integrable w.r.t. h.

It can be seen that hypothesis 1) in Proposition 21 is checked by the Lebesgue–Stieltjes integrable function $\overline{M}_R(s) = \|f(s,x_0)\| + L_R R$.

Next, let $(x_n)_n$ be a pointwise convergent sequence of regulated functions, bounded in $\|\cdot\|_C$-norm (thus, contained in a ball of radius R) and let x be its limit. Hypothesis i) implies that for all $s \in [0,1]$, $f(s,x_n(s)) \to f(s,x(s))$.

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As for each $s \in [0,1]$:

$$\|f(s, x_n(s))\| \le \|f(s, x_0)\| + L_R\|x_n(s) - x_0\| \le \|f(s, x_0)\| + L_R R, \forall\, n \in \mathbb{N},$$

the dominated convergence theorem implies that

$$\int_0^t f(s, x_n(s))dh(s) \to \int_0^t f(s, x(s))dh(s),\ \forall t \in [0,1].$$

As a consequence of [28, Theorem VI.8.1], the KS-integral coincides with the Lebesgue–Stieltjes integral since h is left-continuous, thus Theorem 10 gives us that the sequence $\{f(\cdot, x_n(\cdot)), n \in \mathbb{N}\}$ is asymptotically KS-equiintegrable on $[0,t]$ for every $t \in [0,1]$.

Obviously, in this setting

$$\int_0^1 \overline{M}_{R_0}(s)dh(s) = \int_0^1 \|f(s, x_0)\|dh(s) + L_R R(h(1) - h(0))$$

and so, for R_0 (which can be supposed to be large enough) satisfying the hypothesis $L_{R_0}(h(1) - h(0)) < 1$, we have

$$\int_0^1 \overline{M}_{R_0}(s)dh(s) \le R_0. \qquad \square$$

Let us now pass to the closure results. Consider thus the problem

$$x(t) = x_0 + \int_0^t f(s, x(s))dg(s), \tag{13}$$

let $g_n : [0,1] \to \mathbb{R}$ be a sequence of regulated functions convergent pointwise to the regulated function $g : [0,1] \to \mathbb{R}$ and consider also the approximating problem

$$x_n(t) = x_0 + \int_0^t f(s, x_n(s))dg_n(s). \tag{14}$$

The closure result associated to the existence Theorem 19 states as follows.

Theorem 23. *Let $g_n, g : [0,1] \to \mathbb{R}$ be regulated, $g_n \to g$ pointwise and $f : [0,1] \times X \to X$ satisfy the following assumptions:*
i1) $f(s,\cdot)$ is continuous, for each $s \in [0,1]$
i2) $f(\cdot, x(\cdot))$ is KS-integrable w.r.t. g_k for each $k \in \mathbb{N}$ and for each $x \in G([0,1], X)$;
ii1) for any pointwise convergent and uniformly bounded sequence $(x_n)_n$, the sequence $\{f(\cdot, x_n(\cdot)), n \in \mathbb{N}\}$ is asymptotically KS-equiintegrable w.r.t. each g_k on any interval $[0,t]$ and
ii2) for any $R > 0$ and each k,

$$\left\{ \int_0^{\cdot} f(s, x(s))dg_k(s),\ x \in \overline{B_R(x_0)} \right\}$$

is equiregulated and pointwise relatively compact;

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iii) There exists $R_0 > 0$ such that the constants $M^k_{R_0}$ whose existence is stated in Lemma 18 satisfy $M^k_{R_0} \leq R_0$.

Suppose also that for any pointwise convergent and uniformly bounded sequence $(x_n)_n$, the sequence $\{f(\cdot, x_n(\cdot)), n \in \mathbb{N}\}$ is asymptotically KS-equiintegrable w.r.t. $(g_n)_n$ on any interval $[0, t]$.

Then the problems (14) have regulated solutions for each n (by Theorem 19) and let $x_n : [0, 1] \to X$ be such a solution.

If there exists a regulated function $x : [0, 1] \to X$ such that $x_n \to x$ pointwise, then x is a solution of problem (13).

