

AUTHOR QUERY FORM

 ELSEVIER	Journal: Nonlinear Analysis Article Number: 9048	Please e-mail or fax your responses and any corrections to: E-mail: corrections.esch@elsevier.river-valley.com Fax: +44 1392 285879
---	--	--

Dear Author,

Please check your proof carefully and mark all corrections at the appropriate place in the proof (e.g., by using on-screen annotation in the PDF file) or compile them in a separate list.

For correction or revision of any artwork, please consult <http://www.elsevier.com/artworkinstructions>.

Any queries or remarks that have arisen during the processing of your manuscript are listed below and highlighted by flags in the proof. Click on the 'Q' link to go to the location in the proof.

Location in article	Query / remark click on the Q link to go Please insert your reply or correction at the corresponding line in the proof
Q1	An extra closing parenthesis has been inserted. Please check, and correct if necessary.

Thank you for your assistance.



Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Bounded Palais–Smale sequences for non-differentiable functions

P. Candito^{a,*}, R. Livrea^b, D. Motreanu^c^a Dipartimento D.I.M.E.T Facoltà di Ingegneria, Università di Reggio Calabria Via Graziella, Località Feo di Vito, 89100 Reggio Calabria, Italy^b Dipartimento di Patrimonio Architettonico e Urbanistico Facoltà di Architettura, Università di Reggio Calabria Salita Melissari, 89100 Reggio Calabria, Italy^c Département de Mathématiques, Université de Perpignan, Avenue Paul Alduy 52, 66860 Perpignan Cedex, France

ARTICLE INFO

Article history:

Received 26 August 2010

Accepted 10 May 2011

Communicated by Enzo Mitidieri

Keywords:

Critical points

Non-smooth functions

Bounded Palais–Smale sequences

Mountain pass geometry

Deformation

ABSTRACT

The existence of bounded Palais–Smale sequences (briefly BPS) for functionals depending on a parameter belonging to a real interval and which are the sum of a locally Lipschitz continuous term and of a convex, proper, lower semicontinuous function, is obtained when the parameter runs in a full measure subset of the given interval. Specifically, for this class of non-smooth functions, we obtain BPS related to mountain pass and to global infima levels. This is done by developing a unifying approach, which applies to both cases and relies on a suitable deformation lemma.

© 2011 Published by Elsevier Ltd

1. Introduction

Since its appearing, the Mountain Pass Theorem (briefly MPT) of Ambrosetti and Rabinowitz [1], for C^1 functionals f defined on a Banach space X , had a high impact in the topological study of various differential problems. This fact thoroughly motivated not only the search for new applications to differential equations, but also the extension of the MPT to very general situations. Regarding the so-called ‘mountain pass geometry’, two aspects have attracted particular attention: (a) the regularity of f ; (b) the Palais–Smale condition (briefly (PS)), which represented, with every improvement, a crucial compactness assumption. The study of part (a) refers to the weakening of the C^1 assumption so that the functional can also be non-differentiable. This allows us to handle important concrete topics like variational inequalities and elliptic equations with discontinuous nonlinearities, that, otherwise could not be approached directly through the MPT. Concerning the part (b), it is worth to notice that, whenever applying the MPT, the first step to prove the (PS) consists in verifying that a Palais–Smale sequence is bounded, and only afterward a convergent subsequence is extracted (see, e.g., [2, pag. 79] or [3, Lemma 3.2]). The discussion above poses a theoretical investigation: to determine large classes of functions admitting bounded Palais–Smale sequence, not necessarily requiring the usual (PS). In this respect, it is shown in [4] how the perturbation methods, combined with Ekeland’s variational principle, can be used for C^1 functionals to construct, in a natural way, bounded Palais–Smale sequences, sometimes in cases where the underlying geometry is even more general than that of the MPT (see also [5,6] for certain non-smooth versions). For dealing with this question, it seems that the dependence of f on a real parameter can be particularly useful. Starting from some developments in [2] and exploiting monotonicity arguments, for a family $f(\cdot, \lambda)$ of C^1 functionals on X depending linearly on λ in a positive real interval, it is shown in [7] that, for almost every λ , the functional $f(\cdot, \lambda)$ admits a bounded Palais–Smale sequence $\{x_n\}$, provided the mountain pass geometry occurs. Moreover, there holds $f(x_n, \lambda) \rightarrow c_\lambda$ as $n \rightarrow \infty$, where c_λ stands for the classical mountain pass level of the function $f(\cdot, \lambda)$. The abstract results in [7] are improved in [8] by dropping the linear dependence on λ and weakening the monotonicity requirement (see also [9]

* Corresponding author.

E-mail addresses: pasquale.candito@unirc.it (P. Candito), roberto.livrea@unirc.it (R. Livrea), motreanu@univ-perp.fr (D. Motreanu).

for another extension of [7]). It is relevant noting that the techniques related to the presence of parameters still work if, in place of the mountain pass geometry, the functionals $f(\cdot, \lambda)$ are bounded from below. This has been done by Schechter and Zou [10], again under a linear dependence of λ , obtaining that for every λ there exists a bounded Palais–Smale sequence $\{x_n\}$ satisfying $f(x_n, \lambda) \rightarrow \alpha_\lambda$, where α_λ denotes the global infimum of $f(\cdot, \lambda)$. A careful analysis of the proofs of all these results reveals that in both main settings (the mountain pass geometry and boundedness from below), the key facts seem to be the following: the involved levels, c_λ and α_λ , respectively, are generated through an infimum process and that deformation arguments are available.

