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Irreducibility of Hurwitz Spaces of Coverings with Monodromy Groups Weyl Groups of Type $W(B_d)$

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Dedicated to my father on his 60th

Sunto. – *Rivestimenti di* $\mathbb{P}^1$ *con gruppo di monodromia un gruppo di Weyl di tipo* $W(D_d)$ *sono stati studiati da Biggers e Fried che hanno provato l'irriducibilità dei corrispondenti spazi di Hurwitz. In questo articolo viene dimostrata l'irriducibilità degli spazi di Hurwitz che parametrizzano rivestimenti di una curva proiettiva complessa, connessa, non singolare di genere* ≥ 0*, il cui gruppo di monodromia è un gruppo di Weyl di tipo* $W(B_d)$*.*

Summary. – *Let Y be a smooth, connected, projective complex curve of genus* ≥ 0*. Biggers and Fried proved the irreducibility of the Hurwitz spaces which parametrize coverings of* $\mathbb{P}^1$ *whose monodromy group is a Weyl of type* $W(D_d)$*. Here we prove the irreducibility of Hurwitz spaces that parametrize coverings of Y with monodromy group a Weyl group of type* $W(B_d)$*.*

Introduction.

Let X, X' and Y be smooth, connected, projective complex curves of genus ≥ 0. Let $H_{d,n}(Y)$ be the Hurwitz space which parametrizes degree d coverings of Y simply branched in n points. The irreducibility of the spaces $H_{d,n}(\mathbb{P}^1)$ is proved by Hurwitz in [9]. Coverings of $\mathbb{P}^1$ simply branched in all but one point of the discriminant were studied by Natanzon [13], Kluitmann [11] and Mochizuchi [12], who proved the irreducibility of the corresponding Hurwitz spaces. More generally one can consider sequences of coverings of type $X \xrightarrow{\pi} X' \xrightarrow{f} \mathbb{P}^1$. This is what Biggers and Fried did in [2]. They proved the irreducibility of Hurwitz spaces which parametrize equivalence classes of sequences of coverings $f \circ \pi$, where π is an unramified cyclic covering of degree l and f is a covering simply branched of $\mathbb{P}^1$ of degree m. Only recently Graber, Harris and Starr considered the Hurwitz spaces $H^o_{d,n}(Y)$ parametrizing coverings with full monodromy group S_d of curves of genus ≥ 1. They proved in [8] the irreducibility of these spaces for $n \geq 2d$. Kanev sharpened this result proving in [10] the irreducibility of $H^o_{d,n}(Y)$ in the case $n \geq max\{2, 2d-4\}$ if $g \geq 1$ and $n \geq max\{2, 2d-6\}$ if $g = 1$. Moreover Kanev extended the result to coverings which are simply branched in all but one point of the discriminant. Fixing the branching data of special point, i.e., a partition $\underline{e} = (e_1, \ldots, e_r)$ of d where $e_1 \geq \cdots \geq e_r$, he obtained the Hurwitz spaces $H^o_{d,n,\underline{e}}(Y)$ parametrizing coverings with monodromy group S_d, simply branched in n points and ramified with multiplicities $e_1, \ldots, e_r$ over one additional point. Kanev proved they are irreducible if $n \geq 2d-2$. We sharpened the latter result proving in [15] the irreducibility of $H^o_{d,n,\underline{e}}(Y)$ for $n + |\underline{e}| \geq 2d$, where $|\underline{e}| = \sum_{i=1}^{r} (e_i - 1)$.

Here we consider sequences of coverings $X \xrightarrow{\pi} X' \xrightarrow{f} Y$ where π is a branched covering of degree 2 and f of degree $d \geq 3$. Denote by $D_\pi,\ D_f,\ D$ respectively the branch locus of π, f and $f \circ \pi$. Let $b_0 \in Y - D$ and let $\phi : (f \circ \pi)^{-1}(b_0) \to$ $\to \{-d, \ldots, -1, 1, \ldots, d\}$ be a bijection. In this paper we are interested in Hurwitz spaces that parametrize equivalence classes of pairs $[f \circ \pi, \phi]$ of sequences of coverings $f \circ \pi$ and bijections ϕ satisfying the following: π is a covering as above and either f is a covering simply branched of degree d of Y with $n_2 > 0$ branch points and monodromy group S_d, or f is a covering of degree d of Y with $n_2 > 0$ points of simple branching and one special points c, whose local monodromy has cyclic type given by the partition $\underline{e}$ of d and furthermore f has full monodromy group S_d. Denote these spaces respectively by $H_{W(B_d),(n_1;n_2)}(Y, b_0)$ and $H_{W(B_d),(n_1;n_2,\underline{e})}(Y, b_0)$, $H_{W(B_d),(n_1;n_2,[j_1,\ldots,j_v])}(Y, b_0)$. We prove their irreducibility when $Y \cong \mathbb{P}^1$ and then we extend the result to smooth, connected, projective complex curves of genus ≥ 1. Here we follow the standard approach. Let g be the genus of Y, associate to $[f \circ \pi, \phi]$ an ordered $(n+2g)$-tuple of elements of S_{2d}, $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$, satisfying the following: for each $i = 1, \ldots, n$, $t_i \neq id$, $t_1 \cdots t_n = [\lambda_1, \mu_1] \cdots [\lambda_g, \mu_g]$, n_1 among the t_i are transpositions of type $(j\ {-j})$ and n_2 are permutations of type either $(j\ h)(-j\ \ {-h})$ or $(j\ {-h})(-j\ h)$. Moreover the group generated by t_i, λ_k, μ_k is the Weyl group of type B_d. Observe that when $g = 0$ the $(n+2g)$-tuples $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ are of the form $(t_1, \ldots, t_n)$ and $t_1 \cdots t_n = id$. So we prove the irreducibility of our Hurwitz spaces by proving the transitivity of the action of the braid group $\pi_1((Y - b_0)^{(n)} - \Delta, D)$ on the set of ordered $(n+2g)$-tuple as above. In order to prove the transitivity of the action of $\pi_1((\mathbb{P}^1 - b_0)^{(n)} - \Delta, D)$, we prove that applying elementary transformations of the Artin's braid group it is possible to bring each $(t_1, \ldots, t_n)$ as above to a given normal form. Once this is proved, to extend the result to curve of genus ≥ 1, it is sufficient to prove that applying braid moves it is possible to replace every $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ by $(\bar{t}_1, \ldots, \bar{t}_n; id, id, \ldots, id, id)$. In order to prove this we use the results obtained by Kanev in [10] and by the author in [15]. So we prove the irreducibility of the Hurwitz space $H_{W(B_d),(n_1;n_2)}(Y, b_0)$ for $n_2 \geq 2d-2$ and the irreducibility of the spaces $H_{W(B_d),(n_1;n_2,\underline{e})}(Y, b_0)$ and $H_{W(B_d),(n_1;n_2,[j_1,\ldots,j_v])}(Y, b_0)$ for $n_2 + |\underline{e}| \geq 2d$.

Moreover here we prove the irreducibility of Hurwitz spaces that parametrize equivalence classes $[f \circ \pi]$ where f is a branched covering of degree $d \geq 3$ of $\mathbb{P}^1$, with n_2 points of simple branching and one special point c and π is a branched covering of degree 2 such that $D_\pi \subset f^{-1}(c)$.

CONVENTIONS. – In this paper we work with sequences of branched coverings $X \xrightarrow{\pi} X' \xrightarrow{f} Y$. Two degree d branched coverings $h_1 : X_1 \to Y$ and $h_2 : X_2 \to Y$ are called equivalent if there exists a biholomorphic map $p : X_1 \to X_2$ such that $h_2 \circ p = h_1$. Two sequences of coverings $X_1 \xrightarrow{\pi_1} X_1' \xrightarrow{f_1} Y$ and $X_2 \xrightarrow{\pi_2} X_2' \xrightarrow{f_2} Y$ are called equivalent if there exist two biholomorphic maps $p : X_1 \to X_2$ and $p' : X_1' \to X_2'$ such that $p' \circ \pi_1 = \pi_2 \circ p$ and $f_2 \circ p' = f_1$. The equivalence class containing the covering $f \circ \pi$ is denoted by $[f \circ \pi]$. We denote by D the branch locus of $f \circ \pi$ and by b_0 a point in $Y - D$. We number the fiber $(f \circ \pi)^{-1}(b_0)$ through a bijection $\phi : (f \circ \pi)^{-1}(b_0) \to \{-d, \ldots, -1, 1, \ldots, d\}$ and denote by $m : \pi_1(Y - D, b_0) \to S_{2d}$ the monodromy homomorphism associated to $[f \circ \pi, \phi]$. Moreover here the natural action of S_d on $\{1, \ldots, d\}$ is on the right and multiplication of permutation is by $\sigma \cdot \tau = \tau \circ \sigma$, e.g., $(12)(13) = (123)$.

1. – Preliminaries.

Let $d \geq 3$ be an integer. In § 1.1 we recall some results on Weyl groups of type B_d. We use the notation of [4].

1.1. Let $\{\varepsilon_1, \ldots, \varepsilon_d\}$ be the standard base of $\mathbb{R}^d$ and let R be the root system $\{\pm \varepsilon_i,\ \pm \varepsilon_i \pm \varepsilon_j : 1 \leq i,j \leq d\}$. Let us denote by $W(B_d)$ the Weyl group of type B_d. $W(B_d)$ is generated by the reflections s_{ε_i}, $i = 1, \ldots, d$, and $s_{\varepsilon_i - \varepsilon_j}$, $1 \leq i < j \leq d$. The reflection $s_{\varepsilon_i - \varepsilon_j}$ changes ε_i and ε_j, $-\varepsilon_i$ and $-\varepsilon_j$, leaving unchanged ε_h for each $h \neq \pm i, \pm j$. The reflection s_{ε_i} changes ε_i and $-\varepsilon_i$ while unchanging each ε_h with $h \neq \pm i$. We identify $\{\pm \varepsilon_i : i = 1, \ldots, d\}$ with $\{-d, \ldots, -1, 1, \ldots, d\}$ by the map $\pm \varepsilon_i \to \pm i$. So the action of $W(B_d)$ over $\{\pm \varepsilon_i : i = 1, \ldots, d\}$ allows us to define an injective homomorphism τ from $W(B_d)$ into S_{2d} such that

$$\tau(s_{\varepsilon_i - \varepsilon_j}) = (i\ j)(-i\ \ -j), \qquad \tau(s_{\varepsilon_i}) = (i\ \ -i)$$

and

$$\tau(s_{\varepsilon_i + \varepsilon_j}) = \tau(s_{\varepsilon_i} s_{\varepsilon_j} s_{\varepsilon_i - \varepsilon_j}) = (i\ -j)(-i\ j).$$

Then, by ignoring the sign-changes, each element $w \in W(B_d)$ determines a permutation of the indexes $1, \ldots, d$. This permutation can be expressed in the usual way as a product of disjoint cycles. Let $(i_1 i_2 \ldots i_e)$ be such a cycle. Then w sends ε_{i_1} to $\pm \varepsilon_{i_2}$, $\pm \varepsilon_{i_j}$ to $\pm \varepsilon_{i_{j+1}}$, $j = 2, \ldots, e-1$, and $\pm \varepsilon_{i_e}$ to $\pm \varepsilon_{i_1}$. The cycle $(i_1 \ldots i_e)$ is called positive if $w^e(\varepsilon_{i_1}) = \varepsilon_{i_1}$, and negative if $w^e(\varepsilon_{i_1}) = -\varepsilon_{i_1}$. A positive e-cycle of the form $(i_1 \ldots i_e)$ corresponds in S_{2d} to a product of two disjoint e-cycles, ss', which move the indexes $\{\pm i_1, \ldots, \pm i_e\}$ and are such that if s sends i_j to i_{j+1} (i_j to $-i_{j+1}$) then s' sends $-i_j$ to $-i_{j+1}$ (resp. $-i_j$ to i_{j+1}), where $\pm i_{e+1} := \pm i_1$. Instead a negative e-cycle of the form $(i_1 i_2 \ldots i_e)$ corresponds in S_{2d} to a $2e$-cycle of type $(i_1 \pm i_2 \ldots \pm i_e - i_1 \mp i_2 \ldots \mp i_e)$. The lengths of the cycles together with their signs give a set of positive or negative integers called the signed cycle-type of w. It is easy to see that two elements of $W(B_d)$ are conjugate if and only if they have the same signed cycle-type, [5].

Let us denote by G_1 the subgroup of $W(B_d)$ generated by the reflections with respect to the long roots $\varepsilon_i - \varepsilon_j$, $1 \leq i < j \leq d$. The homomorphism $\tau_1 : G_1 \to S_d$ that corresponds to the action of G_1 over the set $\{\varepsilon_i : i = 1, \ldots, d\}$ is bijective and it sends $s_{\varepsilon_i - \varepsilon_j}$ to $(i\ j)$. Let G_2 be the subgroup of $W(B_d)$ generated by the reflections with respect to the short roots ε_i, $i = 1, \ldots, d$, and let $(\boldsymbol{Z}_2)^d$ be the set of the functions from $\{1, \ldots, d\}$ into $\boldsymbol{Z}_2$. Throughout this paper we denote by $\bar{1}_j$ and by z_{ij} the functions of $(\boldsymbol{Z}_2)^d$ so defined

$$\bar{1}_j(j) = \bar{1} \ \text{ and } \ \bar{1}_j(h) = \bar{0} \ \text{ for each } \ h \neq j$$

and

$$z_{ij}(i) = z_{ij}(j) = z \ \text{ and } \ z_{ij}(h) = \bar{0} \ \text{ for each } \ \mathrm{h} \neq \mathrm{i}, \mathrm{j} \ \text{ and } \ \mathrm{z} \in \boldsymbol{Z}_2.$$

Moreover we denote by $\bar{1}_{i,\ldots,h}$ the function of $(\boldsymbol{Z}_2)^d$ that sends to $\bar{1}$ only the indexes $i, \ldots, h$. Let $\tau_2 : G_2 \to (\boldsymbol{Z}_2)^d$ be the homomorphism that maps s_{ε_i} into $\bar{1}_i$. It is easy to prove that τ_2 is an isomorphism. Let Φ be the homomorphism from S_d in $Aut\,((\boldsymbol{Z}_2)^d)$ which assigns to $t \in S_d$ $\Phi(t) \in Aut\,((\boldsymbol{Z}_2)^d)$ where

$$[\Phi(t)\ z']\ (j) \ := z'(j^t) \ \text{ for each } \ z' \in (\boldsymbol{Z}_2)^d.$$

Let $(\boldsymbol{Z}_2)^d \times^s S_d$ be the semidirect product of $(\boldsymbol{Z}_2)^d$ and S_d through the homomorphism Φ. Given $(z'; t_1), (z''; t_2) \in (\boldsymbol{Z}_2)^d \times^s S_d$ we put

$$(z'; t_1) \cdot (z''; t_2) := (z' + \Phi(t_1)(z''); t_1 t_2).$$

Let $\Psi : (\boldsymbol{Z}_2)^d \times^s S_d \to W(B_d)$ be the map so defined

$$\Psi((z'; t_1)) = \tau_2^{-1}(z')\tau_1^{-1}(t_1).$$

It is easy to check Ψ is an isomorphism which sends $(0; (i\ j))$ to $s_{\varepsilon_i - \varepsilon_j}$, $(\bar{1}_i; id)$ to s_{ε_i} and $(\bar{1}_{ij}; (i\ j))$ to $s_{\varepsilon_i} s_{\varepsilon_j} s_{\varepsilon_i - \varepsilon_j} = s_{\varepsilon_i + \varepsilon_j}$.