Proof. One can write

$$x_n(t) = x_0 + \int_0^t f(s, x_n(s))dg_n(s).$$

Since f is continuous w.r.t. the second argument and $x_n \to x$ pointwise,

$$f(s, x_n(s)) \to f(s, x(s)),\ \forall s \in [0, 1].$$

It is by hypothesis asymptotically KS-equiintegrable w.r.t. g_n on any interval $[0, t]$ (because the sequence $(x_n)_n$ is contained in the ball centered at x_0, of radius R_0 of the space $G([0, 1], X)$), so we can apply Theorem 10 and one gets

$$\int_0^t f(s, x_n(s))dg_n(s) \to \int_0^t f(s, x(s))dg(s)$$

for every $t \in [0, 1]$. Thus,

$$x(t) = x_0 + \int_0^t f(s, x(s))dg(s). \qquad \square$$

Example 24. Since this is, as far as we know, the first closure result for integral equations governed by regulated functions, it can be applied (unlike the existing results, available for BV functions only, e.g. [13], [14], [25]) to problems governed, for instance, by

$$g_n(t) = \begin{cases} \left(1 + \frac{1}{n}\right) t \sin \frac{\pi}{t} & \text{if } \ 0 < t \leq 2; \\ 0 & \text{if } \ t = 0, \end{cases}$$

$$g(t) = \begin{cases} t \sin \frac{\pi}{t} & \text{if } \ 0 < t \leq 2; \\ 0 & \text{if } \ t = 0, \end{cases}$$

which are continuous (therefore regulated), but not of bounded variation on $[0, 2]$ (see [17], page 50).

As a consequence of Theorem 23, we can present a closure result associated to Proposition 21 (with less restrictive assumptions comparing to other closure results in literature, such as Theorem 6.3 in [13]).

Proposition 25. *Let $g_n, g : [0, 1] \to \mathbb{R}$ be left-continuous nondecreasing functions, $g_n \to g$ pointwise and $f : [0, 1] \times \mathbb{R}^k \to \mathbb{R}^k$ satisfy the following assumptions:*

1) f is measurable w.r.t. the first argument, continuous w.r.t. the second one and for any $R > 0$ there exists a constant $\overline{M}_R > 0$ such that

$$\|f(s, x(s))\| \leq \overline{M}_R, \ \forall s \in [0,1], x \in \overline{B_R(x_0)};$$

2) for any pointwise convergent sequence $(x_n)_n \subset G([0,1], \mathbb{R}^k)$ bounded in the norm $\|\cdot\|_C$, the sequence $\{f(\cdot, x_n(\cdot)), n \in \mathbb{N}\}$ is asymptotically KS-equiintegrable w.r.t. each g_k on $[0,t]$ for every $t \in [0,1]$.

If one can find $R_0 > 0$ such that

$$\overline{M}_{R_0}(g_n(1) - g_n(0)) \leq R_0, \forall n \in \mathbb{N}$$

then the problems (14) have regulated solutions for each n (by Proposition 21) and let $x_n : [0,1] \to \mathbb{R}^k$ be such a solution.

Suppose also that for any pointwise convergent and uniformly bounded sequence $(x_n)_n$, the sequence $\{f(\cdot, x_n(\cdot)), n \in \mathbb{N}\}$ is asymptotically KS-equiintegrable w.r.t. $(g_n)_n$ on any interval $[0,t]$.

If there exists a regulated function $x : [0,1] \to \mathbb{R}^k$ such that $x_n \to x$ pointwise, then x is a solution of measure problem (13).

By stretching condition $ii2$) we can easily get a continuous dependence result:

Proposition 26. *Let $g_n, n \in \mathbb{N}$, g and f satisfy assumptions i), ii1), iii) and:*

ii2') for any $R > 0$,

$$\left\{ \int_0^{\cdot} f(s, x(s)) dg_n(s), \ x \in \overline{B_R(x_0)}, n \in \mathbb{N} \right\}$$

is equiregulated and pointwise relatively compact.

Suppose also that for any pointwise convergent and uniformly bounded sequence $(x_n)_n$, the sequence $\{f(\cdot, x_n(\cdot)), n \in \mathbb{N}\}$ is asymptotically KS-equiintegrable w.r.t. $(g_n)_n$ on any interval $[0,t]$.

Then the problem (14) has regulated solutions (by Theorem 19) and let $x_n : [0,1] \to X$ be such a solution.

There exists a subsequence uniformly convergent to a regulated function $x : [0,1] \to X$ and x is a solution of problem (13).