The aim of the present paper is to extend the abstract results contained in [7,8,10] on the existence of bounded Palais–Smale sequences in the parametric framework to the class of non-differentiable functionals satisfying the structure assumption

(H_f) $f(x, \lambda) := \Phi(x, \lambda) + \psi(x)$ for all $(x, \lambda) \in X \times \Lambda$, with an interval $\Lambda \subseteq \mathbf{R}$, a function $\Phi : X \times \Lambda \rightarrow \mathbf{R}$ such that $\Phi(\cdot, \lambda)$ is locally Lipschitz continuous for all $\lambda \in \Lambda$, and a function $\psi : X \rightarrow \mathbf{R} \cup \{+\infty\}$ which is convex, proper (i.e., $\neq +\infty$), and lower semicontinuous.

Our main results covering both the mountain pass geometry and the boundness from below are presented in Sections 3 and 4, respectively. The essential tool in proving them is a general deformation lemma, which is set forth in Section 2. This deformation property has actually a unifying effect on the two different situations leading through the same approach to the main results (Theorems 3.3 and 4.3). Just for the special case of C^1 functionals, our results extend and improve the corresponding statements in [7,8,10] (see hypothesis (i₂) and Remarks 3.1 and 4.1).

2. A deformation result

Let $(X, \|\cdot\|)$ be a reflexive Banach space, Λ a real interval and $f : X \times \Lambda \rightarrow \mathbf{R} \cup \{+\infty\}$ a function satisfying the structure hypothesis (H_f) in Section 1. We recall that, if $x, z \in X$ and $\lambda \in \Lambda$, the generalized directional derivative of the locally Lipschitz function $\Phi(\cdot, \lambda)$ at the point x in the direction z , which we denote $\Phi_x^0(x, \lambda; z)$, is defined by

$$\Phi_x^0(x, \lambda; z) := \limsup_{w \rightarrow x, t \rightarrow 0^+} \frac{\Phi(w + tz, \lambda) - \Phi(w, \lambda)}{t},$$

while the generalized gradient of $\Phi(\cdot, \lambda)$ at x , denoted $\partial_x \Phi(x, \lambda)$, is the set

$$\partial_x \Phi(x, \lambda) := \{x^* \in X^* : \langle x^*, z \rangle \leq \Phi_x^0(x, \lambda; z), \forall z \in X\}.$$

Related to the function $\psi : X \rightarrow \mathbf{R} \cup \{+\infty\}$, we set $D_\psi := \{x \in X : \psi(x) < +\infty\}$. Since ψ turns out to be continuous on $\text{int}(D_\psi)$ (see, e.g., [11, Exercise 1, p. 296]), the same holds for f . Given $\lambda \in \Lambda$, for every $\varepsilon, r > 0$ and $a \in \mathbf{R}$ we introduce the set

$$F_{a,\varepsilon}^r(\lambda) := \{x \in X : \|x\| \leq r + 1, a - \varepsilon \leq f(x, \lambda) \leq a + \varepsilon\}. \quad (1)$$

The next lemma is the basic step in proving our main results.

Lemma 2.1. *Let f be a function satisfying (H_f). Given $\lambda \in \Lambda$, assume that there exist constants $\varepsilon > 0, r > 0$ and $a \in \mathbf{R}$ such that $F_{a,\varepsilon}^r(\lambda) \neq \emptyset$,*

$$F_{a,\varepsilon}^r(\lambda) \subseteq \text{int}(D_\psi) \quad (2)$$

and

$$\inf \{\|x^* + z^*\| : x^* \in \partial_x \Phi(x, \lambda), z^* \in \partial \psi(x), x \in F_{a,\varepsilon}^r(\lambda)\} > 2\varepsilon. \quad (3)$$

Then there exists a homeomorphism $\chi : X \rightarrow X$ with the following properties:

- (a) $\chi(x) = x$ whenever $|f(x, \lambda) - a| \geq 2\varepsilon$;
- (b) $f(\chi(x), \lambda) \leq f(x, \lambda)$ for all $x \in X$;
- (c) $f(\chi(x), \lambda) \leq a - \varepsilon$ for all $x \in X$ provided $\|x\| \leq r$ and $f(x, \lambda) \leq a + \varepsilon$.