Let us denote by r_j the reflection with respect to $\varepsilon_{i_1} \pm \varepsilon_{i_j}$. The image via the injective homomorphism τ of $(r_2 \cdots r_e)$ is a product of two e-cycles of type ss'. Since $s_{\varepsilon_i - \varepsilon_j} s_{\varepsilon_i} = s_{\varepsilon_j} s_{\varepsilon_i - \varepsilon_j}$, $s_{\varepsilon_i}^2 = id$ and $s_{\varepsilon_i + \varepsilon_j} = s_{\varepsilon_i} s_{\varepsilon_j} s_{\varepsilon_i - \varepsilon_j}$ one has

$$(r_2 \cdots r_l) = (c_h \cdots c_k)(r'_2 \ldots r'_l)$$

where r'_j is the reflection with respect to the long root $\varepsilon_{i_1} - \varepsilon_{i_j}$, $c_h, \ldots, c_k$ are reflections with respect to the short roots $\varepsilon_{i_h}, \ldots, \varepsilon_{i_k}$ and the indexes $i_h, \ldots, i_k \in \{i_1, \ldots, i_e\}$ are an even number. Hence $\Psi^{-1}(r_2 \cdots r_e) = = (\bar{1}_{i_h \ldots i_k}; (i_1 i_2 \ldots i_e))$, where $\bar{1}_{i_h \ldots i_k} \in (\boldsymbol{Z}_2)^d$ sends to $\bar{1}$ only an even number of indexes.

Throughout this paper we denote by $(a; \xi)$ an element in $(\boldsymbol{Z}_2)^d \times^s S_d$ such that ξ is a product of r disjoint cycles, $\xi_1, \ldots, \xi_r$, with ξ_i e_i-cycle and $a \in (\boldsymbol{Z}_2)^d$ a function which sends to $\bar{1}$ only an even number of indexes moved by each ξ_i. The element $(a;\xi)$ corresponds in S_{2d} to a product of $2r$ disjoint cycles of the form $s_1 s_1' \cdots s_r s_r'$. Note that an element of type $c_e r_2 \cdots r_e$, instead, corresponds in S_{2d} to a $2e$-cycle of type $(i_1 \pm i_2 \cdots \pm i_e \;\; -i_1 \mp i_2 \cdots \mp i_e)$ and in $(\boldsymbol{Z}_2)^d \times^s S_d$ to $(\bar{1}_{i_{h'} \ldots i_{k'}}; (i_1 \ldots i_e))$ where $\bar{1}_{i_{h'} \ldots i_{k'}} \in (\boldsymbol{Z}_2)^d$ sends to $\bar{1}$ only an odd number of indexes moved by $(i_1 \ldots i_e)$. From now on let us denote by $(a';\xi) \in (\boldsymbol{Z}_2)^d \times^s S_d$ an element satisfying the following: ξ is product of r disjoint cycles $\xi_1, \ldots, \xi_r$, with ξ_i e_i-cycle, and $a' \in (\boldsymbol{Z}_2)^d$ is a function that sends to $\bar{1}$ an even number of indexes moved by each ξ_i, $i \notin \{j_1, \ldots, j_v\} \subset \{1, \ldots, r\}$, and an odd number of indexes moved by each cycle ξ_j, $j \in \{j_1, \ldots, j_v\}$.

Observation 1.2. – Let $(z_{ij};(ij))$, $(z'_{ih};(ih))$, $(\bar{1}_i; id)$, $(a;\xi) \in (\boldsymbol{Z}_2)^d \times^s S_d$, with $\xi = (\ldots hi \ldots)(\ldots k \ldots) \cdots$. Then

(i) $(z_{ij};(ij))(z'_{ih};(ih))(z_{ij};(ij))^{-1} = (z_{ij} + z'_{jh};(ij)(ih))(z_{ij};(ij))$
$= (z_{ij} + z'_{jh} + z_{hi};(ij)(ih)(ij)) = ((z+z')_{jh};(jh))$.

(ii) $(\bar{1}_i; id)(z_{ij};(ij))(\bar{1}_i; id)^{-1} = (\bar{1}_i + z_{ij};(ij))(\bar{1}_i; id) = (\bar{1}_i + z_{ij} + \bar{1}_j;(ij))$
$= ((\bar{1}+z)_{ij};(ij))$.

(iii) $(z_{ij};(ij))(\bar{1}_i; id)(z_{ij};(ij))^{-1} = (z_{ij} + \bar{1}_j;(ij))(z_{ij};(ij)) = (z_{ij} + \bar{1}_j + z_{ij}; id)$
$= (\bar{1}_j; id)$.

(iv) $(\bar{1}_i; id)(a;\xi)(\bar{1}_i; id)^{-1} = (\bar{1}_i + a;\xi)(\bar{1}_i; id) = (\bar{1}_i + a + \bar{1}_h;\xi)$.

(v) $(z_{jk};(jk))(a;\xi)(z_{jk};(jk))^{-1} = (z_{jk} + \Phi((jk))a;(jk)\xi)(z_{jk};(jk))$
$= (z_{jk} + \Phi((jk))a + z_{j'k'};(jk)\xi(jk))$

where $\Phi(jk)a$ is a function which sends to $\bar{1}$ the same number of indexes sent to $\bar{1}$ by a. Therefore $z_{jk} + \Phi((jk))a + z_{j'k'}$ sends to $\bar{1}$ only an even number of indexes moved by each cycles $(jk)\xi_i(jk)$.

1.3. Let X, X' and Y be smooth, connected, projective complex curves of genus ≥ 0. Let $d \geq 3$ be an integer and let $\underline{e} = (e_1, \ldots, e_r)$ be a partition of d where $e_1 \geq \cdots \geq e_r \geq 1$. We associate to $\underline{e}$ the following element in S_d having cycle type $\underline{e}$,

$$(12 \ldots e_1)(e_1 + 1 \ldots e_1 + e_2) \cdots ((e_1 + \cdots + e_{r-1}) + 1 \ldots d). \tag{1}$$

We denote by C_1 and C_2 the conjugate classes of $W(B_d)$ containing respectively reflections with respect to short roots, i.e, elements of type $(\bar{1}_i; id)$ and reflections with respect to long roots, i.e, elements of type $(z_{ij};(ij))$. Moreover let $C_{\underline{e}}$ and $\bar{C}_{\underline{e}}$ be conjugate classes of $W(B_d)$ containing respectively elements of the form $(a;\xi)$ and of the form $(a';\xi)$, where ξ is product of r disjoint cycles $\xi_1 \cdots \xi_r$ with ξ_i e_i-cycle.

Definition 1. – *An ordered sequence* $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ *of permutations of* S_{2d} *such that* $t_i \neq id$ *for each* $i = 1, \ldots, n$ *and* $t_1 t_2 \cdots t_n = [\lambda_1, \mu_1] \cdots [\lambda_g, \mu_g]$ *is called a* Hurwitz system. *The subgroup* $G \subseteq S_{2d}$ *generated* *by t_i, λ_k, μ_k with $i = 1, \dots, n$ and $k = 1, \dots, g$ is called the* monodromy group *of the Hurwitz system.*

Note that if $g = 0$ the Hurwitz systems $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ are of the form $(t_1, \dots, t_n)$ and $t_1 \cdots t_n = id$.

DEFINITION 2. – *We call $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ a* Hurwitz system with values in $(\boldsymbol{Z}_2)^d \times^s S_d$, *if t_i, λ_k, $\mu_k \in (\boldsymbol{Z}_2)^d \times^s S_d$, $t_i \neq (0; id)$ for each i and $t_1 \cdots t_n = [\lambda_1, \mu_1] \cdots [\lambda_g, \mu_g]$.*

Let us denote by $A_{(n_1;n_2),g}$ the set of all Hurwitz systems, $(t_1, \dots, t_{n_1+n_2}; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ with values in $(\boldsymbol{Z}_2)^d \times^s S_d$, with monodromy group $(\boldsymbol{Z}_2)^d \times^s S_d$, such that n_1 among the t_l belong to C_1 and n_2 to C_2. Moreover we denote by $A_{(n_1;n_2,\underline{e}),g}$ $(A_{(n_1;n_2,[j_1,\dots,j_v]),g})$ the set of all Hurwitz systems, $(t_1, \dots, t_{n_1+n_2+1}; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ with values in $(\boldsymbol{Z}_2)^d \times^s S_d$, with monodromy group all $(\boldsymbol{Z}_2)^d \times^s S_d$, such that n_1 among the t_l belong to C_1, n_2 to C_2 and one belong to $C_{\underline{e}}$ (resp. $\bar{C}_{\underline{e}}$). Note that when $g = 0$ we put $A_{(n_1;n_2),0} := A_{(n_1;n_2)}$, $A_{(n_1;n_2,\underline{e}),0} := A_{(n_1;n_2,\underline{e})}$ and $A_{(n_1;n_2,[j_1,\dots,j_v]),0} := A_{(n_1;n_2,[j_1,\dots,j_v])}$.

Let g be the genus of Y and let $b_0 \in Y$. By Riemann's existence theorem there is a natural one-to-one correspondence between the following sets:

– the set of equivalence classes of pairs $[h : X \to Y, \phi]$ where h is a degree $2d$ covering unramified in b_0 and with branch locus D and $\phi : h^{-1}(b_0) \to \to \{-d, \dots, -1, 1, \dots, d\}$ a bijection, and

– the set of homomorphisms $m : \pi_1(Y - D, b_0) \to S_{2d}$ whose monodromy group is transitive.

Let $D = \{b_1, \dots, b_n\}$, we choose loops γ_i around b_i and closed arcs α_k, β_k oriented counterclockwise so that $\gamma_1, \dots, \gamma_n, \alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ is a standard generating system of the fundamental group $\pi_1(Y - D, b_0)$. The images via the monodromy homomorphisms m of $\gamma_1, \dots, \gamma_n, \alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ determine Hurwitz systems whose monodromy groups are transitive subgroups of S_{2d}. Then chosen a standard generating system of $\pi_1(Y - D, b_0)$, to each class $[h : X \to Y, \phi]$ corresponding a Hurwitz system with transitive monodromy group.

Let us denote by $H_{W(B_d),(n_1;n_2)}(Y, b_0)$ be the Hurwitz space that parametrizes equivalence classes of pairs $[h : X \to Y, \phi]$ where h is a degree $2d$ covering unramified in b_0, with $n_1 + n_2$ branch points and Hurwitz system belonging to $A_{(n_1;n_2),g}$. By $H_{W(B_d),(n_1;n_2,\underline{e})}(Y, b_0)$ and $H_{W(B_d),(n_1;n_2,[j_1,\dots,j_v])}(Y, b_0)$ we denote the Hurwitz spaces that parametrize equivalence classes of pairs $[h : X \to Y, \phi]$ as above, only this time h is branched in $n_1 + n_2 + 1$ points and has Hurwitz system belonging respectively to $A_{(n_1;n_2,\underline{e}),g}$ and to $A_{(n_1;n_2,[j_1,\dots,j_v]),g}$. In reality in this paper we work with sequences of coverings $X \xrightarrow{\pi} X' \xrightarrow{f} Y$ where π is a branched covering of degree 2 and f is a degree d branched covering with monodromy group S_d. We denote by D_π the branch locus of π, by D_f the one of f and by D we denote the branch locus of $f \circ \pi$.

DEFINITION 3. – *A sequence of coverings* $X \xrightarrow{\pi} X' \xrightarrow{f} Y$ *where* π *is a branched covering of degree* 2 *and* f *is a degree* d *branched covering with monodromy group* S_d *is called:*

a) of type $(n_1; n_2)$ *if* π *is branched in* n_1 *points,* f *is a simple covering with* n_2 *branch points and* $f(D_\pi) \cap D_f = \emptyset$*;*

b) of type $(n_1; n_2, \underline{e})$ *if* f *is branched in* $n_2 + 1$ *points,* n_2 *of which are points of simple branching and one is a special point* c *whose local monodromy has cyclic type* $\underline{e}$*, and again* π *has* n_1 *branch points and* $f(D_\pi) \cap D_f = \emptyset$*;*

c) of type $(n_1; n_2, [j_1, \ldots, j_v])$ *if* f *is a covering as in b),* π *is a covering with* $n_1 + v > 1$ *branch points such that* $D_\pi \cap f^{-1}(c) = \{c_{j_1}, \ldots, c_{j_v}\}$ *and moreover* D_π *is not contained in* $f^{-1}(c)$*.*

Note that throughout this paper we will work with sequences such that if $x \in D_\pi$ and $f(x) = y$ with $y \neq c$ then $f^{-1}(y) \cap D_\pi = \{x\}$.

Let $b_0 \in Y - D$ and let $[f \circ \pi, \phi]$ be the equivalence class of a sequence of coverings $f \circ \pi$ and a bijection $\phi : (f \circ \pi)^{-1}(b_0) \to \{-d, \ldots, -1, 1, \ldots, d\}$ such that if $f^{-1}(b_0) = \{y_1, \ldots, y_d\}$ and $\pi^{-1}(y_i) = \{x_i, x_{-i}\}$, $\phi(x_i) = i$ and $\phi(x_{-i}) = -i$. We want to understand what is the Hurwitz system associated to a class of this type. At first we suppose that $f \circ \pi$ is a sequence of type $(n_1; n_2)$. Let $[\gamma] \in \pi_1(Y - D, b_0)$ and let γ be a closed arc that bounds a region containing an unique point $b \in D$. If $b \in f(D_\pi)$, lifting γ through f we obtain d closed arcs, one of which bounds a point in D_π. The lifting of this arc through π is an arc $\bar{\gamma}$ with $\bar{\gamma}(0) = x_i$ and $\bar{\gamma}(1) = x_{-i}$ for some $i \in \{1, \ldots, d\}$. So $m(\gamma) \in S_{2d}$ is a transposition that sends i to $-i$, i.e., the local monodromy $m(\gamma)$ is a reflection with respect to the short root ε_i. If instead $b \in D_f$, lifting γ through f we obtain $d - 2$ closed arcs and one arc $\bar{\gamma}$ with $\bar{\gamma}(0) = y_i$ and $\bar{\gamma}(1) = y_j$. Lifting $\bar{\gamma}$ through π we obtain two distinct arcs of X having starting points in the set $\{x_i, x_{-i}\}$ and ending points in $\{x_j, x_{-j}\}$. So $m(\gamma)$ is a permutation of S_{2d} that transforms the set $\{i, -i\}$ into the set $\{j, -j\}$, i.e., the local monodromy $m(\gamma)$ is a reflection with respect to a long root. Then the Hurwitz system $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ associated to the equivalence class $[f \circ \pi, \phi]$ is such that n_1 among the t_l belong to C_1 and n_2 to C_2. Moreover since the monodromy group of f is all S_d and at least one among the t_l is of the form $(\bar{1}_i; id)$, the monodromy group of $f \circ \pi$ is all $(\boldsymbol{Z}_2)^d \times^s S_d$. This implies that the Hurwitz system of a pair $[f \circ \pi, \phi]$ where $f \circ \pi$ is a sequence of type $(n_1; n_2)$ belong to $A_{(n_1;n_2),g}$.