Proof. By hypothesis $ii2'$), the sequence $(x_n)_n$ is equiregulated and pointwise relatively compact, therefore, by [16, Theorem 6.2], it is relatively compact in the space of regulated functions. It follows that there exists a subsequence uniformly convergent to a regulated function x and so, the result is a consequence of Theorem 23. □

5. Existence and closure results in the set-valued setting

First, we prove an existence result, via the following nonlinear alternative of Leray–Schauder type.

Theorem 27. *([26, Theorem 1.1]) Let D be an open subset of a Banach space E such that $0 \in D$ and let $T : \overline{D} \to \mathcal{P}_{cc}(E)$ be a compact operator with closed Graph. Then either*

i) T has a fixed point in $\overline{D}$

or

ii) there exists $x \in \partial D$ such that $\lambda x \in T(x)$ for some $\lambda > 1$.

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Theorem 28. *Let $h : [0, 1] \to \mathbb{R}$ be a regulated function and let $F : [0, 1] \times X \to \mathcal{P}_{cc}(X)$ satisfy the following hypothesis:*

1) *$F(t, \cdot)$ is upper semi-continuous for every $t \in [0, 1]$;*
2) *For every $R > 0$, there is $M_R > 0$ s.t. for every $x \in \overline{B_R}$, the map $F(\cdot, x(\cdot))$ has bounded variation with respect to the Hausdorff–Pompeiu distance and*
$$var(F(\cdot, x(\cdot))) \leq M_R;$$
3) *For every $R > 0$, there is a multifunction $G_R : [0, 1] \to \mathcal{P}_{kc}(X)$ such that*
$$F(t, x) \subset G_R(t), \text{ for all } t \in [0, 1] \text{ and } x \in \overline{B_R};$$
4) *any pointwise convergent sequence of selections of G_R with equibounded variation is KS-equiintegrable w.r.t. h.*

If moreover there exists R_0 such that $\|x\|_C \neq R_0$ for any regulated solution x of

$$x(t) \in \lambda \left(x_0 + \int_0^t F(s, x(s)) dh(s) \right)$$

for all $\lambda \in (0, 1)$, then the integral inclusion (2) possess regulated solutions with $\|x\|_C \leq R_0$.

Proof. Let $N : \overline{B_{R_0}} \to \mathcal{P}_{cc}(G([0, 1], X))$ be the operator defined by

$$N(x)(t) = \left\{ x_0 + \int_0^t f(s) dh(s), f \text{ selection of } F(\cdot, x(\cdot)), var(f) \leq M_{R_0} \right\}.$$

We will check the hypothesis of Theorem 27.

Let us note first that the values of N are contained in the space of regulated functions (see Proposition 3), are convex and non-empty since, by [3, Theorem 2], one can find at least one selection whose variation is not greater than the variation of the multifunction.

Let us prove that the values are closed. Fix then $x \in \overline{B_{R_0}}$, consider a sequence

$$\left(x_0 + \int_0^{\cdot} f_n(s) dh(s) \right)_n \subset N(x)$$

convergent to $y \in G([0, 1], X)$. The sequence $(f_n)_n$ satisfies the Helly selection theorem, so there exists a subsequence $(f_{n_k})_k$ of $(f_n)_n$ pointwise convergent towards a selection f of $F(\cdot, x(\cdot))$ with variation smaller than M_{R_0}. It follows by hypothesis 4) and Theorem 10 that

$$\int_0^t f_{n_k}(s) dh(s) \to \int_0^t f(s) dh(s) \text{ for every } t \in [0, 1],$$

therefore $y(t) = x_0 + \int_0^t f(s) dh(s)$ for any $t \in [0, 1]$.

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In the sequel, let us prove that N is compact. Take $(y_n)_n \subset \bigcup\{N(x), x \in \overline{B_{R_0}}\}$, so

$$y_n(t) = x_0 + \int_0^t f_n(s)dh(s), \ \forall t \in [0,1]$$

where f_n is a selection with variation smaller than M_{R_0} of $F(\cdot, x_n(\cdot))$ for some $x_n \in \overline{B_{R_0}}$.

As before, we are able to find a subsequence $(f_{n_k})_k$ pointwise convergent to a function f with variation smaller than M_{R_0}, whence $\int_0^t f_{n_k}(s)dh(s) \to \int_0^t f(s)dh(s)$, $\forall t \in [0,1]$ and, by Lemma 6 and Theorem 5, the convergence is uniform. In conclusion, $(y_n)_n$ has a subsequence convergent in the topology of $G([0,1], X)$, so the operator is compact.