Proof. The assumptions ensure that $F_{a,\varepsilon}^r(\lambda)$ is a nonempty paracompact space because it is closed. Arguing as in [12], we construct an open locally finite covering $\mathcal{W} = \{W_i : i \in I\}$ of $F_{a,\varepsilon}^r(\lambda)$ such that, for every $i \in I$, W_i is convex,

$$f|_{W_i}(\cdot, \lambda) \text{ is Lipschitz continuous,} \quad (4)$$

$$a - 2\varepsilon < f(x, \lambda) < a + 2\varepsilon \quad \text{for all } x \in W_i, \quad (5)$$

and there exists $\xi_i \in X$, with $\|\xi_i\| = 1$ satisfying

$$\langle x^* + z^*, \xi_i \rangle > 2\varepsilon, \quad \forall x^* \in \partial_x \Phi(x', \lambda), \forall z^* \in \partial \psi(x''), \forall x', x'' \in W_i. \quad (6)$$

Set $W = \bigcup_{i \in I} W_i$ and

$$d_i(x) = d(x, X \setminus W_i), \quad \rho_i(x) = \frac{d_i(x)}{\sum_{j \in I} d_j(x)}, \quad \forall x \in W, \forall i \in I,$$

$$\Theta(x) = \begin{cases} \sum_{i \in I} \rho_i(x) \xi_i & \text{if } x \in W \\ 0 & \text{otherwise,} \end{cases}$$

$$l(x) = \frac{d(x, X \setminus W)}{d(x, X \setminus W) + d(x, F_{a, \varepsilon}^r(\lambda))}, \quad \forall x \in X,$$

and

$$V(x) = l(x)\Theta(x), \quad \forall x \in X.$$

We observe that $V : X \rightarrow X$ is locally Lipschitz continuous and

$$\|V(x)\| \leq 1 \quad \forall x \in X. \quad (7)$$

The basic **existence–uniqueness** theorem for ordinary differential equations provides a mapping $\eta \in C^0(\mathbf{R} \times X, X)$ such that

$$\frac{d\eta(t, x)}{dt} = -V(\eta(t, x)), \quad \forall (t, x) \in \mathbf{R} \times X, \quad \eta(0, x) = x.$$

We claim that

$$\text{for every } x \in X, \text{ the function } t \mapsto f(\eta(t, x), \lambda) \text{ is non-increasing on } \mathbf{R}. \quad (8)$$

In order to check (8), we distinguish two cases:

- (α) $x \in X \setminus W$,
- (β) $x \in W$.

In the case (α), $V(x) = 0$, thus $\eta(\cdot, x)$ is constant and (8) holds true.

In the case (β), we start by noting that $\eta(\mathbf{R}, x) \subset W$. Indeed, setting $T = \sup\{\sigma > 0 : \eta([- \sigma, \sigma], x) \subseteq W\}$, assume by contradiction that $T < +\infty$. Hence we have $w_x = \eta(T, x) = \lim_{t \rightarrow T^-} \eta(t, x) \in \partial W$. Then the Cauchy problem

$$\vartheta'(t) = -V(\vartheta(t)) \quad \forall t \in \mathbf{R}, \quad \vartheta(T) = w_x,$$

admits the constant solution $\tilde{\eta}(\cdot) \equiv w_x$, as does $t \mapsto \eta(t, x)$ for $t \leq T$, which is against the uniqueness of solutions.

Fix $t \in \mathbf{R}$. We know that $\eta(t, x) \in W$ and, due to the local finiteness of $\mathcal{W}$, the set $J = \{i \in I : \eta(t, x) \in W_i\}$ is finite. It follows that $\tilde{W} = \bigcap_{i \in J} W_i$ is a convex, open neighborhood of $\eta(t, x)$, and there exists $\delta > 0$ such that

$$\eta([t - \delta, t + \delta], x) \subseteq \tilde{W} \quad \text{and} \quad \eta([t - \delta, t + \delta], x) \cap \left(\bigcup_{i \in I \setminus J} W_i \right) = \emptyset. \quad (9)$$

For arbitrary $t', t'' \in [t - \delta, t + \delta]$ with $t' < t''$, Lebourg's mean value theorem provides $y \in]\eta(t', x), \eta(t'', x)[$, $x^* \in \partial_x \Phi(y, \lambda)$, $z^* \in \partial \psi(y)$ satisfying

$$\begin{aligned} f(\eta(t''), x, \lambda) - f(\eta(t'), x, \lambda) &= \langle x^* + z^*, \eta(t''), x - \eta(t'), x \rangle \\ &= - \int_{t'}^{t''} \langle x^* + z^*, V(\eta(\tau, x)) \rangle d\tau \\ &< -2\varepsilon \int_{t'}^{t''} l(\eta(\tau, x)) \sum_{j \in J} \rho_j(\eta(\tau, x)) d\tau \\ &= -2\varepsilon \int_{t'}^{t''} l(\eta(\tau, x)) d\tau, \end{aligned}$$

thanks to (6) and (9).

Given $p, q \in [0, 1]$ with $p < q$, a standard compactness argument and the above estimate enable us to find $t_1, t_2, \dots, t_s \in [0, 1]$ with $p = t_1 < t_2 < \dots < t_s = q$ such that

$$f(\eta(t_i, x), \lambda) - f(\eta(t_{i-1}, x), \lambda) < -2\varepsilon \int_{t_{i-1}}^{t_i} l(\eta(\tau, x)) d\tau$$

for all $i = 0, 1, \dots, s$. It turns out that

$$\begin{aligned} f(\eta(q, x), \lambda) - f(\eta(p, x), \lambda) &= \sum_{i=1}^s [f(\eta(t_i, x), \lambda) - f(\eta(t_{i-1}, x), \lambda)] \\ &< -2\varepsilon \int_p^q l(\eta(\tau, x)) d\tau, \end{aligned} \quad (10)$$

which establishes (8).