Now we suppose that $f \circ \pi$ is a sequence of type $(n_1; n_2, \underline{e})$. Again let $[\gamma] \in \pi_1(Y - D, b_0)$ with γ closed arc that bounds a region containing an unique point $b \in D$. Let $b = c$ and let $f^{-1}(c) = \{c_1, \ldots, c_r\}$ where c_i has multiplicity e_i, $i = 1, \ldots, r$. Since c_i has multiplicity e_i, there are e_i lifting of γ in X',

$\gamma_{1_i}, \gamma_{2_i}, \ldots, \gamma_{(e_i)_i}$, such that $\gamma_{1_i}(0) = y_{1_i}$, $\gamma_{j_i}(1) = y_{(j+1)_i} = \gamma_{(j+1)_i}(0)$, for each $j = 1, \ldots, (e_i - 1)$ and $\gamma_{(e_i)_i}(1) = y_{1_i}$. Lifting each γ_{j_i} by π we obtain two distinct arcs with starting points in $\{x_{j_i}, x_{-j_i}\}$ and ending points in the set $\{x_{(j+1)_i}, x_{-(j+1)_i}\}$ where $(e_i + 1)_i := 1_i$. So $m(\gamma) \in S_{2d}$ transforms $\{j_i, -j_i\}$ into $\{(j+1)_i, -(j+1)_i\}$ for each $j = 1, \ldots, e_i$ and $i = 1, \ldots, r$. Hence $m(\gamma)$ is product of $2r$ disjoint cycles, $s_1 s_1' \cdots s_r s_r'$, with s_i, s_i' e_i-cycles such that if s_i sends k to h (k to $-h$) then s_i' sends $-k$ to $-h$ (resp. $-k$ to h), i.e., $m(\gamma)$ is an element of $(\boldsymbol{Z}_2)^d \times^s S_d$ of the form $(a; \xi)$. This assure, with what we observed about sequence of type $(n_1; n_2)$, that the Hurwitz system associated to the equivalence class $[f \circ \pi, \phi]$ where $f \circ \pi$ is a sequence of type $(n_1; n_2, \underline{e})$ belong to $A_{(n_1;n_2,\underline{e}),g}$.

In the end let $[f \circ \pi, \phi]$ be a sequence of type $(n_1; n_2, [j_1, \ldots, j_v])$. Again we choose $[\gamma] \in \pi_1(Y - D, b_0)$ so that γ is a closed arc that bounds a region containing a unique point $b \in D$. Let $b = c$ and let $f^{-1}(c) = \{c_1, \ldots, c_r\}$ where c_i has multiplicity e_i, $i = 1, \ldots, r$. Lifting γ thought f we obtain arcs $\gamma_{1_i}, \gamma_{2_i}, \ldots, \gamma_{(e_i)_i}$, $i = 1, \ldots, r$, as above. If $c_j \in f^{-1}(c) \cap D_\pi$, lifting by π the arcs $\gamma_{1_j}, \ldots, \gamma_{(e_j)_j}$, we obtain one arc with starting point x_{1_j} and ending point x_{-1_j}. Then $m(\gamma) \in S_{2d}$ is such that $(1_j)^\omega = -1_j$, where $\omega = (m(\gamma))^{e_j}$ and $m(\gamma)$ transforms the set $\{h_j, -h_j\}$ into the set $\{(h+1)_j, -(h+1)_j\}$ for each $h = 1, \ldots, e_j$ and $j \in \{j_1, \ldots, j_v\}$. Hence $m(\gamma)$ is product of $2(r - v) + v$ disjoint cycles, $s_1 s_1' \cdots s_{r-v} s_{r-v}' q_{j_1} \cdots q_{j_v}$, satisfying the following: q_{j_k}, for each $k = 1, \ldots, v$, is a $2e_{j_k}$-cycle of type $(h_1 \ldots h_{e_{j_k}} - - h_1 \ldots - h_{e_{j_k}})$ where the indexes $h_2, \ldots, h_{e_{j_k}}$ can be either positive or negative and the cycles s_l, s_l' are as above. Then $m(\gamma)$ corresponds in $(\boldsymbol{Z}_2)^d \times^s S_d$ to an element of the form $(a'; \xi)$. So we can assert that the Hurwitz system associated to the class $[f \circ \pi, \phi]$ where $f \circ \pi$ is a sequence of type $(n_1; n_2, [j_1, \ldots, j_v])$ belong to $A_{(n_1;n_2,[j_1,\ldots,j_v]),g}$.

From what we said above and by Riemann's existence theorem we can identify the space of the pairs $[f \circ \pi, \phi]$ where $f \circ \pi$ is a sequence of type $(n_1; n_2)$ by $H_{W(B_d),(n_1;n_2)}(Y, b_0)$, the space of the pairs $[f \circ \pi, \phi]$ where $f \circ \pi$ is a sequence of type $(n_1; n_2, \underline{e})$ by $H_{W(B_d),(n_1;n_2,\underline{e})}(Y, b_0)$ and the space of the pairs $[f \circ \pi, \phi]$ where $f \circ \pi$ is a sequence of type $(n_1; n_2, [j_1, \ldots, j_v])$ by $H_{W(B_d),(n_1;n_2,[j_1,\ldots,j_v])}(Y, b_0)$.

Let $H_{W(B_d),(n_1;n_2)}(Y)$, $H_{W(B_d),(n_1;n_2,\underline{e})}(Y)$ and $H_{W(B_d),(n_1;n_2,[j_1,\ldots,j_v])}(Y)$ be the Hurwitz spaces that parametrize equivalence classes $[f \circ \pi]$ where $f \circ \pi$ is a sequence of branched coverings of type respectively $(n_1; n_2)$, $(n_1; n_2, \underline{e})$ and $(n_1; n_2, [j_1, \ldots, j_v])$.

Let $Y^{(n)}$ be the n-fold symmetric product of Y and let $\varDelta$ be the codimension 1 locus of $Y^{(n)}$ consisting of non simple divisors. Let $\delta_1 : H_{W(B_d),(n_1;n_2)}(Y, b_0) \to \to (Y - b_0)^{(n_1+n_2)} - \varDelta$, $\delta_2 : H_{W(B_d),(n_1;n_2,\underline{e})}(Y, b_0) \to (Y - b_0)^{(n_1+n_2+1)} - \varDelta$ and $\delta_3 : H_{W(B_d),(n_1;n_2,[j_1,\ldots,j_v])}(Y, b_0) \to (Y - b_0)^{(n_1+n_2+1)} - \varDelta$ be the maps which assign to each $[f \circ \pi, \phi]$ the branch locus D of $f \circ \pi$. By Riemann's existence theorem we can identify the fiber of $\delta_1, \delta_2, \delta_3$ over D respectively with $A_{(n_1;n_2),g}$, $A_{(n_1;n_2,\underline{e}),g}$, $A_{(n_1;n_2,[j_1,\ldots,j_v]),g}$. There is an unique topology on $H_{W(B_d),(n_1;n_2)}(Y, b_0)$, $H_{W(B_d),(n_1;n_2,\underline{e})}(Y, b_0)$ and $H_{W(B_d),(n_1;n_2,[j_1,\ldots,j_v])}(Y, b_0)$ such that $\delta_1, \delta_2, \delta_3$ are topological covering maps (see [7]). Therefore the braid group $\pi_1((Y - b_0)^{(n_1+n_2)} - \Delta, D)$ acts on $A_{(n_1;n_2),g}$ and the braid group $\pi_1((Y - b_0)^{(n_1+n_2+1)} - \Delta, D)$ acts on $A_{(n_1;n_2,\underline{e}),g}$ and on $A_{(n_1;n_2,[j_1,\dots,j_v]),g}$.

If these actions are transitive then the Hurwitz spaces $H_{W(B_d),(n_1;n_2)}(Y, b_0)$, $H_{W(B_d),(n_1;n_2,\underline{e})}(Y, b_0)$ and $H_{W(B_d),(n_1;n_2,[j_1,\dots,j_v])}(Y, b_0)$ are connected.

1.4. Shortly we recall some notion on braid groups. The generators of $\pi_1((Y - b_0)^{(n)} - \Delta, D)$ are the elementary braids σ_i with $i = 1, \dots, n-1$ and the braids $\rho_{jk},\ \tau_{jk}$ with $1 \leq j \leq n$ and $1 \leq k \leq g$ (see [3], [6], [14]). The calculation of the action of the elementary braids σ_i on Hurwitz systems is due to Hurwitz [9].

The elementary moves σ_i', relative to the elementary braids σ_i, bring $(t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ to

$$(t_1, \dots, t_{i-1}, t_i t_{i+1} t_i^{-1}, t_i, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g).$$

Therefore their inverses bring $(t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ to

$$(t_1, \dots, t_{i-1}, t_{i+1}, t_{i+1}^{-1} t_i t_{i+1}, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g).$$

The braid moves that correspond to the generators ρ_{ik}, τ_{ik} were studied by Graber, Harris, Starr in [8] and by Kanev in [10]. We make use of some results proved in [10]. In this paper to each generator ρ_{ik} or τ_{ik} is associated a pair of braid moves $\rho_{ik}',\ \rho_{ik}'' = (\rho_{ik}')^{-1}$ and $\tau_{ik}',\ \tau_{ik}'' = (\tau_{ik}')^{-1}$ respectively.

Let $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ be a Hurwitz system. The braid move ρ_{ik}' leaves unchanged λ_l for each l, t_j for each $j \neq i$ and μ_l for each $l \neq k$, while changing t_i and μ_k. Analogously the braid move τ_{ik}'' changes t_i and λ_k, leaving unchanged μ_l for each l, λ_l for each $l \neq k$ and t_j for each $j \neq i$.

We use the following result.

PROPOSITION 1 ([10] Corollary 1.9). – *Let $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ be a Hurwitz system. Let $u_k = [\lambda_1, \mu_1] \cdots [\lambda_k, \mu_k]$ for $k = 1, \dots, g$ and let $u_0 = id$. The following formulae hold:*

i) *For ρ_{1k}':*

$$\rho_{1k}' : \mu_k \to \mu_k' = (b_1^{-1} t_1^{-1} b_1)\mu_k,$$

where $b_1 = u_{k-1}\lambda_k$

ii) *For τ_{1k}'':*

$$\tau_{1k}'' : \lambda_k \to \lambda_k'' = (u_{k-1}^{-1} t_1^{-1} u_{k-1})\lambda_k.$$

In particular

$$\tau_{11}'' : \lambda_1 \to t_1^{-1}\lambda_1.$$

2. – Irreducibility of $H_{W(B_d),(n_1;n_2)}(Y,b_0)$, $H_{W(B_d),(n_1;n_2,\underline{e})}(Y,b_0)$ and $H_{W(B_d),(n_1;n_2,[j_1,\dots,j_v])}(Y,b_0)$.

In this section we will prove the irreducibility of the Hurwitz spaces $H_{W(B_d),(n_1;n_2)}(Y,b_0)$, $H_{W(B_d),(n_1;n_2,\underline{e})}(Y,b_0)$ and $H_{W(B_d),(n_1;n_2,[j_1,\dots,j_v])}(Y,b_0)$. Since these spaces are smooth in order to prove their irreducibility it suffices to prove they are connected. We observed that if $\pi_1((Y-b_0)^{(n_1+n_2)}-\Delta,D)$ acts transitively on $A_{(n_1;n_2),g}$, $H_{W(B_d),(n_1;n_2)}(Y,b_0)$ is connected. Analogously if $\pi_1((Y-b_0)^{(n_1+n_2+1)}-\Delta,D)$ acts transitively on $A_{(n_1;n_2,\underline{e}),g}$ $(A_{(n_1;n_2,[j_1,\dots,j_v]),g})$ the Hurwitz space $H_{W(B_d),(n_1;n_2,\underline{e})}(Y,b_0)$ (resp. $H_{W(B_d),(n_1;n_2,[j_1,\dots,j_v])}(Y,b_0)$) is connected. In order to prove the transitivity of these actions we will prove that, acting by braid moves, it is possible to bring each Hurwitz system respectively in $A_{(n_1;n_2),g}$, $A_{(n_1;n_2,\underline{e}),g}$, $A_{(n_1;n_2,[j_1,\dots,j_v]),g}$ to a given normal form.

DEFINITION 4. – *We call two Hurwitz systems with values in* $(\boldsymbol{Z}_2)^d \times^s S_d$ braid equivalent *if one is obtained from the other by a finite sequence of braid moves* $\sigma'_i, \rho'_{jk}, \tau'_{jk}, (\sigma'_i)^{-1}, \rho''_{jk}, \tau''_{jk}$ *where* $1 \le i \le n-1$, $1 \le j \le n$ *and* $1 \le k \le g$. *We denote the braid equivalence by* $\sim$.

DEFINITION 5. – *Two ordered n-tuples (or sequences) of elements in* $(\boldsymbol{Z}_2)^d \times^s S_d$, $(t_1,\dots,t_n)$ *and* $(\widetilde{t}_1,\dots,\widetilde{t}_n)$, *are called* braid equivalent *if* $(\widetilde{t}_1,\dots,\widetilde{t}_n)$ *is obtained from* $(t_1,\dots,t_n)$ *by a finite sequence of braid moves of type* σ'_i, $(\sigma'_i)^{-1}$. *Note that if* $t_1\cdots t_n = s$ *then* $\widetilde{t}_1\cdots\widetilde{t}_n = s$.

We use the following result.

LEMMA 1 ([12] Lemma 2.4). – *Let* $(t'_1,\dots,t'_n)$ *be a sequence of transpositions of* S_d *such that* $G=\langle t'_1,\dots,t'_n\rangle$ *is transitive. Then* $(t'_1,\dots,t'_n)$ *is braid equivalent to* $(\dots,(ij))$ *where* (ij) *is an arbitrary transposition of G.*

THEOREM 1. – *The Hurwitz space* $H_{W(B_d),(n_1;n_2)}(\mathbb{P}^1,b_0)$ *is irreducible.*

PROOF. – The theorem follows if we prove that each Hurwitz system in $A_{(n_1;n_2)}$ is braid equivalent to the normal form

$$((0;(12)),(0;(12)),(0;(13)),(0;(13)),\dots,(0;(1d-1)),(0;(1d-1)),(0;(1d)),\dots,$$
$$(0;(1d)),(\bar{1}_1;id),\dots,(\bar{1}_1;id))$$

where each $(0;(1i))$, $2\le i\le d-1$, appears twice, $(0;(1d))$ appears $(n_2-2(d-2))$-times and $(\bar{1}_1;id)$ appears n_1-times.

Step 1. – We claim that each Hurwitz system $(t_1,\dots,t_{n_1+n_2})\in A_{(n_1;n_2)}$ is braid equivalent to a Hurwitz system of type $(\widetilde{t}_1,\dots,\widetilde{t}_{n_2},(\bar{1}_1;id),\dots,(\bar{1}_1;id))$. If among the t_j there are elements of type $(\bar{1}_1; id)$, acting by elementary moves σ'_j, we move them to the right, obtaining the sequence $(\bar{t}_1, \dots, \bar{t}_h, (\bar{1}_1; id), \dots, (\bar{1}_1; id))$. Acting again by appropriate braid moves σ'_j, we moves to the right the other elements of our Hurwitz system of type $(\bar{1}_*; id)$, obtaining the new system

$$(\bar{\bar{t}}_1, \dots, \bar{\bar{t}}_{n_2}, (\bar{1}_h; id), \dots, (\bar{1}_k; id), (\bar{1}_1; id), \dots, (\bar{1}_1; id)).$$

Let $\bar{\bar{t}}_j = (z_{ij}; t'_j)$, $j = 1, \dots, n_2$. $(t'_1, \dots, t'_{n_2})$ is the Hurwitz system of a covering of degree $d \geq 3$ of $\mathbb{P}^1$ with n_2 points of simple branching and monodromy group S_d. So by Lemma 1 we can replace $(\bar{\bar{t}}_1, \dots, \bar{\bar{t}}_{n_2})$ by a new braid equivalent sequence which has at the place n_2 $(z_{1h}; (1h))$. Applying σ'_{n_2} twice time, by (iii) and (ii), one has

$$(\dots, (z_{1h}; (1h)), (\bar{1}_h; id), \dots) \sim (\dots, (\bar{1}_1; id), (z_{1h}; (1h)), \dots) \sim$$

$$(\dots, (z'_{1h}; (1h)), (\bar{1}_1; id), \dots).$$

Now we move the new element of type $(\bar{1}_1; id)$ so obtained near by the others. Proceeding in this way for each $(\bar{1}_*; id) \neq (\bar{1}_1; id)$ we obtain the claim.