Let us now check that it has closed Graph. To this aim, let $(x_n, y_n)_n \subset Graph(N)$ converge uniformly to (x, y) and prove that $(x, y) \in Graph(N)$.

As before,

$$y_n(t) = x_0 + \int_0^t f_n(s)dh(s), \ \forall t \in [0,1]$$

where f_n is a selection with variation smaller than M_{R_0} of $F(\cdot, x_n(\cdot))$ for each n.

The sequence $(f_n)_n$ satisfies the Helly selection theorem, so it has a subsequence $(f_{n_k})_k$ pointwise convergent to a function f with variation smaller than M_{R_0}. Using Theorem 10,

$$\int_0^t f_{n_k}(s)dh(s) \to \int_0^t f(s)dh(s), \ \forall t \in [0,1]$$

so

$$y(t) = x_0 + \int_0^t f(s)dh(s), \ \forall t \in [0,1].$$

Finally, hypothesis 1) implies that f is a selection of $F(\cdot, x(\cdot))$ since for each s and $\varepsilon > 0$ there exists $N_{\varepsilon,s} \in \mathbb{N}$ such that for any $n > N_{\varepsilon,s}$: $F(s, x_n(s)) \subset F(s, x(s)) + \varepsilon B$, where B is the unit open ball of X. Thus, the closed Graph property is verified.

The conditions of Theorem 27 are satisfied and, as the alternative is excluded by hypothesis, it follows that the operator N has fixed points and our inclusion has regulated solutions. □

Remark 29. The KS-equiintegrability assumption could be replaced by the asymptotical KS-equiintegrability on any interval $[0, t]$, but in this case we would also need to impose the equiregulatedness of the primitives (as in Theorem 19) since Lemma 6 cannot be applied.

Remark 30. The strong assumption 2) in our existence result is justified by the fact that g is a very general function, being only regulated; however, when imposing stronger assumption on g, e.g. to have a bounded variation, condition 2) could be replaced by the following (more natural) condition:

For every $R > 0$, there is $M_R > 0$ s.t. for every function x with $var(x) \leq R$, the map $F(\cdot, x(\cdot))$ has bounded variation with respect to the Hausdorff–Pompeiu distance and $var(F(\cdot, x(\cdot))) \leq M_R$.

For a related result in this particular setting, we refer the reader to [31, Section 3.1].

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Consider in what follows the problem

$$x(t) \in x_0 + \int_0^t F(s, x(s))dg(s), \tag{15}$$

let $g_n : [0,1] \to \mathbb{R}$ be a sequence of regulated functions convergent pointwise to the regulated function $g : [0,1] \to \mathbb{R}$ and consider the approximating problem

$$x_n(t) \in x_0 + \int_0^t F(s, x_n(s))dg_n(s). \tag{16}$$

The closure result can now be proved using the tools of Theorem 28.

Theorem 31. *Let $g_n : [0,1] \to \mathbb{R}$ be a sequence of regulated functions pointwise convergent to a regulated function $g : [0,1] \to \mathbb{R}$ and let $F : [0,1] \times X \to \mathcal{P}_{cc}(X)$ satisfy the hypothesis 1), 2) and 3) in the previous result and:*

4') any pointwise convergent sequence of selections of G_R with equibounded variation is KS-equiintegrable w.r.t. each g_k;
4") any pointwise convergent sequence of selections of G_R with equibounded variation is KS-equiintegrable w.r.t. $(g_n)_n$.

Suppose that there exists R_0 such that $\|x\|_C \neq R_0$ for any regulated solution x of

$$x(t) \in \lambda \left(x_0 + \int_0^t F(s, x(s))dg_n(s) \right)$$

for all $n \in \mathbb{N}$ and $\lambda \in (0,1)$.

Then, by Theorem 28, the inclusions (16) possess regulated solutions with $\|x\|_C \leq R_0$ (let x_n be such solutions). If there exists a regulated function x such that $x_n \to x$ pointwise, then x is a regulated solution for inclusion (15).

Remark 32. Theorems 23 and 31 contain closure results for Stieltjes integral problems (which, in the particular case of BV functions g_n and g become measure integral problems) under convergence assumptions on the functions g_n and g driving the equations. In other related works, the assumptions were given in terms of convergence of measures (see [37] or [31]).

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