Now we define

$$\chi(x) = \eta(1, x) \quad \forall x \in X.$$

From the general theory of ordinary differential equations it is well known that $\chi : X \rightarrow X$ is a homeomorphism. Since (5) renders $\{x \in X : |f(x, \lambda) - a| \geq 2\varepsilon\} \subseteq X \setminus W$, we deduce property (a). From (8) we readily derive (b).

In order to prove (c), let $x \in X$ with $\|x\| \leq r$ and $f(x, \lambda) \leq a + \varepsilon$. We examine separately the cases:

- (A) $f(x, \lambda) \leq a - \varepsilon$,
 (B) $a - \varepsilon < f(x, \lambda)$.

If (A) holds, then (c) follows immediately from (b). In the case (B), we argue by contradiction which amounts to say

$$a - \varepsilon < f(\chi(x), \lambda) = f(\eta(1, x), \lambda) \leq f(\eta(t, x), \lambda) \leq f(x, \lambda) \leq a + \varepsilon \quad (11)$$

for all $t \in [0, 1]$, where (8) has been used. In addition, through (7), for all $t \in [0, 1]$ we get

$$\begin{aligned} \|\eta(t, x)\| &\leq \|x\| + \|\eta(t, x) - x\| \leq r + \left\| \int_0^t \frac{d\eta(\tau, x)}{d\tau} d\tau \right\| \\ &\leq r + \int_0^t \|V(\eta(\tau, x))\| d\tau \leq r + 1. \end{aligned}$$

Consequently, $\eta([0, 1], x) \subseteq F_{a, \varepsilon}^r(\lambda)$, which forces $l(\eta(\cdot, x))|_{[0, 1]} \equiv 1$. Then (10) with $p = 0$ and $q = 1$ reads as

$$f(\chi(x), \lambda) - f(x, \lambda) < -2\varepsilon. \quad (12)$$

Combining (11) and (12) gives

$$a - \varepsilon < f(\chi(x), \lambda) < f(x, \lambda) - 2\varepsilon \leq a - \varepsilon.$$

This contradiction justifies (c), which completes the proof. $\square$

3. Main result for functionals in a mountain pass setting

Fix

$$v_0, v_1 \in D_\psi \quad (13)$$

and consider the following set of paths

$$\Gamma := \{\gamma \in C^0([0, 1], X) : \gamma(0) = v_0, \gamma(1) = v_1\}$$

and a function $f : X \times \Lambda \rightarrow \mathbf{R} \cup \{+\infty\}$ which verifies hypothesis (H_f) . Corresponding to every $\lambda \in \Lambda$ we set

$$c_\lambda := \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} f(\gamma(t), \lambda). \quad (14)$$

We assume the following condition related to the ‘mountain pass geometry’:

$$(i_1) \quad \max\{f(v_0, \lambda), f(v_1, \lambda)\} < c_\lambda \quad \forall \lambda \in \Lambda.$$

Since the path $t \mapsto v_0 + t(v_1 - v_0)$ belongs to Γ and ψ is convex, we see from (13) and (i_1) that $\lambda \mapsto c_\lambda$ is a real valued function.

For a later use, let us introduce

$$D := \left\{ \lambda \in \Lambda : \liminf_{\mu \rightarrow \lambda^-} \frac{c_\mu - c_\lambda}{\lambda - \mu} < +\infty \right\}. \quad (15)$$

Lemma 3.1. *The set D defined in (15) has full measure in Λ .*

Proof. We note that $\Lambda \setminus D$ consists of points $\lambda \in \Lambda$ at which both left Dini derivatives of the function c_λ are equal to $-\infty$. Then we can conclude from Denjoy–Young–Saks’s theorem [13]. $\square$

According to (15), for every $\lambda \in D$ we have

$$M(\lambda) := \liminf_{\mu \rightarrow \lambda^-} \frac{c_\mu - c_\lambda}{\lambda - \mu} \in [-\infty, +\infty]. \quad (16)$$

We suppose the following additional hypotheses:

(i₂) For every $\lambda \in D$ there exists $k = k(\lambda) > 0$ such that for each sequence $\{(x_n, \lambda_n)\}$ in $X \times \Lambda$ satisfying $\lambda_n \uparrow \lambda$, $-f(x_n, \lambda)$ and $f(x_n, \lambda_n)$ are bounded above, and

$$\limsup_{n \rightarrow \infty} \frac{\Phi(x_n, \lambda_n) - \Phi(x_n, \lambda)}{\lambda - \lambda_n} \leq M(\lambda) + 1, \quad (17)$$

one has $\|x_n\| \leq k$ for n sufficiently large, and

$$\liminf_{n \rightarrow +\infty} [\Phi(x_n, \lambda) - \Phi(x_n, \lambda_n)] \leq 0. \quad (18)$$

(i₃) For every $\lambda \in D$ there exists $\varepsilon_0 > 0$ such that

$$F_{c_\lambda, \varepsilon_0}^k(\lambda) \subseteq \text{int}(D_\psi),$$

with $k = k(\lambda)$ in (i₂) and $F_{c_\lambda, \varepsilon_0}^k(\lambda)$ in (1).

Proposition 3.2. Assume that the function $f(x, \lambda)$ fulfills (H_f), (13), (i₁), and (i₂). Then, for all $\lambda \in D$ and $\varepsilon > 0$, the set $F_{c_\lambda, \varepsilon}^k(\lambda)$ is nonempty.