Step 2. – Starting by $(t_1, \dots, t_{n_1+n_2})$ and applying Step 1 we obtain $(\widetilde{t}_1, \dots, \widetilde{t}_{n_2}, (\bar{1}_1; id), \dots, (\bar{1}_1; id))$. Because $H_{d,n_2}(\mathbb{P}^1)$ is irreducible (see [9]), applying appropriate braid moves σ'_j and their inverses, we can bring the sequence $(\widetilde{t}_1, \dots, \widetilde{t}_{n_2})$ to the form

$$((a^2_{12}; (12)), (b^2_{12}; (12)), \dots, (a^{d-1}_{1d-1}; (1d-1)), (b^{d-1}_{1d-1}; (1d-1)), (c^1_{1d}; (1d)),$$
$$(c^2_{1d}; (1d)), \dots, (c^h_{1d}; (1d))),$$

where $h = (n_2 - 2(d-2))$ and $a^j,\ b^j,\ c^k \in \boldsymbol{Z}_2$. Seeing that

$$(a^2_{12}; (12))(b^2_{12}; (12)) \cdots (a^{d-1}_{1d-1}; (1d-1))(b^{d-1}_{1d-1}; (1d-1))(c^1_{1d}; (1d))(c^2_{1d}; (1d)) \cdots$$
$$(c^h_{1d}; (1d))(\bar{1}_1; id) \cdots (\bar{1}_1; id) = (0; id),$$

one has

$$(a^2_{12} + b^2_{12} + \dots + a^{d-1}_{1d-1} + b^{d-1}_{1d-1} + c^1_{1d} + \dots + c^h_{1d} + \bar{1}_1 + \dots + \bar{1}_1) = 0.$$

That implies

$a^j + b^j \equiv \bar{0}\ (mod\ 2)$ for each $j = 2, \dots, d-1$ and so $a^j = b^j$, and $c^1 + \dots + c^h \equiv \bar{0}\ (mod\ 2)$, then the $c^k = \bar{1}$ are an even number.

If in our Hurwitz system there are elements of type $(0; (1d))$, applying braid moves of type σ'_j we moves them near by the elements $(\bar{1}_1; id)$. Since the number of $(0; (1d))$ is even, by elementary moves σ'_j we can move to the left one element of type $(\bar{1}_1; id)$ obtaining the Hurwitz system

$$((a^2_{12}; (12)), (a^2_{12}; (12)), \dots, (a^{d-1}_{1d-1}; (1d-1)), (a^{d-1}_{1d-1}; (1d-1)), (\bar{1}_{1d}; (1d)), \dots,$$
$$(\bar{1}_{1d}; (1d)), (\bar{1}_1; id), (0; (1d)), \dots, (0; (1d)), (\bar{1}_1; id), \dots),$$

where the elements $(\bar{1}_{1d}; (1d))$ occupy the places $2(d-2)+1,\ 2(d-2)+2, \dots, k$.

Applying successively the braid moves $(\sigma'_k)^{-1}, (\sigma'_{k-1})^{-1}, \dots, (\sigma'_{2(d-2)+1})^{-1}$ by (ii) we obtain

$$((\bar{1}_{1d};(1d)),\dots,(\bar{1}_{1d};(1d)),(\bar{1}_1;id)) \sim ((\bar{1}_{1d};(1d)),\dots,(\bar{1}_{1d};(1d)),(\bar{1}_1;id),$$
$$(0;(1d))) \sim \cdots \sim ((\bar{1}_1;id),(0;(1d)),\dots,(0;(1d))).$$

Now if $a^{d-1}_{1d-1} = \bar{1}$, acting by $(\sigma'_{2(d-2)})^{-1},\ (\sigma'_{2(d-2)-1})^{-1}$, one has

$$((\bar{1}_{1d-1};(1d-1)),(\bar{1}_{1d-1};(1d-1)),(\bar{1}_1;id)) \sim ((\bar{1}_1;id),(0;(1d-1)),(0;(1d-1))).$$

If instead $a^{d-1}_{1d-1} = \bar{0}$, we use $\sigma'_{2(d-2)},\ \sigma'_{2(d-2)-1}$ so we replace

$$((0;(1d-1)),(0;(1d-1)),(\bar{1}_1;id)) \quad \text{by} \quad ((\bar{1}_1;id),(0;(1d-1)),(0;(1d-1))).$$

Proceeding in this way for each a^j, $j = 2,\dots,d-2$, after $(d-2)$-steps we obtain the system

$$((\bar{1}_1;id),(0;(12)),(0;(12)),(0;(13)),(0;(13)),\dots,(0;(1d-1)),(0;(1d-1)),(0;$$
$$(1d)),\dots,(0;(1d)),(\bar{1}_1;id),\dots,(\bar{1}_1;id)).$$

Now the proof follows by applying in the order the sequence of elementary moves $(\sigma'_1)^{-1},\ (\sigma'_2)^{-1},\dots,(\sigma'_{n_2})^{-1}$.

REMARK 2.1. – The irreducibility of $H_{W(B_d),(n_1;n_2)}(\mathbb{P}^1, b_0)$ also follows by observing the map $H_{W(B_d),(n_1;n_2)}(\mathbb{P}^1) \to H_{d,n_2}(\mathbb{P}^1)$ which sends $[X \xrightarrow{\pi} X' \xrightarrow{f} \mathbb{P}^1]$ to $[X' \xrightarrow{f} \mathbb{P}^1]$ has fibers given by Hurwitz spaces of type $H_{2,n_1}(X')$. Since the spaces $H_{d,n_2}(\mathbb{P}^1)$ and $H_{2,n_1}(X')$ are irreducible, $H_{W(B_d),(n_1;n_2)}(\mathbb{P}^1)$ is irreducible. $H_{W(B_d),(n_1;n_2)}(\mathbb{P}^1)$ parametrizes equivalence classes of coverings of $\mathbb{P}^1$ with monodromy group $G = W(B_d)$. Conjugating by elements of G we leave each braid orbit invariant (see [1] or [16] Lemma 9.4). This implies that the irreducible components of $H_{W(B_d),(n_1;n_2)}(\mathbb{P}^1)$ are in one-to-one correspondence by ones of $H_{W(B_d),(n_1;n_2)}(\mathbb{P}^1, b_0)$ and so the Hurwitz space $H_{W(B_d),(n_1;n_2)}(\mathbb{P}^1, b_0)$ is irreducible.

2.2. Let $(t'_1,\dots,t'_n)$ be a sequence of transpositions of S_d such that $t'_1\cdots t'_n = s$ and $\langle t'_1,\dots,t'_n\rangle$ is transitive. Let $s = s_1\cdots s_r$ be a factorization of s into a product of independent cycles and let $\Gamma_1,\dots,\Gamma_r$ be the domains of transitivity of s. If $\sharp\Gamma_i = e_i$ for each $1 \le i \le r$ and 1_i is the minimal number in Γ_i, then we write $s_i = (1_i\ 2_i \dots (e_i)_i)$. Let us order the Γ_i so that $1_1 < 1_2 < \cdots < 1_r$ and denote by Z_i the sequence $((1_i\ 2_i),(1_i\ 3_i),\dots,(1_i\ (e_i)_i))$. Let Z be the concatenation $Z_1Z_2\dots Z_r$. The sequence Z consists of $N = \sum_{i=1}^{r}(e_i-1)$ transpositions. We use the following result.

PROPOSITION 2 ([11] or [12] pp. 369-370). – *Let $(t'_1,\dots,t'_n)$ be a sequence of transpositions such that $t'_1\cdots t'_n = s$ and $\langle t'_1,\dots,t'_n\rangle$ is transitive. Then $(t'_1,\dots,t'_n)$ is braid equivalent to*

$$(Z, t''_{N+1},\dots,t''_n)$$

where $n - N \equiv 0(mod\ 2)$ *and*

(i) *if* $r = 1$ $t''_i = (1_1\ 2_1)$ *for each* $i \geq N+1$

(ii) *if* $r > 1$ *then*

$$(t''_{N+1}, \dots, t''_n) = ((1_1\ 1_2), (1_1\ 1_2), (1_1\ 1_3), (1_1\ 1_3), \dots, (1_1\ 1_r), \dots, (1_1\ 1_r))$$

where each $(1_1\ 1_i)$ *appears twice if* $2 \leq i \leq r-1$ *and* $(1_1\ 1_r)$ *appears an even number of times.*

From now on let us denote the permutation Eq. (1) by

$$\varepsilon = (1_1 2_1 \dots (e_1)_1)(1_2 2_2 \dots (e_2)_2) \dots (1_r 2_r \dots (e_r)_r).$$

THEOREM 2. – *The Hurwitz space* $H_{W(B_d),(n_1;n_2,\underline{e})}(\mathbb{P}^1, b_0)$ *is irreducible.*

PROOF. – To prove the irreducibility of $H_{W(B_d),(n_1;n_2,\underline{e})}(\mathbb{P}^1, b_0)$ it is sufficient to check that each Hurwitz system in $A_{(n_1;n_2,\underline{e})}$ is braid equivalent to the normal form

$$((0;(1_1 2_1)), (0;(1_1 3_1)), \dots, (0;(1_1(e_1)_1)), (0;(1_2 2_2)), \dots, (0;(1_2(e_2)_2)), \dots,$$
$$(0;(1_r 2_r)), \dots, (0;(1_r(e_r)_r)), (0;t''_{N+1}), \dots, (0;t''_{n_2}), (\bar{1}_1;id), \dots, (\bar{1}_1;id), (0;\varepsilon^{-1}))$$

where $(\bar{1}_1;id)$ appears n_1-times, $N = \sum_{i=1}^r e_i - r$, $n_2 - N \equiv 0(mod\ 2)$ and

i) if $r = 1$, $(0;t''_j) = (0;(1_1 2_1))$ for each $j = N+1, ..., n_2$,

ii) if $r > 1$,

$$((0;t''_{N+1}), \dots, (0;t''_{n_2})) = ((0;(1_1 1_2)), (0;(1_1 1_2)), (0;(1_1 1_3)), (0;(1_1 1_3)), \dots,$$
$$(0;(1_1 1_r)), \dots, (0;(1_1 1_r)))$$

where each $(0;(1_1 1_i))$, $2 \leq i \leq r-1$, appears twice and $(0;(1_1 1_r))$ an even number of times.

Step 1. – We prove that, applying elementary moves σ'_j and their inverses, it is possible replace $(t_1, \dots, t_{n_1+n_2+1}) \in A_{(n_1;n_2,\underline{e})}$ by a new system which has $(0;\varepsilon^{-1})$ at the place $(n_1 + n_2 + 1)$. Using inverses of braid moves σ'_j, we bring to the first n_2 places of our Hurwitz system the t_j of type $(z_{ih};(ih))$ and to the place (n_2+1) the element $(a;\xi)$, obtaining

$$(\widetilde{t}_1, \dots, \widetilde{t}_{n_2}, (a;\xi), (\bar{1}_k;id), \dots, (\bar{1}_l;id)).$$

Let $\widetilde{t}_j = (z_{ih};t'_j)$, $(t'_1, \dots, t'_{n_2}, \xi)$ is the Hurwitz system of a covering of degree $d \geq 3$ of $\mathbb{P}^1$, with monodromy group S_d, branched in n_2+1 points, n_2 of whose are points of simple branching and one is a special point which local monodromy has cycle type $\underline{e}$. Since $H_{d,n_2,\underline{e}}(\mathbb{P}^1, b_0)$ is irreducible (see [13], [11] or [12]), acting by appropriate braid moves $\sigma'_j, (\sigma'_j)^{-1}$, we can replace the sequence $(t'_1, \dots, t'_{n_2}, \xi)$ by a new sequence of type $(t''_1, \dots, t''_{n_2}, \varepsilon^{-1})$. In this way our Hurwitz system results braid equivalent to $(\bar{t}_1, \dots, \bar{t}_{n_2}, (b;\varepsilon^{-1}), \dots)$, where $\varepsilon^{-1} = q_1 \cdots q_r$, with the q_i

disjoint cycles, and $b \in (\boldsymbol{Z}_2)^d$ is, by (v), a function that sends to $\bar{1}$ only an even number of indexes moved by each cycle q_i. By braid moves σ'_j we move $(b; \varepsilon^{-1})$ to the last place of our Hurwitz system.

Let $i^1, ..., i^{s_i}$ the indexes moved by q_i, $i = 1, \ldots, r$, which b sends to $\bar{1}$. We assume these indexes are place in the order $(\ldots i^1 \ldots i^2 \ldots i^{s_i} \ldots)$.

Since $\langle t''_1, \ldots, t''_{n_2}, \varepsilon^{-1} \rangle = S_d$ and $t''_1 \cdots t''_{n_2} = \varepsilon$, $\langle t''_1, \ldots, t''_{n_2} \rangle = S_d$. Note that we can ever assume that among the elements of type $(\bar{1}_*; id)$ in our Hurwitz system there is $(\bar{1}_{i^j}; id)$, for each $1 \le i \le r$, $1 \le j \le s_i$. In fact, applying Lemma 1 we can replace $(\ldots, \bar{t}_{n_2}, (\bar{1}_{k'}; id), \ldots, (\bar{1}_{l'}; id), (b; \varepsilon^{-1}))$ by

$$(\ldots, (z_{k' i^j}; (k' i^j)), (\bar{1}_{k'}; id), \ldots, (\bar{1}_{l'}; id), (b; \varepsilon^{-1})).$$

In the end using the elementary move σ'_{n_2}, by (iii), we obtain a new Hurwitz system with $(\bar{1}_{i^j}; id)$ at the place n_2.

At first we check that our Hurwitz system is braid equivalent to a system which has $(\bar{b}; \varepsilon^{-1})$ at the last place, where $\bar{b} \in (\boldsymbol{Z}_2)^d$ sends to $\bar{0}$ all indexes moved by q_1 and the indexes moved by q_i, $i = 2, \ldots, r$, distinct by $i^1, \ldots, i^{s_i}$. By elementary moves σ'_j we move $(\bar{1}_{1^{s_1}}; id)$ to the left of $(b; \varepsilon^{-1})$ and after we apply $\sigma'_{n_1+n_2}$ obtaining by (iv) the Hurwitz system

$$(\ldots, (\bar{1}_{1^{s_1}} + b + \bar{1}_\star; \varepsilon^{-1}), (\bar{1}_{1^{s_1}}; id))$$

where $\star$ is the index that precedes 1^{s_1} in q_1.

If $\star = 1^{s_1 - 1}$, the only indexes moved by q_1 that $b' = \bar{1}_{1^{s_1}} + b + \bar{1}_\star$ sends to $\bar{1}$ are $1^1, ..., 1^{s_1-2}$. If instead $\star \neq 1^{s_1-1}$, b' sends to $\bar{1}$ also 1^{s_1-1} and $\star$.

Let l be the number of indexes moved by q_1 included between 1^{s_1-1} and 1^{s_1}. Reasoning as above after $(l+1)$-steps we obtain a new system having at the last place $(\widehat{b}; \varepsilon^{-1})$ where $\widehat{b} = \bar{1}_* + \widetilde{b} + \bar{1}_{1^{s_1-1}}$, $*$ is the index which follows 1^{s_1-1} in q_1 and the only indexes moved by q_1 sent to $\bar{1}$ by $\widetilde{b}$ are $1^1, \ldots, 1^{s_1-1}, *$.

In this way we replaced b by a new function of $(\boldsymbol{Z}_2)^d$ which sends to $\bar{1}$ the same indexes sent to $\bar{1}$ by b, except the two indexes 1^{s_1} e 1^{s_1-1}. So proceeding also for $(\bar{1}_{1^{s_1-2}}; id), (\bar{1}_{1^{s_1-4}}; id), \ldots, (\bar{1}_{1^2}; id)$, after $\frac{s_1}{2}$ steps, we obtain a new system having as required at the $(n_1 + n_2 + 1)$ place $(\bar{b}; \varepsilon^{-1})$.