Proof. Fix $\lambda \in D$ and $\varepsilon > 0$. By (16) there exists a sequence $\{\lambda_n\}$ in Λ such that $\lambda_n \uparrow \lambda$ and

$$M(\lambda) = \lim_{n \rightarrow \infty} \frac{c_{\lambda_n} - c_\lambda}{\lambda - \lambda_n}. \quad (19)$$

By the definition of c_{λ_n} , there is $\gamma_n \in \Gamma$ such that

$$c_{\lambda_n} \leq \sup_{t \in [0, 1]} f(\gamma_n(t), \lambda_n) < c_{\lambda_n} + \frac{\lambda - \lambda_n}{2}. \quad (20)$$

Using $c_\lambda \leq \sup_{t \in [0, 1]} f(\gamma_n(t), \lambda)$ and (20), we can find $t_n \in [0, 1]$ with

$$c_\lambda - \frac{\lambda - \lambda_n}{2} < f(\gamma_n(t_n), \lambda) \quad \text{and} \quad f(\gamma_n(t_n), \lambda_n) < c_{\lambda_n} + \frac{\lambda - \lambda_n}{2}. \quad (21)$$

From (19) and (21), we infer that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\Phi(\gamma_n(t_n), \lambda_n) - \Phi(\gamma_n(t_n), \lambda)}{\lambda - \lambda_n} &= \limsup_{n \rightarrow \infty} \frac{f(\gamma_n(t_n), \lambda_n) - f(\gamma_n(t_n), \lambda)}{\lambda - \lambda_n} \\ &\leq \lim_{n \rightarrow \infty} \frac{c_{\lambda_n} - c_\lambda}{\lambda - \lambda_n} + 1 = M(\lambda) + 1, \end{aligned}$$

thus (17) is satisfied. Then hypothesis (i₂) implies that

$$\|\gamma_n(t_n)\| \leq k \quad \text{for all } n \text{ sufficiently large} \quad (22)$$

and, due to (18), a relabeled subsequence can be found to have

$$f(\gamma_n(t_n), \lambda) - f(\gamma_n(t_n), \lambda_n) = \Phi(\gamma_n(t_n), \lambda) - \Phi(\gamma_n(t_n), \lambda_n) < \frac{1}{n}. \quad (23)$$

Combining (21) and (23) yields

$$\begin{aligned} c_\lambda - \frac{\lambda - \lambda_n}{2} &< f(\gamma_n(t_n), \lambda) < f(\gamma_n(t_n), \lambda_n) + \frac{1}{n} < c_{\lambda_n} + \frac{\lambda - \lambda_n}{2} + \frac{1}{n} \\ &= \frac{c_{\lambda_n} - c_\lambda}{\lambda - \lambda_n} (\lambda - \lambda_n) + c_\lambda + \frac{\lambda - \lambda_n}{2} + \frac{1}{n} \quad \text{for all } n. \end{aligned} \quad (24)$$

If $M(\lambda) = -\infty$, as $c_{\lambda_n} < c_\lambda$ for every n large enough (see (19)), (24) leads to

$$c_\lambda - \frac{\lambda - \lambda_n}{2} < f(\gamma_n(t_n), \lambda) < c_\lambda + \frac{\lambda - \lambda_n}{2} + \frac{1}{n}. \quad (25)$$

On the basis of (22) and (25), we get the desired conclusion. If $M(\lambda)$ is finite, then (19), (22), (24) and (25) permit also to conclude, which completes the proof. $\square$

Now we can state our main result of this section.

Theorem 3.3. Let $f(x, \lambda)$ be a function fulfilling hypotheses (H_f) , (13), and (i_1) – (i_3) . Then, for every $\lambda \in D$, there exists a bounded sequence $\{x_n\}$ in X such that

$$(j_1) \quad f(x_n, \lambda) \rightarrow c_\lambda \text{ as } n \rightarrow \infty,$$

$$(j_1) \quad \Phi_x^0(x_n, \lambda; x - x_n) + \psi(x) - \psi(x_n) \geq -\varepsilon_n \|x - x_n\| \quad \forall x \in X, \quad \forall n \in \mathbb{N}, \text{ with } \varepsilon_n \rightarrow 0^+.$$

Proof. Let $\lambda \in D$. Then there exist the positive constants $k = k(\lambda)$ and ε_0 with the properties in (i_2) – (i_3) . Setting

$$\hat{\varepsilon} = \min \left\{ \varepsilon_0, \frac{1}{2} [c_\lambda - \max\{f(v_0, \lambda), f(v_1, \lambda)\}] \right\}, \quad (26)$$

we claim that for every $\varepsilon \in]0, \hat{\varepsilon}[$ there holds

$$\inf \{ \|x^* + z^*\| : x^* \in \partial_x \Phi(x, \lambda), z^* \in \partial \psi(x), x \in F_{c_\lambda, \varepsilon}^k(\lambda) \} \leq 2\varepsilon. \quad (27)$$