Reasoning so also with the indexes moved by $q_2, \ldots, q_r$ that $\bar{b}$ sends to $\bar{1}$ after $(\sum_{i=2}^{r} \frac{s_i}{2})$ steps we obtain the claim.

S*tep* 2. – Starting by the Hurwitz system $(t_1, \ldots, t_{n_1+n_2+1})$ and applying Step 1 we have obtained the system $(\widetilde{t}_1, \ldots, \widetilde{t}_{n_1+n_2}, (0; \varepsilon^{-1}))$. Since the group generated by transpositions corresponding to the elements $(z_{ih}; (ih))$ is all S_d, we can proceed as in Step 1, Theorem 1 and so we can bring our system to the form

$$(\widehat{t}_1, \ldots, \widehat{t}_{n_2}, (\bar{1}_1; id), \ldots, (\bar{1}_1; id), (0; \varepsilon^{-1})).$$

By Proposition 2 this system is braid equivalent to the system

$$((a^1_{1_1 2_1};(1_1 2_1)),\ (b^1_{1_1 3_1};\ (1_1 3_1)),\dots,\ (e^1_{1_1(e_1)_1};(1_1(e_1)_1)),\ (a^2_{1_2 2_2};(1_2 2_2)),\ \dots,$$
$$(e^2_{1_2(e_2)_2};(1_2(e_2)_2)),\dots,(a^r_{1_r 2_r};(1_r 2_r)),\dots,(e^r_{1_r(e_r)_r};(1_r(e_r)_r)),(z^{N+1};t''_{N+1}),\dots,$$
$$(z^{n_2};t''_{n_2}),(\bar{1}_1;id),\dots)$$

where $a^i,\ b^i,\dots,\ e^i,\ z^j \in \boldsymbol{Z}_2$.

At first we analyze the case $r>1$. Put

$$(z^{N+1},\dots,z^{n_2}) = (z^2_{1_1 1_2},(z^2)'_{1_1 1_2},\dots,z^{r-1}_{1_1 1_{r-1}},(z^{r-1})'_{1_1 1_{r-1}},z^r_{1_1 1_r},\dots,(z^r)^v_{1_1 1_r}).$$

Seeing that

$$(a^1_{1_1 2_1};(1_1 2_1))\cdots(e^1_{1_1(e_1)_1};(1_1(e_1)_1))\cdots(a^r_{1_r 2_r};(1_r 2_r))\cdots(e^r_{1_r(e_r)_r};(1_r(e_r)_r))$$
$$(z^2_{1_1 1_2};(1_1 1_2))((z^2)'_{1_1 1_2};(1_1 1_2))\cdots(z^r_{1_1 1_r};(1_1 1_r))\cdots((z^r)^v_{1_1 1_r};(1_1 1_r))\,(\bar{1}_1;id)\cdots$$
$$(\bar{1}_1;id)=(0;\varepsilon),$$

one has

$$(a^1_{1_1 2_1}+b^1_{2_1 3_1}+\cdots+e^1_{(e_1-1)_1(e_1)_1}+\cdots+a^r_{1_r 2_r}+\cdots+e^r_{(e_r-1)_r(e_r)_r}+z^2_{(e_1)_1(e_2)_2}+$$
$$(z^2)'_{(e_1)_1(e_2)_2}+\cdots+z^r_{(e_1)_1(e_r)_r}+\cdots+(z^r)^v_{(e_1)_1(e_r)_r}+\bar{1}_{(e_1)_1}+\cdots+\bar{1}_{(e_1)_1})=0.$$

That implies

$a^i=\bar{0}$ for each i and so $b^i=c^i=\ldots=e^i=\bar{0}$ for each i,

$z^j+(z^j)'\equiv\ \bar{0}\ (mod\ 2)$ for each $j=2,\dots,r-1$ and then $z^j=(z^j)'$,

$z^r+(z^r)^1+\cdots+(z^r)^v\equiv\ \bar{0}\ (mod\ 2)$ and therefore the $(z^r)^h=\bar{1}$ are an even number.

If in our Hurwitz system there are elements of type $(0;(1_1 1_r))$, acting by suitable elementary moves σ'_j, we move them to the right obtaining

$$((0;(1_1 2_1)),\dots,(0;(1_1(e_1)_1)),\dots,(0;(1_r 2_r)),\dots,(0;(1_r(e_r)_r)),(z^2_{1_1 1_2};(1_1 1_2)),$$
$$(z^2_{1_1 1_2};\ (1_1 1_2)),\dots,\ (z^{r-1}_{1_1 1_{r-1}};(1_1 1_{r-1})),\ (z^{r-1}_{1_1 1_{r-1}};(1_1 1_{r-1})),\ (\bar{1}_{1_1 1_r};(1_1 1_r)),\ \dots,$$
$$(\bar{1}_{1_1 1_r};(1_1 1_r)),(0;(1_1 1_r)),\dots,(0;(1_1 1_r)),(\bar{1}_1;id),\dots)$$

where the elements $(\bar{1}_{1_1 1_r};(1_1 1_r))$ appear at the places $(\Sigma_i e_i+r-4)+1,\dots,k$.

Since the elements $(0;(1_1 1_r))$ are an even number, we can move using elementary moves σ'_j one element $(\bar{1}_1;id)$ near by the elements of type $(\bar{1}_{1_1 1_r};(1_1 1_r))$ and then we apply the braid moves $(\sigma'_k)^{-1},(\sigma'_{k-1})^{-1},\dots,(\sigma'_{(\Sigma_i e_i+r-4)+1})^{-1}$. So by (ii) we can replace $((\bar{1}_{1_1 1_r};(1_1 1_r)),\dots,(\bar{1}_{1_1 1_r};(1_1 1_r)),(\bar{1}_1;id))$ by

$$((\bar{1}_1;id),(0;(1_1 1_r)),\dots,(0;(1_1 1_r))).$$

Now if $z^{r-1}=\bar{1}$ we act by the braid moves $(\sigma'_{(\Sigma_i e_i+r-4)})^{-1},\ (\sigma'_{(\Sigma_i e_i+r-5)})^{-1}$, if instead $z^{r-1}=\bar{0}$ we use $\sigma'_{(\Sigma_i e_i+r-4)},\sigma'_{(\Sigma_i e_i+r-4)-1}$.

The proof follows by proceeding in the same way for each $z^j, j=2,\dots,r-2$ and then applying in order the elementary braid moves

$$(\sigma'_{(\Sigma_i e_i-r+1)})^{-1},\ (\sigma'_{(\Sigma_i e_i-r+2)})^{-1},\dots,(\sigma'_{n_2})^{-1}.$$

Let now $r = 1$. This time we have

$$(a^1_{1_1 2_1} + b^1_{2_1 3_1} + \cdots + e^1_{(e_1-1)_1 (e_1)_1} + z^{N+1}_{1_1 (e_1)_1} + \cdots + z^{n_2}_{1_1 (e_1)_1} + \bar{1}_{(e_1)_1} + \cdots + \bar{1}_{(e_1)_1}) = 0.$$

This implies

$$\begin{aligned} &a^1 + z^{N+1} + \cdots + z^{n_2} \equiv \ \bar{0}\ (mod\ 2),\\ &a^1 + b^1 \equiv \ \bar{0}\ (mod\ 2) \ \text{ and then } a^1 = b^1,\\ &b^1 + c^1 \equiv \ \bar{0}\ (mod\ 2) \text{ and so } b^1 = c^1,\\ &\qquad\vdots\\ &(e-1)^1 + e^1 \equiv \ \bar{0}\ (mod\ 2) \quad \text{and then} \quad (e-1)^1 = e^1. \end{aligned}$$

Therefore $a^1 = b^1 = c^1 = \cdots = e^1$. If $a^1 = b^1 = c^1 = \cdots = e^1 = \bar{0}$ the elements z^j equal to $\bar{1}$ are an even number. So to obtain the wanted normal form we act as in the case $r > 1$.

If instead $a^1 = b^1 = c^1 = \cdots = e^1 = \bar{1}$, the number of $z^j = \bar{1}$ is odd. By appropriate elementary moves we bring our system to the form

$$\begin{gathered}((\bar{1}_{1_1 2_1}; (1_1 2_1)), \ldots, (\bar{1}_{1_1 (e_1)_1}; (1_1 (e_1)_1)), (\bar{1}_{1_1 2_1}; (1_1 2_1)), \ldots, (\bar{1}_{1_1 2_1}; (1_1 2_1)),\\ (0; (1_1 2_1)), \ldots, (0; (1_1 2_1)), (\bar{1}_1; id), \ldots, (\bar{1}_1; id), (0; \varepsilon^{-1}))\end{gathered}$$

where the elements of type $(0; (1_1 2_1))$, that are an odd number, occupy the places $k+1, \ldots, n_2$. Now acting in the order by $\sigma'_{n_2}, \ldots, \sigma'_{k+1}$, by the (iii), we obtain the system

$$\begin{gathered}(\ldots, (\bar{1}_{1_1 (e_1)_1}; (1_1 (e_1)_1)), (\bar{1}_{1_1 2_1}; (1_1 2_1)), \ldots, (\bar{1}_{1_1 2_1}; (1_1 2_1)), (\bar{1}_{2_1}; id), (0; (1_1 2_1)),\\ \ldots, (0; (1_1 2_1)), (\bar{1}_1; id), \ldots, (\bar{1}_1; id), (0; \varepsilon^{-1})).\end{gathered}$$

Applying the braid moves $(\sigma'_k)^{-1}, \ldots, (\sigma'_{(e_1)_1})^{-1}$ we can replace the sequence $((\bar{1}_{1_1 (e_1)_1}; (1_1 (e_1)_1)), (\bar{1}_{1_1 2_1}; (1_1 2_1)), \ldots, (\bar{1}_{1_1 2_1}; (1_1 2_1)), (\bar{1}_{2_1}; id))$ by

$$((\bar{1}_{1_1 (e_1)_1}; (1_1 (e_1)_1)), (\bar{1}_{2_1}; id), (0; (1_1 2_1)), \ldots, (0; (1_1 2_1))).$$

Now we move $(\bar{1}_{2_1}; id)$ to the right of $(\bar{1}_{1_1 2_1}; (1_1 2_1))$ and then we use σ'_1 obtaining the system

$$((\bar{1}_1; id), (\bar{1}_{1_1 2_1}; (1_1 2_1)), \ldots, (\bar{1}_{1_1 (e_1)_1}; (1_1 (e_1)_1)), (0; (1_1 2_1)), \ldots).$$

The theorem follows by applying the elementary moves

$$\sigma'_1, \ldots, \sigma'_{e_1 - 1}, (\sigma'_{e_1})^{-1}, \ldots, (\sigma'_{n_2})^{-1}.$$

THEOREM 3. – *The Hurwitz space* $H_{W(B_d),(n_1; n_2, [j_1, \ldots, j_v])}(\mathbb{P}^1, b_0)$ *is irreducible.*

PROOF. – The theorem follows if we prove that, acting by elementary moves σ'_j and their inverses, it is possible to bring any Hurwitz system of $A_{(n_1; n_2, [j_1, \ldots, j_v])}$ to the normal form

$$((0;(1_1 2_1)),\dots,(0;(1_1(e_1)_1)),(0;(1_2 2_2)),\dots,(0;(1_2(e_2)_2)),\dots,(0;(1_r 2_r)),\dots,$$
$$(0;(1_r(e_r)_r)),(z^{N+1};t''_{N+1}),\dots,(z^{n_2};t''_{n_2}),(\bar{1}_1;id),\dots,(\bar{1}_1;id),(\bar{1}_{1_{j_1}\dots 1_{j_v}};\varepsilon^{-1})),$$

where $(\bar{1}_1;id)$ appears n_1-times, $N=\sum_{i=1}^r e_i - r$, $n_2 - N \equiv 0\ (mod\ 2)$ and

i) if $r=1$ then $v=1, j_1=1$ and $(z^j;t''_j)=(0;(1_1 2_1))$ for each $j=N+1,\dots,n_2$,

ii) if $r>1$,

$$((z^{N+1};t''_{N+1}),\dots,(z^{n_2};t''_{n_2})) \;=\; ((z^2_{1_1 1_2};(1_1 1_2)),\;((z^2)^1_{1_1 1_2};(1_1 1_2)),$$
$$(z^3_{1_1 1_3};(1_1 1_3)),((z^3)^1_{1_1 1_3};(1_1 1_3)),\dots,(z^r_{1_1 1_r};(1_1 1_r)),\dots,((z^r)^l_{1_1 1_r};(1_1 1_r))),$$

where if $j\in\{j_1,\dots,j_v\}$

$$((z^j_{1_1 1_j};(1_1 1_j)),((z^j)^1_{1_1 1_j};(1_1 1_j))) = ((\bar{1}_{1_1 1_j};(1_1 1_j)),(0;(1_1 1_j))) \tag{2}$$

if instead $j\notin\{j_1,\dots,j_v\}$, $2\le j\le r$,

$$((z^j_{1_1 1_j};(1_1 1_j)),((z^j)^1_{1_1 1_j};(1_1 1_j))) = ((0;(1_1 1_j)),(0;(1_1 1_j))). \tag{3}$$

Moreover $((z^r)^m_{1_1 1_r};(1_1 1_r)) = (0;(1_1 1_r))$ for each $m=2,\dots,l$.

Step 1. – We claim that, acting by elementary moves and their inverses, it is possible to replace $(t_1,\dots,t_{n_1+n_2+1})\in A_{(n_1;n_2,[j_1,\dots,j_v])}$ by a new Hurwitz system having $(\bar{1}_{1_{j_1}\dots 1_{j_v}};\varepsilon^{-1})$ at the last place.

Proceeding as in Step 1 of Theorem 2, we can replace our Hurwitz system by one braid equivalent which has at the last place $(\bar{1}_{h_{j_1}\dots h_{j_v}};\varepsilon^{-1})$, where $\varepsilon^{-1}=q_1\cdots q_r$, with q_k e_k-cycle, h_{j_i}, $i=1,\dots,v$, is an index moved by the cycle q_{j_i} and $\bar{1}_{h_{j_1}\dots h_{j_v}}\in(\boldsymbol{Z}_2)^d$ sends to $\bar{1}$ only the indexes $h_{j_1},\dots,h_{j_v}$. The claim follows if $h_{j_i}=1_{j_i}$ for each $i=1,\dots,v$. So we suppose $h_{j_1}\neq 1_{j_1}$. Applying suitable elementary moves and then Lemma 1, we obtain a new system in which there is $(\bar{1}_{h_{j_1}};id)$. By braid moves of type σ'_j we move $(\bar{1}_{h_{j_1}};id)$ to the left of $(\bar{1}_{h_{j_1}\dots h_{j_v}};\varepsilon^{-1})$ and then we apply the braid move $\sigma'_{n_1+n_2}$. So we have

$$(\dots,(\bar{1}_{h_{j_1}};id),(\bar{1}_{h_{j_1}\dots h_{j_v}};\varepsilon^{-1}))\sim(\dots,(\bar{1}_{h_{j_1}}+\bar{1}_{h_{j_1}\dots h_{j_v}}+\bar{1}_{(h-1)_{j_1}};\varepsilon^{-1}),(\bar{1}_{h_{j_1}};id))$$

where $(h-1)_{j_1}$ is the index that comes first of h_{j_1} in q_{j_1} and $\bar{a}=\bar{1}_{h_{j_1}}+$ $+\bar{1}_{h_{j_1}\dots h_{j_v}}+\bar{1}_{(h-1)_{j_1}}$ is a function that sends to $\bar{1}$ only the indexes $(h-1)_{j_1},h_{j_2},\dots,h_{j_v}$. Let w be the indexes of q_{j_1} between h_{j_1} and 1_{j_1}. Reasoning as above after $(w+1)$-steps we obtain a new system with $(\bar{1}_{1_{j_1}h_{j_2}\dots h_{j_v}};\varepsilon^{-1})$ at the last place. Proceeding in this way for each $h_{j_i}\neq 1_{j_i}$, after a finite number of steps, we have the claim.

Step 2. – Starting by a Hurwitz system $(t_1,\dots,t_{n_1+n_2+1})$ and applying Step 1 we obtain the system $(\widetilde{t}_1,\dots,\widetilde{t}_{n_1+n_2},(\bar{1}_{1_{j_1}1_{j_2}\dots 1_{j_v}};\varepsilon^{-1}))$. Now acting by suitable elementary moves and applying Lemma 1 (see Theorem 1, Step 1) we can replace our Hurwitz system with one braid equivalent of the form

$$(\widehat{t}_1,\dots,\widehat{t}_{n_2-1},(\bar{1}_1;id),\dots,(\bar{1}_1;id),(\bar{1}_{1_{j_1}1_{j_2}\dots 1_{j_v}};\varepsilon^{-1})).$$

By Proposition 2 this Hurwitz system is braid equivalent to

$$((a^1_{1_12_1};\ (1_12_1)),\ (b^1_{1_13_1};(1_13_1)),\dots,(e^1_{1_1(e_1)_1};\ (1_1(e_1)_1)),\ (a^2_{1_22_2};(1_22_2)),\dots,$$
$$(e^2_{1_2(e_2)_2};(1_2(e_2)_2)),\dots,(a^r_{1_r2_r};(1_r2_r)),\dots,(e^r_{1_r(e_r)_r};(1_r(e_r)_r)),(z^{N+1};t''_{N+1}),\dots,$$
$$(z^{n_2};t''_{n_2}),(\bar{1}_1;id),\dots,(\bar{1}_1;id),(\bar{1}_{1_{j_1}1_{j_2}\dots 1_{j_v}};\varepsilon^{-1})),$$

where $a^i,\ b^i,\dots,\ e^i,\ z^j\in\boldsymbol{Z}_2$.

At first we suppose $r>1$. Put

$$(z^{N+1},\dots,z^{n_2})=(z^2_{1_11_2},(z^2)^1_{1_11_2},\dots,z^{r-1}_{1_11_{r-1}},(z^{r-1})^1_{1_11_{r-1}},z^r_{1_11_r},\dots,(z^r)^l_{1_11_r}).$$

Since

$$(a^1_{1_12_1};(1_12_1))\cdots(e^1_{1_1(e_1)_1};(1_1(e_1)_1))\cdots(a^r_{1_r2_r};(1_r2_r))\cdots(e^r_{1_r(e_r)_r};(1_r(e_r)_r))$$
$$(z^2_{1_11_2};(1_11_2))((z^2)^1_{1_11_2};(1_11_2))\cdots(z^r_{1_11_r};(1_11_r))\cdots((z^r)^l_{1_11_r};(1_11_r))(\bar{1}_1;id)\cdots$$
$$(\bar{1}_1;id)\ =\ ((\bar{1}_{1_{j_1}1_{j_2}\dots 1_{j_v}};\varepsilon^{-1}))^{-1}\ =\ (\bar{1}_{(e_{j_1})_{j_1}(e_{j_2})_{j_2}\dots(e_{j_v})_{j_v}};\varepsilon),$$

one has

$$(a^1_{1_12_1}+b^1_{2_13_1}+\dots+e^1_{(e_1-1)_1(e_1)_1}+a^2_{1_22_2}+b^2_{2_23_2}+\dots+e^2_{(e_2-1)_2(e_2)_2}+\dots+$$
$$a^r_{1_r2_r}+\dots+e^r_{(e_r-1)_r(e_r)_r}+z^2_{(e_1)_1(e_2)_2}+(z^2)^1_{(e_1)_1(e_2)_2}+z^3_{(e_1)_1(e_3)_3}+(z^3)^1_{(e_1)_1(e_3)_3}+$$
$$\dots+z^r_{(e_1)_1(e_r)_r}+\dots+(z^r)^l_{(e_1)_1(e_r)_r}+\bar{1}_{(e_1)_1}+\dots+\bar{1}_{(e_1)_1})=\bar{1}_{(e_{j_1})_{j_1}(e_{j_2})_{j_2}\cdots(e_{j_v})_{j_v}}.$$

This implies

$a^i=\bar{0}$ for each i and so $b^i=c^i=\ldots=e^i=\bar{0}$ for each i,

$z^j+(z^j)^1\equiv\bar{0}\ (mod\ 2)$ for $j\notin\{j_1,\dots,j_v\}$, $2\le j\le r-1$, and then $z^j=(z^j)^1$,

$z^j+(z^j)^1\equiv\bar{1}\ (mod\ 2)$ for $j\in\{j_1,\dots,j_v\}$, $2\le j\le r-1$, so either $z^j=\bar{1}$ and $(z^j)^1=\bar{0}$ or $(z^j)^1=\bar{1}$ and $z^j=\bar{0}$,

$z^r+(z^r)^1+\dots+(z^r)^l\equiv\bar{0}\ (mod\ 2)$ if $r\notin\{j_1,\dots,j_v\}$ and so the number of $(z^r)^h=\bar{1}$ is even, while if $r\in\{j_1,\dots,j_v\}$, $z^r+(z^r)^1+\dots+(z^r)^l\equiv\ \bar{1}\ (mod\ 2)$ and then the $(z^r)^h=\bar{1}$ are an odd number.

Note that, applying the braid move σ'_h, it is possible to replace $(\bar{t}_h=(0;(1_11_j)),\ \bar{t}_{h+1}=(\bar{1}_{1_11_j};(1_11_j)))$ by $((\bar{1}_{1_11_j};(1_11_j)),\ (0;(1_11_j)))$, so we can suppose $z^j=\bar{1},(z^j)^1=\bar{0}$ for each $j\in\{j_1,\dots,j_v\}$, $2\le j\le r-1$.

By braid move σ'_j we move the elements of type $(0;(1_11_r))$ near by ones of type $(\bar{1}_1;id)$. So we obtain a new system in which the elements $(\bar{1}_{1_11_r};(1_11_r))$ are at the places $(\Sigma_ie_i+r-4)+1,\dots,k$.

Applying the moves $\sigma'_{n_2},\dots,\sigma'_{k+1}$, we bring to the $(k+1)$-place $(\bar{1}_1;id)$ if $r\notin\{j_1,\dots,j_v\}$, $(\bar{1}_{1_r};id)$ if $r\in\{j_1,\dots,j_v\}$.

If $r\notin\{j_1,\dots,j_v\}$, we act by $(\sigma'_k)^{-1},\dots,(\sigma'_{(\Sigma_ie_i+r-4)+1})^{-1}$, to replace the sequence $((\bar{1}_{1_11_r};(1_11_r)),\dots,(\bar{1}_{1_11_r};(1_11_r)),(\bar{1}_1;id))$ by

$$((\bar{1}_1;id),(0;(1_11_r)),\dots,(0;(1_11_r))).$$

If instead $r \in \{j_1, \ldots, j_v\}$ we apply the sequence of braid moves $(\sigma'_k)^{-1}, \ldots, (\sigma'_{(\Sigma_i e_i+r-4)+2})^{-1}, \sigma'_{(\Sigma_i e_i+r-4)+1}$, so we have

$$((\bar{1}_{1_1 1_r}; (1_1 1_r)), \ldots, (\bar{1}_{1_1 1_r}; (1_1 1_r)), (\bar{1}_{1_r}; id)) \sim ((\bar{1}_1; id), (\bar{1}_{1_1 1_r}; (1_1 1_r)), (0; (1_1 1_r)), \ldots, (0; (1_1 1_r))).$$

Now if $z^{r-1} = (z^{r-1})^1 = \bar{1}$, applying $(\sigma'_{(\Sigma_i e_i+r-4)})^{-1}, (\sigma'_{(\Sigma_i e_i+r-4)-1})^{-1}$ one has

$$((\bar{1}_{1_1 1_{r-1}}; (1_1 1_{r-1})), (\bar{1}_{1_1 1_{r-1}}; (1_1 1_{r-1})), (\bar{1}_{1_1}; id)) \sim$$

$$((\bar{1}_1; id), (0; (1_1 1_{r-1})), (0; (1_1 1_{r-1}))).$$

If instead either $z^{r-1} = (z^{r-1})^1 = \bar{0}$ or $z^{r-1} = \bar{1}$ and $(z^{r-1})^1 = \bar{0}$ we use the moves $\sigma'_{(\Sigma_i e_i+r-4)}$, $\sigma'_{(\Sigma_i e_i+r-4)-1}$ to replace $((z^j_{1_1 1_{r-1}}; (1_1 1_{r-1})), ((z^j)^1_{1_1 1_{r-1}}; (1_1 1_{r-1})), (\bar{1}_1; id))$ by the sequence

$$((\bar{1}_1; id), (z^j_{1_1 1_{r-1}}; (1_1 1_{r-1})), ((z^j)^1_{1_1 1_{r-1}}; (1_1 1_{r-1}))).$$

Proceeding in the same way for each $j = 2, \ldots, r-2$ and then applying successively the braid moves $(\sigma'_{(\Sigma_i e_i-r+1)})^{-1}, (\sigma'_{(\Sigma_i e_i-r+2)})^{-1}, \ldots, (\sigma'_{n_2})^{-1}$ we obtain the theorem.

Now let $r = 1$. This time one has

$(a^1_{1_1 2_1} + b^1_{2_1 3_1} + \cdots + e^1_{(e_1-1)_1 (e_1)_1} + z^{N+1}_{1_1 (e_1)_1} + \cdots + z^{n_2}_{1_1 (e_1)_1} + \bar{1}_{(e_1)_1} + \cdots + \bar{1}_{(e_1)_1}) = \bar{1}_{(e_1)_1}$.

Therefore $a^1 = b^1 = c^1 = \cdots = e^1$ and $e^1 + z^{N+1} + \cdots + z^{n_2} \equiv \bar{0} \ (mod\ 2)$ (see Theorem 2, Step 2, case $r = 1$). Moreover $e^1 + z^{N+1} + \cdots + z^{n_2} + \bar{1} + \cdots + \bar{1} \equiv \equiv \bar{1} \ (mod\ 2)$ and then the number of elements $(\bar{1}_1; id)$ in our system is odd.

Observe that if $a^1 = b^1 = c^1 = \cdots = e^1 = \bar{0}$, the $z^j = \bar{1}$ are even, if instead $a^1 = b^1 = c^1 = \cdots = e^1 = \bar{1}$ the $z^j = \bar{1}$ are odd.

By suitable braid moves we bring our Hurwitz system to the form

$$((a^1_{1_1 2_1}; (1_1 2_1)), (b^1_{1_1 3_1}; (1_1 3_1)), \ldots, (e^1_{1_1 (e_1)_1}; (1_1 (e_1)_1)), (\bar{1}_{1_1 2_1}; (1_1 2_1)), \ldots, (\bar{1}_{1_1 2_1}; (1_1 2_1)), (0; (1_1 2_1)), \ldots, (0; (1_1 2_1)), (\bar{1}_1; id), \ldots, (\bar{1}_1; id), (\bar{1}_1; \varepsilon^{-1}))$$

where the elements $(0; (1_1 2_1))$ occupy the places $k+1, \ldots, n_2$.

Using in order the elementary moves $\sigma'_{n_2}, \ldots, \sigma'_{k+1}$ we bring to the place $(k+1)$ $(\bar{1}_1; id)$ if $a^1 = \cdots = e^1 = \bar{0}$, $(\bar{1}_{2_1}; id)$ if $a^1 = \cdots = e^1 = \bar{1}$. Now if $a^1 = \cdots = e^1 = \bar{0}$ we obtain the desired form applying the sequence of inverse braid moves $(\sigma'_k)^{-1}, \ldots, (\sigma'_{(e_1)_1})^{-1}$, $(\sigma'_{(e_1)_1})^{-1}, \ldots, (\sigma'_{n_2})^{-1}$. When instead $a^1 = \cdots = = e^1 = \bar{1}$ the proof follows by applying the moves $(\sigma'_k)^{-1}, \ldots, (\sigma'_{(e_1)_1})^{-1}$, $\sigma'_{(e_1-1)_1}, \ldots, \sigma'_1$, $\sigma'_1, \ldots,$ $\sigma'_{(e_1-1)_1}, (\sigma'_{(e_1)_1})^{-1}, \ldots, (\sigma'_{n_2})^{-1}$ (see Theorem 2, Step 2, case $r = 1$).

The next result follows from Theorem 2 and Theorem 3.

THEOREM 4. – *The Hurwitz spaces* $H_{W(B_d),(n_1;n_2,\underline{e})}(\mathbb{P}^1)$ *and* $H_{W(B_d),(n_1;n_2,[j_1,\ldots,j_v])}(\mathbb{P}^1)$ *are irreducible.*

PROOF. – The map forgetful $H_{W(B_d),(n_1;n_2,\underline{e})}(\mathbb{P}^1, b_0) \rightarrow H_{W(B_d),(n_1;n_2,\underline{e})}(\mathbb{P}^1)$ given by $[f \circ \pi, \phi] \rightarrow [f \circ \pi]$ is a morfism and it has image given by a dense subset H of $H_{W(B_d),(n_1;n_2,\underline{e})}(\mathbb{P}^1)$. Since, by Theorem 2, $H_{W(B_d),(n_1;n_2,\underline{e})}(\mathbb{P}^1, b_0)$ is irreducible also H is irreducible and then $H_{W(B_d),(n_1;n_2,\underline{e})}(\mathbb{P}^1)$ is irreducible.

Analogously, using Theorem 3, one prove that also the Hurwitz space $H_{W(B_d),(n_1;n_2,[j_1,\dots,j_v])}(\mathbb{P}^1)$ is irreducible.

2.3. Let Y be a smooth, connected, projective, complex curve of genus $g \geq 1$. Let $b_0 \in Y$ and let $|\underline{e}| = \sum_{i=1}^{r}(e_i - 1)$.

THEOREM 5. – *If* $n_2 \geq 2d - 2$ *the Hurwitz space* $H_{W(B_d),(n_1;n_2)}(Y, b_0)$ *is irreducible.*

PROOF. – To prove the irreducibility of $H_{W(B_d),(n_1;n_2)}(Y, b_0)$ it is sufficient to check that each Hurwitz system $(t_1, \dots, t_{n_1+n_2}; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g) \in A_{(n_1;n_2),g}$ is braid equivalent to a system of the form $(\widehat{t}_1, \dots, \widehat{t}_{n_1+n_2}; (0; id), \dots, (0; id))$. In fact $(\widehat{t}_1, \dots, \widehat{t}_{n_1+n_2})$ is the Hurwitz system of a covering in $H_{W(B_d),(n_1;n_2)}(\mathbb{P}^1, b_0)$, so the proof follows by Theorem 1.