To show this, we argue by contradiction. If (27) were not true, then making use Proposition 3.2 and assumption (i_3) , we would find $\varepsilon \in]0, \hat{\varepsilon}[$ for which Lemma 2.1 can be applied with $a = c_\lambda$ and $r = k$. So there would exist a homeomorphism $\chi : X \rightarrow X$ with the properties (a)–(c) formulated in Lemma 2.1. As in the proof of Proposition 3.2, we construct two sequences $\{\lambda_n\}$ in Λ , with $\lambda_n \uparrow \lambda$, and $\{\gamma_n\}$ in Γ satisfying (19) and (20). Due to (a) and the definition of $\hat{\varepsilon}$, it is straightforward to verify that $\chi(\gamma_n(\cdot)) \in \Gamma$ for every n . Hence, in view of (14), we may consider a sequence $\{s_n\}$ in $[0, 1]$ such that

$$c_\lambda - \frac{\lambda - \lambda_n}{2} < f(\chi(\gamma_n(s_n)), \lambda) \quad \text{and} \quad f(\gamma_n(s_n), \lambda_n) < c_{\lambda_n} + \frac{\lambda - \lambda_n}{2}. \quad (28)$$

From (b), (28) and (19) it follows that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\Phi(\gamma_n(s_n), \lambda_n) - \Phi(\gamma_n(s_n), \lambda)}{\lambda - \lambda_n} &= \limsup_{n \rightarrow \infty} \frac{f(\gamma_n(s_n), \lambda_n) - f(\gamma_n(s_n), \lambda)}{\lambda - \lambda_n} \\ &\leq \lim_{n \rightarrow \infty} \frac{c_{\lambda_n} - c_\lambda}{\lambda - \lambda_n} + 1 = M(\lambda) + 1. \end{aligned}$$

This allows to invoke assumption (i_2) , which implies

$$\|\gamma_n(s_n)\| \leq k \quad (29)$$

and

$$f(\gamma_n(s_n), \lambda) - f(\gamma_n(s_n), \lambda_n) = \Phi(\gamma_n(s_n), \lambda) - \Phi(\gamma_n(s_n), \lambda_n) < \frac{1}{n}$$

along a relabeled subsequence. The last inequality, in conjunction with (28) and (19), leads to

$$f(\gamma_n(s_n), \lambda) \leq c_{\lambda_n} + \frac{\lambda - \lambda_n}{2} + \frac{1}{n} \leq c_\lambda + \varepsilon, \quad (30)$$

for all n sufficiently large. Fix n to have $(\lambda - \lambda_n)/2 < \varepsilon$ and (30). Exploiting (28)–(30) and (c), we achieve the contradiction

$$c_\lambda - \varepsilon < c_\lambda - (\lambda - \lambda_n)/2 < f(\chi(\gamma_n(s_n)), \lambda) \leq c_\lambda - \varepsilon,$$

thereby (27) holds true.

By virtue of (27), there exist $x_n \in F_{c_\lambda, 1/n}^k(\lambda)$, $x_n^* \in \partial_x \Phi(x_n, \lambda)$ and $z_n^* \in \partial \psi(x_n)$ such that

$$\|x_n^* + z_n^*\|_{X^*} < \frac{3}{n}.$$

This guarantees that

$$\|x_n\| \leq k + 1,$$

$$c_\lambda - \frac{1}{n} \leq f(x_n, \lambda) \leq c_\lambda + \frac{1}{n}$$

and

$$\begin{aligned} \Phi_x^0((x_n, \lambda); x - x_n) + \psi(x) - \psi(x_n) &\geq \langle x_n^* + z_n^*, x - x_n \rangle \\ &\geq -\|x_n^* + z_n^*\|_{X^*} \|x - x_n\| \\ &> -\frac{3}{n} \|x - x_n\| \end{aligned}$$

for all $x \in X$ and all $n \in \mathbb{N}$, which completes the proof. $\square$

Remark 3.1. Theorem 3.3 extends Theorem 2.1 in [8], where the functions $f(\cdot, \lambda)$ are supposed to be of class C^1 on X for every λ in a compact interval. Moreover, we relax even in the smooth case hypothesis (H) in [8, Theorem 2.1], where it is required

$$\limsup_{n \rightarrow +\infty} [\Phi(x_n, \lambda) - \Phi(x_n, \lambda_n)] \leq 0.$$

Theorem 3.3 extends also Theorem 1.1 of [7] that deals with a parametric family of C^1 functionals fulfilling certain monotonicity conditions.

The following example sets forth a large class of functionals satisfying the crucial assumption (i_2) .

Example 3.1. Let S be a nonempty, closed and convex set in a Banach space X and let $f : X \times \Lambda \rightarrow \mathbf{R} \cup \{+\infty\}$, with a real interval Λ , be defined by

$$f(x, \lambda) = \Phi(x, \lambda) + \psi(x), \quad \forall (x, \lambda) \in X \times \Lambda,$$

where ψ is the indicator function of the set S , while Φ satisfies the conditions:

(h_1) $\Phi(\cdot, \lambda)$ is locally Lipschitz continuous for every $\lambda \in \Lambda$;

(h_1) there exist $\xi : X \times \Lambda \rightarrow \mathbf{R}$ such that, corresponding to $\lambda \in D$ there is $\nu = \nu(\lambda) > 1$ for which $\{x \in S : \xi(x, \lambda) \leq M(\lambda) + \nu\}$ is bounded, and there exists $\sigma_0 > 0$ such that

$$\Phi(x, \mu) - \Phi(x, \lambda) \geq \xi(x, \lambda)(\lambda - \mu), \quad \forall x \in S, \forall \lambda \in \Lambda, \forall \mu \in [\lambda - \sigma_0, \lambda]; \quad (31)$$

(h_1) $\limsup_{\mu \rightarrow \lambda^-} (\Phi(x, \lambda) - \Phi(x, \mu)) \leq 0$ uniformly with respect to x on bounded subsets of S .