With inverses of elementary moves σ'_j we move to the left the t_j of type $(z_{ih}; (ih))$, obtaining the system

$$(\widetilde{t}_1, \dots, \widetilde{t}_{n_2}, (\bar{1}_h; id), \dots, (\bar{1}_k; id); \lambda_1, \mu_1, \dots, \lambda_g, \mu_g).$$

Let $\lambda_k = (a_k; \lambda'_k)$, $\mu_k = (b_k; \mu'_k)$ and $\widetilde{t}_j = (z_{ih}; t'_j)$ where $t'_j = (ih)$. Note that $(t'_1, \dots, t'_{n_2}; \lambda'_1, \mu'_1, \dots, \lambda'_g, \mu'_g)$ is the Hurwitz system of a covering of degree $d \geq 3$ of Y, with n_2 points of simple branching and with monodromy group S_d. Since for $n_2 \geq 2d - 2$ the Hurwitz space $H^o_{d,n_2}(Y, b_0)$ is irreducible (see [10]), acting by suitable braid moves $\sigma'_i, \rho'_{jk}, \tau'_{jk}$, $1 \leq i \leq n_2 - 1$, $1 \leq j \leq n_2$, $1 \leq k \leq g$ and their inverses, we can bring $(t'_1, \dots, t'_{n_2}; \lambda'_1, \mu'_1, \dots, \lambda'_g, \mu'_g)$ to the form $(t''_1, \dots, t''_{n_2}; id, \dots, id)$. Therefore our Hurwitz system is braid equivalent to the system $(\bar{t}_1, \dots, \bar{t}_{n_2}, (\bar{1}_h; id), \dots, (\bar{1}_k; id); (a'_1; id), (b'_1; id), \dots, (a'_g; id), (b'_g, id))$ where $a'_k, b'_k \in (\boldsymbol{Z}_2)^d$.

The theorem follows if $a'_k = 0$ and $b'_k = 0$ for each $k = 1, \dots, g$. So let $a'_1 \neq 0$ and let l be the indexes sent to $\bar{1}$ by a'_1. Let i be one of these indexes. As we saw in Theorem 2, Step 1, it is not restrictive to suppose that in our system there is $(\bar{1}_i; id)$. By inverse braid moves σ'_j we bring to the first place the element $(\bar{1}_i; id)$ and after we apply the braid move τ''_{11} which transforms $(a'_1; id)$ into $(a''_1; id)$ where $(a''_1; id) = (\bar{1}_i; id)(a'_1; id)$ and a''_1 is a function that sends i to $\bar{0}$. So reasoning after $(l - 1)$-steps we obtain a new Hurwitz system in which there is $(0; id)$ at the place $(n_2 + n_1 + 1)$. If $a'_1 = 0$, $b'_1 \neq 0$ and b'_1 sends i to $\bar{1}$, we move to the first place of our system $(\bar{1}_i; id)$ and after we apply the braid move ρ'_{11} that transforms $(b'_1; id)$ into $(b''_1; id) = (\bar{1}_i; id)(b'_1; id)$ where b''_1 sends i to $\bar{0}$. Proceeding as above for all indexes sent to $\bar{1}$ by b''_1 we can replace $(b''_1; id)$ by $(0; id)$. Observe that if $a'_k \neq 0$ and

$a'_l = b'_l = 0$, for each $l \leq k-1$, we reason in the same way this time applying the braid move τ''_{1k}. Analogously if $b'_k \neq 0$ and $a'_l = b'_l = 0$, for each $l \leq k-1$ and $a'_k = 0$ we apply the braid move ρ'_{1k} to transform $(b'_k; id)$ into $(0; id)$.

THEOREM 6. – *If* $n_2 + |\underline{e}| \geq 2d$ *the Hurwitz spaces* $H_{W(B_d),(n_1;n_2,\underline{e})}(Y, b_0)$ *and* $H_{W(B_d),(n_1;n_2,[j_1,\ldots,j_v])}(Y, b_0)$ *are irreducible.*

PROOF. – To prove the irreducibility of $H_{W(B_d),(n_1;n_2,\underline{e})}(Y, b_0)$ (respectively $H_{W(B_d),(n_1;n_2,[j_1,\ldots,j_v])}(Y, b_0)$) it is sufficient to test that, acting by braid moves σ'_i, ρ'_{jk}, τ'_{jk} and their inverses, it is possible to bring each Hurwitz system in $A_{(n_1;n_2,\underline{e}),g}$ (resp. $A_{(n_1;n_2,[j_1,\ldots,j_v]),g}$) to a given normal form. If we check it is possible to replace $(t_1, \ldots, t_{n_1+n_2+1}; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g) \in A_{(n_1;n_2,\underline{e}),g}$ (resp. $A_{(n_1;n_2,[j_1,\ldots,j_v]),g}$) by $(\widehat{t}_1, \ldots, \widehat{t}_{n_1+n_2+1}; (0; id), \ldots, (0; id))$, the proof follows by Theorem 2 (resp. Theorem 3). Acting by appropriate elementary moves σ'_j we can bring our Hurwitz system to the form

$$(\widetilde{t}_1, \ldots, \widetilde{t}_{n_2+1}, (\bar{1}_h; id), \ldots, (\bar{1}_k; id); \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g),$$

where $\lambda_k = (a_k; \lambda'_k)$, $\mu_k = (b_k; \mu'_k)$, for some l $\widetilde{t}_l = (a; t'_l = \xi)$ (resp. $(a'; t'_l = \xi)$) and $\widetilde{t}_j = (*; t'_j)$ for each $j \neq l$. $(t'_1, \ldots, t'_{n_2+1}; \lambda'_1, \ldots, \mu'_g)$ is the Hurwitz system of a branched covering of degree $d \geq 3$ of Y, with monodromy group S_d and with n_2 points of simple branching and one special point whose local monodromy has cycle type $\underline{e}$. Since $n_2 + |\underline{e}| \geq 2d$ the Hurwitz space $H^o_{d,n_2,\underline{e}}(Y, b_0)$ is irreducible (see [15]). Therefore it is possible acting by braid move, to replace $(t'_1, \ldots, t'_{n_2+1}; \lambda'_1, \ldots, \mu'_g)$ by $(t''_1, \ldots, t''_{n_2+1}; id, \ldots, id)$. So our Hurwitz system results braid equivalent to the system $(\bar{t}_1, \ldots, \bar{t}_{n_2+1}, (\bar{1}_h; id), \ldots, (\bar{1}_k; id); (a'_1; id), (b'_1; id), \ldots, (a'_g; id), (b'_g; id))$, where a'_k, $b'_k \in (\boldsymbol{Z}_2)^d$. Now to obtain the desired normal form it is sufficient to proceed as in Theorem 4.

THEOREM 7. – *In the same hypothesis of Theorem* 5 *the Hurwitz space* $H_{W(B_d),(n_1;n_2)}(Y)$ *is irreducible. In the same hypothesis of Theorem* 6 *the Hurwitz spaces* $H_{W(B_d),(n_1;n_2,\underline{e})}(Y)$ *and* $H_{W(B_d),(n_1;n_2,[j_1,\ldots,j_v])}(Y)$ *are irreducible.*

PROOF. – To obtain the thesis it is sufficient to proceed as in Theorem 4 by using respectively Theorem 5 and Theorem 6.

3. – The case $D_\pi \subseteq f^{-1}(c)$.

Let X and X' be smooth, connected, projective, complex curves of genus ≥ 0. Let $X \xrightarrow{\pi} X' \xrightarrow{f} \mathbb{P}^1$ be a sequence of coverings satisfying the following: π is a branched covering of degree 2 with $n_1 > 0$ branch points and f is a branched covering of degree $d \geq 3$, with n_2 points of simple branching and one special point whose local monodromy has cycle type $\underline{e}$. Let us denote by c the special point of f. In this section we are interested in coverings such that $D_\pi \subset f^{-1}(c) = \{c_1, \dots, c_r\}$. Note that n_1 is even follows by Hurwitz formula, therefore $r \geq 2$. Let D_f be the branch locus of f. Then $f \circ \pi$ is a covering of degree $2d$ of $\mathbb{P}^1$, with $n_2 + 1$ branch points and branch locus $D = D_f$. Let $D_\pi = \{c_{j_1}, \dots, c_{j_{n_1}}\}$ where c_{j_i} has multiplicity e_{j_i}, $1 \leq i \leq n_1$ and $j_1 < j_2 < \dots < j_{n_1}$. Let us denote by $H_{W(B_d),(n_2;[j_1,\dots,j_{n_1}])}(\mathbb{P}^1)$ the Hurwitz space that parametrizes equivalence classes $[f \circ \pi]$ where $f \circ \pi$ is a covering as above. Let $\delta : H_{W(B_d),(n_2,[j_1,\dots,j_{n_1}])}(\mathbb{P}^1) \rightarrow (\mathbb{P}^1)^{(n_2+1)} - \Delta$ the map that sends each class $[f \circ \pi]$ to the branch locus of $f \circ \pi$.

DEFINITION 6. – *Two Hurwitz systems,* $(t_1, \dots, t_n)$ *and* $(\bar{t}_1, \dots, \bar{t}_n)$ *with values in* $(\boldsymbol{Z}_2)^d \times^s S_d$, *are called* equivalent *if there exists* $s \in (\boldsymbol{Z}_2)^d \times^s S_d$ *such that* $\bar{t}_j = s^{-1}\ t_j\ s$ *for each* j. *The equivalence class containing* $(t_1, \dots, t_n)$ *is denoted by* $[t_1, \dots, t_n]$.

Let $A_{(n_2,[j_1,\dots,j_{n_1}])}$ be the set of all equivalence classes $[t_1, \dots, t_{n_2+1}]$ of Hurwitz systems with values in $(\boldsymbol{Z}_2)^d \times^s S_d$, with transitive monodromy group, such that n_2 among the t_j are of type $(z_{ih}; (ih))$ and one of type $(a'; \xi)$. By Riemann existence theorem we can identify the fiber of δ over D by $A_{(n_2,[j_1,\dots,j_{n_1}])}$. There is a unique topology on $H_{W(B_d),(n_2,[j_1,\dots,j_{n_1}])}(\mathbb{P}^1)$ such that δ is a topological covering map, [7]. So the braid group $\pi_1((\mathbb{P}^1)^{(n_2+1)} - \Delta, D)$ acts on $A_{(n_2,[j_1,\dots,j_{n_1}])}$. If this action is transitive then $H_{W(B_d),(n_2,[j_1,\dots,j_{n_1}])}(\mathbb{P}^1)$ is connected.

THEOREM 8. – *The Hurwitz space* $H_{W(B_d),(n_2,[j_1,\dots,j_{n_1}])}(\mathbb{P}^1)$ *is irreducible.*

PROOF. – Since $H_{W(B_d),(n_2,[j_1,\dots,j_{n_1}])}(\mathbb{P}^1)$ is smooth in order to prove its irreducibility it suffices to prove it is connected. Therefore the theorem follows if we prove that, acting by elementary moves σ'_j and their inverses, it is possible to bring each $[t_1, \dots, t_{n_2+1}] \in A_{(n_2,[j_1,\dots,j_{n_1}])}$ to the normal form

$$[(0; (1_1 2_1)),\ (0; (1_1 3_1)), \dots, (0; (1_1 (e_1)_1)),\ (0; (1_2 2_2)), \dots,\ (0; (1_2 (e_2)_2)), \dots,$$
$$(0; (1_r 2_r)), \dots, (0; (1_r (e_r)_r)),\ (z^2_{1_1 1_2}; (1_1 1_2)),\ ((z^2)^1_{1_1 1_2}; (1_1 1_2)),\ (z^3_{1_1 1_3}; (1_1 1_3)),$$
$$((z^3)^1_{1_1 1_3}; (1_1 1_3)), \dots, (z^r_{1_1 1_r}; (1_1 1_r)), \dots, ((z^r)^l_{1_1 1_r}; (1_1 1_r)), (\bar{1}_{1_{j_1} \dots 1_{j_{n_1}}}; \varepsilon^{-1})]$$

where the pairs $((z^j_{1_1 1_j}; (1_1 1_j)), ((z^j)^1_{1_1 1_j}; (1_1 1_j)))$, $2 \leq j \leq r$, satisfy either (2) or (3) depending on whether j belongs to $\{j_1, \dots, j_{n_1}\}$ or not. Moreover $((z^r)^m_{1_1 1_r}; (1_1 1_r)) = (0; (1_1 1_r))$ for each $m = 2, \dots, l$.

Step 1. – We check that, acting by elementary moves and their inverses, it is possible to bring each $[t_1, \dots, t_{n_2+1}] \in A_{(n_2,[j_1,\dots,j_{n_1}])}$ to $[\tilde{t}_1, \dots, \tilde{t}_{n_2}, (\bar{1}_{1_{j_1} \dots 1_{j_{n_1}}}; \varepsilon^{-1})]$. By elementary moves σ'_j we move to the last place of our system the element of type $(a'; \xi)$. Because ξ is in the same conjugate class of ε, there is $s \in S_d$ such that

$\varepsilon^{-1} = s^{-1}\ \xi\ s$. So if we conjugate each element of our system by $(0; s)$, we obtain a new system of the class having to the last place $(b'; \varepsilon^{-1})$. Let $\varepsilon^{-1} = q_1 \cdots q_r$ where q_i is an e_i-cycle and let $i^1, i^2, \ldots, i^{s_i}$ be the indexes moved by q_i which b' sends to $\bar{1}$. Let

$$q_i = (\ldots i^1\ (i^1+1) \ldots (i^2-1)\ i^2 \ldots i^{s_i-1}\ (i^{s_i-1}+1) \ldots (i^{s_i}-1)\ i^{s_i} \ldots).$$

Conjugating by $(\bar{1}_{(i^{s_i-1}+1)\ldots(i^{s_i}-1)i^{s_i}}; id)$ we obtain

$$[\ldots, (\bar{1}_{(i^{s_i-1}+1)\ldots(i^{s_i}-1)i^{s_i}} + b' + \bar{1}_{i^{s_i-1}\ldots(i^{s_i}-1)}; \varepsilon^{-1})]$$

where $b'' = \bar{1}_{(i^{s_i-1}+1)\ldots(i^{s_i}-1)i^{s_i}} + b' + \bar{1}_{i^{s_i-1}\ldots(i^{s_i}-1)} \in (\boldsymbol{Z}_2)^d$ is a function that sends to $\bar{1}$ only the indexes $i^1, i^2, \ldots, i^{s_{i-2}}$. If $i \notin \{j_1, \ldots, j_{n_1}\}$, s_i is even. So if we conjugate by

$$(\bar{1}_{(i^{s_i-3}+1)\ldots(i^{s_i-2}-1)i^{s_i-2}}; id)(\bar{1}_{(i^{s_i-5}+1)\ldots(i^{s_i-4}-1)i^{s_i-4}}; id) \cdots$$

$$(\bar{1}_{(i^1+1)\ldots(i^2-1)i^2}; id)$$

we obtain a new system of the class with $(\bar{b}; \varepsilon^{-1})$ at the place (n_2+1), where $\bar{b}$ is a function which sends to $\bar{0}$ all indexes of q_i. If $i \in \{j_1, \ldots, j_{n_1}\}$, s_i is odd and then conjugating by

$$(\bar{1}_{(i^{s_i-3}+1)\ldots(i^{s_i-2}-1)i^{s_i-2}}; id)(\bar{1}_{(i^{s_i-5}+1)\ldots(i^{s_i-4}-1)i^{s_i-4}}; id) \cdots$$

$$(\bar{1}_{(i^2+1)\ldots(i^3-1)i^3}; id)$$

we obtain another system of the class having $(\bar{\bar{b}}; \varepsilon^{-1})$ at the last place, where the only index of q_i mapped in $\bar{1}$ by $\bar{\bar{b}}$ is i^1. So if $i^1 \neq 1_i$ and $q_i = (1_i\ (e_i)_i \ldots (i^1-1)\ i^1 \ldots)$, we conjugate by $(\bar{1}_{(e_i)_i \ldots (i^1-1)i^1}; id)$ to obtain $[\bar{t}_1, \ldots, \bar{t}_{n_2}, (\widetilde{b}; \varepsilon^{-1})]$ where $\widetilde{b} \in (\boldsymbol{Z}_2)^d$ sends 1_i to $\bar{1}$ and to $\bar{0}$ each other indexes moved by the cycle q_i. We obtain the claim reasoning in this way for each $i = 1, \ldots, r$.