Then (i_2) is verified. In order to show this, we notice from (h_2) that for any $\lambda \in D$ there exists $k = k(\lambda)$ such that

$$\{x \in S : \xi(x, \lambda) \leq M(\lambda) + \nu\} \subseteq B(0, k). \quad (32)$$

Let $\{(x_n, \lambda_n)\}$ satisfy $\lambda_n \uparrow \lambda$, $-f(x_n, \lambda)$ and $f(x_n, \lambda_n)$ are bounded above and (17). Then $\{x_n\} \subset S$, and from (17), (31) and (32) it follows that $\|x_n\| \leq k$ provided n is sufficiently large. Then condition (h_3) ensures (18).

Remark 3.2. For instance, (h_2) holds if in (31) the function $\xi(\cdot, \lambda)$ is coercive on S , that is

$$\lim_{x \in S, \|x\| \rightarrow +\infty} \xi(x, \lambda) = +\infty, \quad \forall \lambda \in D.$$

4. Main result for functionals bounded below

In this section we exploit the deformation result in Lemma 2.1 when (i_1) is replaced with

(i'_1) $f(\cdot, \lambda)$ is bounded below for every $\lambda \in \Lambda$.

Assuming (i'_1) , the mapping $\lambda \mapsto \alpha_\lambda := \inf_{x \in X} f(x, \lambda)$ is real valued and we can proceed following the pattern of Section 3. Setting

$$\tilde{D} := \left\{ \lambda \in \Lambda : \liminf_{\mu \rightarrow \lambda^-} \frac{\alpha_\mu - \alpha_\lambda}{\lambda - \mu} < +\infty \right\}, \quad (33)$$

the Denjoy–Young–Saks's theorem yields the following result.

Lemma 4.1. The set $\tilde{D}$ has full measure in Λ .

For every $\lambda \in \Lambda$ let us set

$$\tilde{M}(\lambda) := \liminf_{\mu \rightarrow \lambda^-} \frac{\alpha_\mu - \alpha_\lambda}{\lambda - \mu}. \quad (34)$$

Using (33), we note that for every $\lambda \in \tilde{D}$ one has $\tilde{M}(\lambda) \in [-\infty, +\infty[$.

Next we assume the following hypotheses:

(i'_2) For every $\lambda \in \tilde{D}$ there exists $\tilde{k} = \tilde{k}(\lambda) > 0$ such that for each sequence $\{(x_n, \lambda_n)\}$ in $X \times \Lambda$ satisfying $\lambda_n \uparrow \lambda$, $-f(x_n, \lambda)$ and $f(x_n, \lambda_n)$ are bounded above, and

$$\limsup_{n \rightarrow \infty} \frac{\Phi(x_n, \lambda_n) - \Phi(x_n, \lambda)}{\lambda - \lambda_n} \leq \tilde{M}(\lambda) + 1, \quad (35)$$

there hold $\|x_n\| \leq \tilde{k}$, for n sufficiently large, and (18).

(i₃') For every $\lambda \in \tilde{D}$ there exists $\varepsilon_0 > 0$ such that

$$F_{\alpha_\lambda, \varepsilon_0}^{\tilde{k}}(\lambda) \subseteq \text{int}(D_\psi),$$

with $\tilde{k} = \tilde{k}(\lambda)$ in (i₂') and $F_{\alpha_\lambda, \varepsilon_0}^{\tilde{k}}(\lambda)$ in (1).

Proposition 4.2. Assume (H_f), (i₁') and (i₂'). Then, for all $\lambda \in \tilde{D}$ and $\varepsilon > 0$, the set $F_{\alpha_\lambda, \varepsilon}^{\tilde{k}}(\lambda)$ is nonempty.

Proof. Fix $\lambda \in \tilde{D}$ and $\varepsilon > 0$. By (34) there exists a sequence $\{\lambda_n\}$ in Λ such that $\lambda_n \uparrow \lambda$ and

$$\tilde{M}(\lambda) = \lim_{n \rightarrow +\infty} \frac{\alpha_{\lambda_n} - \alpha_\lambda}{\lambda - \lambda_n}. \quad (36)$$

The definition of α_{λ_n} allows to consider $x_n \in D_\psi$ such that

$$\alpha_{\lambda_n} \leq f(x_n, \lambda_n) < \alpha_{\lambda_n} + (\lambda - \lambda_n) \quad \text{and} \quad \alpha_\lambda \leq f(x_n, \lambda). \quad (37)$$

Using (37) and (36), one has

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{\Phi(x_n, \lambda_n) - \Phi(x_n, \lambda)}{\lambda - \lambda_n} &= \limsup_{n \rightarrow +\infty} \frac{f(x_n, \lambda_n) - f(x_n, \lambda)}{\lambda - \lambda_n} \\ &\leq \lim_{n \rightarrow +\infty} \frac{\alpha_{\lambda_n} - \alpha_\lambda}{\lambda - \lambda_n} + 1 = \tilde{M}(\lambda) + 1, \end{aligned}$$

so (35) is true. Hypothesis (i₂') implies that

$$\|x_n\| \leq \tilde{k} \quad (38)$$

and

$$f(x_n, \lambda) - f(x_n, \lambda_n) = \Phi(x_n, \lambda) - \Phi(x_n, \lambda_n) < \frac{1}{n}, \quad (39)$$

where a relabeled subsequence is considered if necessary. Then, on the basis of (36)–(39), we can proceed as in Proposition 3.2 to complete the proof. $\square$

The main result of the section is as follows.