Step 2. – Starting by $[t_1, \ldots, t_{n_2+1}] \in A_{(n_2, [j_1, \ldots, j_{n_1}])}$ and applying Step 1 we obtain $[\widetilde{t}_1, \ldots, \widetilde{t}_{n_2}, (\bar{1}_{1_{j_1}\ldots 1_{j_{n_1}}}; \varepsilon^{-1})]$. Let $\widetilde{t}_j = (z_{ih}; t'_j)$. Since $\langle t'_1, \ldots, t'_{n_2} \rangle = S_d$, $t'_1 \cdots t'_{n_2} = \varepsilon$ and $r \geq 2$, by Proposition 2 $[\widetilde{t}_1, \ldots, \widetilde{t}_{n_2}, (\bar{1}_{1_{j_1}\ldots 1_{j_{n_1}}}; \varepsilon^{-1})]$ is braid equivalent to

$$[(a^1_{1_1 2_1}; (1_1 2_1)),\ (b^1_{1_1 3_1}; (1_1 3_1)), \ldots,\ (e^1_{1_1 (e_1)_1}; (1_1 (e_1)_1)),\ (a^2_{1_2 2_2}; (1_2 2_2)), \ldots,$$

$$(e^2_{1_2 (e_2)_2};\ (1_2 (e_2)_2)), \ldots, (a^r_{1_r 2_r}; (1_r 2_r)), \ldots,\ (e^r_{1_r (e_r)_r}; (1_r (e_r)_r)),\ (z^2_{1_1 1_2}; (1_1 1_2)),$$

$$((z^2)^1_{1_1 1_2}; (1_1 1_2)), \ldots, (z^r_{1_1 1_r}; (1_1 1_r)), \ldots, ((z^r)^l_{1_1 1_r}; (1_1 1_r)), (\bar{1}_{1_{j_1} 1_{j_2} \ldots 1_{j_{n_1}}}; \varepsilon^{-1})]$$

where $a^i,\ b^i, \ldots,\ e^i,\ z^j, (z^j)^h\ \in \boldsymbol{Z}_2$. Seeing that

$$(a^1_{1_1 2_1}; (1_1 2_1)) \cdots (e^1_{1_1 (e_1)_1}; (1_1 (e_1)_1)) \cdots (a^r_{1_r 2_r}; (1_r 2_r)) \cdots (e^r_{1_r (e_r)_r}; (1_r (e_r)_r))$$

$$(z^2_{1_1 1_2}; (1_1 1_2))((z^2)^1_{1_1 1_2}; (1_1 1_2)) \cdots (z^r_{1_1 1_r}; (1_1 1_r)) \cdots ((z^r)^l_{1_1 1_r}; (1_1 1_r)) =$$

$$= (\bar{1}_{(e_{j_1})_{j_1} (e_{j_2})_{j_2} \ldots (e_{j_{n_1}})_{j_{n_1}}}; \varepsilon)$$

we have

$$(a^1_{1_12_1}+b^1_{2_13_1}+\cdots+e^1_{(e_1-1)_1(e_1)_1}+\cdots+a^r_{1_r2_r}+b^r_{2_r3_r}+\cdots+e^r_{(e_r-1)_r(e_r)_r}+$$

$$+z^2_{(e_1)_1(e_2)_2}+(z^2)^1_{(e_1)_1(e_2)_2}+\cdots+z^r_{(e_1)_1(e_r)_r}+\cdots+(z^r)^l_{(e_1)_1(e_r)_r})=$$

$$=\bar{1}_{(e_{j_1})_{j_1}(e_{j_2})_{j_2}\dots(e_{j_{n_1}})_{j_{n_1}}}.$$

Therefore (see Theorem 3, Step 2, case $r>1$) $a^i=b^i=c^i=\ldots=e^i=\bar{0}$ for each i, $z^j=(z^j)^1$ for each $j\notin\{j_1,\ldots,j_{n_1}\}$, while $z^j=\bar{1}$ and $(z^j)^1=\bar{0}$ for each $j\in\{j_1,\ldots,j_{n_1}\}$. Moreover the number of $(z^r)^h=\bar{1}$ is even if $r\notin\{j_1,\ldots,j_{n_1}\}$, odd otherwise.

Acting by braid moves σ'_j we move to the left of $(\bar{1}_{1_{j_1}\cdots 1_{j_{n_1}}};\varepsilon^{-1})$ the elements of type $(0;(1_11_r))$, obtaining so a new braid equivalent system in which the elements $(\bar{1}_{1_11_r};(1_11_r))$ are at the places $(\Sigma_ie_i+r-4)+1,\ldots,k$. Since $n_1>0$ and n_1 is even, $n_1\ge 2$ and so $|\{j_1,\ldots,j_{n_1}\}|\ge 2$. At first we analyze the case in which there exists at least one $j\in\{j_1,\ldots,j_{n_1}\}$ such that $j\neq 1,r$. Since for each $j\in\{j_1,\ldots,j_{n_1}\}$ $j\neq 1$, we obtain pairs of type $((\bar{1}_{1_11_j};(1_11_j)),(0;(1_11_j)))$, in our Hurwitz system there is one pair of this type with $j\neq r$. Let h and $h+1$ be the places occupy by $(\bar{1}_{1_11_j};(1_11_j))$ and $(0;(1_11_j))$. With elementary moves we bring the pair $((\bar{1}_{1_11_j};(1_11_j)),(0;(1_11_j)))$ to the left of the element $(\bar{1}_{1_11_r};(1_11_r))$ of place $(\Sigma_ie_i+r-4)+1)$. Now, if $r\notin\{j_1,\ldots,j_{n_1}\}$, we apply the moves $\sigma'_{(\sum_i e_i+r-4)}$, $\sigma'_{(\sum_i e_i+r-4)-1}$, $\sigma'_{(\sum_i e_i+r-4)+1}$, $\sigma'_{(\sum_i e_i+r-4)},\ldots,\sigma'_{k-1}$, σ'_{k-2}, so we can replace the sequence

$$((\bar{1}_{1_11_j};(1_11_j)),(0;(1_11_j)),(\bar{1}_{1_11_r};(1_11_r)),\ldots,(\bar{1}_{1_11_r};(1_11_r)))$$

by

$$((0;(1_11_r)),\ldots,(0;(1_11_r)),(\bar{1}_{1_11_j};(1_11_j)),(0;(1_11_j))).$$

If instead $r\in\{j_1,\ldots,j_{n_1}\}$, we use in the order the moves $(\sigma'_{(\sum_i e_i+r-4)})^{-1}$, $(\sigma'_{(\sum_i e_i+r-4)-1})^{-1},\sigma'_{(\sum_i e_i+r-4)+1},\sigma'_{(\sum_i e_i+r-4)},\ldots,\sigma'_{k-1},\sigma'_{k-2}$ obtaining

$$((\bar{1}_{1_11_j};(1_11_j)),(0;(1_11_j)),(\bar{1}_{1_11_r};(1_11_r)),\ldots,(\bar{1}_{1_11_r};(1_11_r)))\sim$$

$$((\bar{1}_{1_11_r};(1_11_r)),(0;(1_11_r)),\ldots,(0;(1_11_r)),(0;(1_j1_r)),(\bar{1}_{1_j1_r};(1_j1_r))).$$

Acting by σ'_{k-2}, σ'_{k-1}, σ'_{k-3}, $\sigma'_{k-2},\ldots$, $\sigma'_{(\sum_i e_i+r-4)-1}$, $\sigma'_{(\sum_i e_i+r-4)}$, we replace if $r\notin\{j_1,\ldots,j_{n_1}\}$ the sequence $((0;(1_11_r)),\ldots,(0;(1_11_r)),(\bar{1}_{1_11_j};(1_11_j)),(0;(1_11_j)))$ by

$$((\bar{1}_{1_11_j};(1_11_j)),(0;(1_11_j)),(0;(1_11_r)),\ldots,(0;(1_11_r))),$$

if instead $r\in\{j_1,\ldots,j_{n_1}\}$ the sequence $((\bar{1}_{1_11_r};(1_11_r)),(0;(1_11_r)),\ldots,(0;(1_11_r)),$

$(0;(1_j1_r)),(\bar{1}_{1_j1_r};(1_j1_r)))$ by

$$((\bar{1}_{1_11_j};(1_11_j)),(0;(1_11_j)),(\bar{1}_{1_11_r};(1_11_r)),(0;(1_11_r)),\dots,(0;(1_11_r))).$$

Now if $z^{r-1}=(z^{r-1})^1=\bar{1}$, applying $(\sigma'_{(\Sigma_i e_i+r-4)-2})^{-1}$, $(\sigma'_{(\Sigma_i e_i+r-4)-1})^{-1}$, $(\sigma'_{(\Sigma_i e_i+r-4)-3})^{-1}$, $(\sigma'_{(\Sigma_i e_i+r-4)-2})^{-1}$ we have

$$((\bar{1}_{1_11_{r-1}};(1_11_{r-1})),(\bar{1}_{1_11_{r-1}};(1_11_{r-1})),(\bar{1}_{1_11_j};(1_11_j)),(0;(1_11_j)))\sim$$

$$((\bar{1}_{1_11_j};(1_11_j)),(0;(1_11_j)),(0;(1_11_{r-1})),(0;(1_11_{r-1}))).$$

If either $z^{r-1}=(z^{r-1})^1=\bar{0}$ or $z^{r-1}=\bar{1}$ and $(z^{r-1})^1=\bar{0}$, we act by $\sigma'_{(\Sigma_i e_i+r-4)-2}$, $\sigma'_{(\Sigma_i e_i+r-4)-1}$, $\sigma'_{(\Sigma_i e_i+r-4)-3}$, $\sigma'_{(\Sigma_i e_i+r-4)-2}$, obtaining either

$$((0;(1_11_{r-1})),(0;(1_11_{r-1})),(\bar{1}_{1_11_j};(1_11_j)),(0;(1_11_j)))\sim$$

$$((\bar{1}_{1_11_j};(1_11_j)),(0;(1_11_j)),(0;(1_11_{r-1})),(0;(1_11_{r-1})))$$

or

$$((\bar{1}_{1_11_{r-1}};(1_11_{r-1})),(0;(1_11_{r-1})),(\bar{1}_{1_11_j};(1_11_j)),(0;(1_11_j)))\sim$$

$$((0;(1_11_j)),(\bar{1}_{1_11_j};(1_11_j)),(\bar{1}_{1_11_{r-1}};(1_11_{r-1})),(0;(1_11_{r-1}))).$$

The proof follows by proceeding in this way for each $j=2,\dots,r-2$ and applying $(\sigma'_{(\Sigma_i e_i-r)+2})^{-1}$, $(\sigma'_{(\Sigma_i e_i-r)+1})^{-1}$, $(\sigma'_{(\Sigma_i e_i-r)+3})^{-1}$, $(\sigma'_{(\Sigma_i e_i-r)+2})^{-1},\dots$, $(\sigma'_h)^{-1}$, $(\sigma'_{h-1})^{-1}$.

Now we analyze the case in which there is not any $j\in\{j_1,\dots,j_{n_1}\}$ such that $j\neq 1,\ r$, this implies $n_1=2$ and $\{j_1,j_2\}=\{1,r\}$. Note that since $r\in\{j_1,\ j_2\}$, the $(z^r)^h=\bar{1}$ are an odd number and so among the elements of type $((z^r)^h_{1_11_r};(1_11_r))$ in our Hurwitz system there is at least one $(0;(1_11_r))$. Recall that the elements $(\bar{1}_{1_11_r};(1_11_r))$ occupy the places $(\Sigma_i e_i+r-4)+1,\dots,k,$ so acting by the braid move $(\sigma'_{(\sum_i e_i+r-4)})^{-1}$ we replace the sequence $((z^{r-1}_{1_11_{r-1}};(1_11_{r-1})),\ (\bar{1}_{1_11_r};(1_11_r)),\dots,(\bar{1}_{1_11_r};(1_11_r)),(0;(1_11_r)))$ by

$$((\bar{1}_{1_11_r};(1_11_r)),(z_{1_{r-1}1_r};(1_{r-1}1_r)),(\bar{1}_{1_11_r};(1_11_r)),\dots,(\bar{1}_{1_11_r};(1_11_r)),(0;(1_11_r)))$$

where z is either $\bar{0}$ or $\bar{1}$ depending on whether z^{r-1} is equal to $\bar{1}$ or $\bar{0}$. Applying the elementary moves $\sigma'_{(\sum_i e_i+r-4)+1},\dots,\sigma'_{k-1},\ (\sigma'_k)^{-1}$ we can replace this sequence by

$$((\bar{1}_{1_11_r};(1_11_r)),\ (z'_{1_11_{r-1}};(1_11_{r-1})),\dots,\ (z'_{1_11_{r-1}};(1_11_{r-1})),$$
$$(0;(1_11_r)),\ (z_{1_11_{r-1}};(1_11_{r-1})))$$

where z' is either $\bar{0}$ or $\bar{1}$ depending on whether z is equal to $\bar{1}$ or $\bar{0}$. Now using the braid moves $\sigma'_{(\sum_i e_i+r-4)},\dots,\sigma'_{k-2},\ (\sigma'_k)^{-1},\ (\sigma'_{k-1})^{-1}$ we obtain that the sequence above is braid equivalent to

$$((z''_{1_{r-1}1_r};(1_{r-1}1_r)),\dots,(z''_{1_{r-1}1_r};(1_{r-1}1_r)),\ (z_{1_11_{r-1}};(1_11_{r-1})),$$
$$(\bar{1}_{1_{r-1}1_r};(1_{r-1}1_r)),\ (0;(1_{r-1}1_r)))$$

where z'' is either $\bar{0}$ or $\bar{1}$ depending on whether z' is equal to $\bar{1}$ or $\bar{0}$. Acting by $(\sigma'_{k-2})^{-1},\dots,(\sigma'_{(\sum_i e_i+r-4)})^{-1}$, $(\sigma'_{(\sum_i e_i+r-4)})^{-1},\dots,(\sigma'_{k-2})^{-1}$ we bring our sequence to the form

$$((0;(1_11_r)),\dots,(0;(1_11_r)),(z_{1_11_{r-1}};(1_11_{r-1})),(\bar{1};(1_{r-1}1_r)),(0;(1_{r-1}1_r))).$$

Now if $z=0$ we act by σ'_{k-1},σ'_k, if instead $z=\bar{1}$ we use $\sigma'_k,\sigma'_{k-1},\sigma'_k$, to obtain

$$((z_{1_11_{r-1}};(1_11_{r-1})),(\bar{1};(1_{r-1}1_r)),(0;(1_{r-1}1_r)))\sim$$

$$((\bar{1};(1_11_r)),(0;(1_11_r)),(z_{1_11_{r-1}};(1_11_{r-1}))).$$

Then applying the elementary moves $(\sigma'_{k-2})^{-1},\dots,(\sigma'_{(\sum_i e_i+r-4)})^{-1}$, $\sigma'_k,\dots,\sigma'_{(\sum_i e_i+r-4)}$ we can replace the sequence $((0;\ (1_11_r)),\ \dots,\ (0;(1_11_r)),$ $(\bar{1};(1_11_r)),\ (0;(1_11_r)),\ ((z_{1_11_{r-1}};(1_11_{r-1}))))$ by

$$((z^{r-1}_{1_11_{r-1}};(1_11_{r-1})),(\bar{1};(1_11_r)),(0;(1_11_r)),\dots,(0;(1_11_r))).$$

Now to obtain the required normal form it is sufficient to proceed as in the case in which there exists at least one j belonging to $\{j_1,\dots,j_{n_1}\}$ such that $j\neq 1,\ r$.

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