Theorem 4.3. Let the function $f(x, \lambda)$ fulfill (H_f), (i₁')–(i₃'). Then, for every $\lambda \in \tilde{D}$, there exists a bounded sequence $\{x_n\}$ in X such that

(j₁') $f(x_n, \lambda) \rightarrow \alpha_\lambda$ as $n \rightarrow \infty$

and (j₂') in Theorem 3.3 are satisfied.

Proof. Fix $\lambda \in \tilde{D}$ and consider $\tilde{k} = \tilde{k}(\lambda)$ and ε_0 given in (i₂')–(i₃'). We claim that for every $\varepsilon \in]0, \varepsilon_0[$ it is true that

$$\inf \left\{ \|x^* + z^*\| : x^* \in \partial_x \Phi(x, \lambda), z^* \in \partial \psi(x), x \in F_{\alpha_\lambda, \varepsilon}^{\tilde{k}}(\lambda) \right\} \leq 2\varepsilon. \quad (40)$$

Suppose by contradiction that (40) is false. Then, bearing in mind Proposition 4.2, there exists $\varepsilon \in]0, \varepsilon_0[$ such that the assumptions of Lemma 2.1 are satisfied with $r = \tilde{k}$ and $a = \alpha_\lambda$. Hence there exists a homeomorphism $\chi : X \rightarrow X$ with the properties (a)–(c). Thanks to Proposition 4.2, we may fix a point $x \in F_{\alpha_\lambda, \varepsilon}^{\tilde{k}}$. Then, through property (c) and the definitions of $F_{\alpha_\lambda, \varepsilon}^{\tilde{k}}$ and α_λ , we are led to

$$\alpha_\lambda \leq f(\chi(x), \lambda) \leq \alpha_\lambda - \varepsilon.$$

The obtained contradiction proves (40). Now it suffices to follow the same lines as for of Theorem 3.3 to complete the proof. $\square$

Remark 4.1. Theorem 4.3 extends, to the non-smooth case and to a more general structure, the statement in Theorem 1.1 of [10] that refers to C^1 functionals of the following form

$$f(x, \lambda) = \frac{1}{2} \lambda I(x) - J(x), \quad x \in X, \lambda \in \Lambda,$$

with an open interval $\Lambda \subset]0, +\infty[$ and where I is required to be non-positive and to have bounded upper levels. Here it is valid that $\tilde{D} = \Lambda$.

Acknowledgment

The authors were partially supported by G.N.A.M.P.A.

References

- [1] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* 14 (1973) 349–381.
- [2] M. Struwe, Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, 2nd ed., in: *Ergeb. Math. Grenzgeb.* (3), vol. 34, Springer-Verlag, Berlin, 1996.
- [3] P. Candito, S.A. Marano, D. Motreanu, Critical pointy for a class of non-differentiable functions and applications, *Discrete Contin. Dyn. Syst.* 13 (2005) 175–194.
- [4] N. Ghoussoub, Duality and Perturbation Methods in Critical Point Theory, in: *Cambridge Tracts in Math.*, vol. 107, Cambridge Univ. Press, Cambridge, 1993.
- [5] R. Livrea, S.A. Marano, Existence and classification of critical points for non-differentiable functions, *Adv. Differential Equations* 9 (2004) 961–978.
- [6] P. Candito, R. Livrea, D. Motreanu, $\mathbb{Z}_2$ -symmetric critical point theorems for nondifferentiable funtions, *Glasg. Math. J.* 50 (2008) 447–466.
- [7] L. Jeanjean, On the existence of bounded Palais–Smale sequences and application to a Landesman–Lazer type problem set on $\mathbb{R}^n$, *Proc. Roy. Soc. Edinburgh* 129 (1999) 787–809.
- [8] L. Jeanjean, J.F. Toland, Bounded Palais–Smale mountain-pass sequences, *C. R. Acad. Sci. Paris* 327 (1998) 23–28.
- [9] P.J. Rabier, Bounded Palais–Smale sequences for functionals with a mountain pass geometry, *Arch. Math.* 88 (2007) 143–152.
- [10] M. Schechter, W. Zou, On functionals bounded below, *Funct. Anal. Appl.* 37 (2003) 142–146.
- [11] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.
- [12] S.A. Marano, D. Motreanu, A deformation theorem and some critical point results for non-differentiable functions, *Topol. Methods Nonlinear Anal.* 22 (2003) 139–158.
- [13] G. Alberti, M. Csörnyei, M. Laczkovich, D. Preiss, Denjoy–Young–Saks theorem for approximate derivatives revisited, *Real Anal. Exchange* 26 (2000/2001) 485–488.