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POSITIVE SOLUTIONS FOR NONLINEAR ROBIN PROBLEMS

DIEGO AVERNA, NIKOLAOS S. PAPAGEORGIOU, ELISABETTA TORNATORE

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Abstract. We consider a parametric Robin problem driven by the p-Laplacian with an indefinite potential and with a superlinear reaction term which does not satisfy the Ambrosetti-Rabinowitz condition. We look for positive solutions. We prove a bifurcation-type theorem describing the nonexistence, existence and multiplicity of positive solutions as the parameter varies. We also show the existence of a minimal positive solution $\tilde{u}_\lambda$ and establish the monotonicity and continuity of the map $\lambda \to \tilde{u}_\lambda$.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear parametric Robin problem

$$\begin{gathered} -\Delta_p u(z) + (\xi(z)+\lambda)u(z)^{p-1} = f(z,u(z)) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0, \quad \text{on } \partial\Omega, \\ u \geq 0. \end{gathered} \tag{1.1}$$

In this problem Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|Du|^{p-2}Du\right)$$

for all $u \in W_0^{1,r}(\Omega)$, $1 < p < \infty$.

Also $\lambda > 0$ is a parameter and $\xi \in L^\infty(\Omega)$ is an indefinite (that is, sign-changing) potential function. The reaction term $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z,x)$ is continuous). We assume that $f(z,\cdot)$ exhibits $(p-1)$-superlinear growth near $+\infty$, but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short). In the boundary condition $\frac{\partial u}{\partial n_p}$ denotes the generalized normal derivative defined by extension of the map

$$C^1(\overline{\Omega}) \ni u \to |Du|^{p-2}(Du,n)_{\mathbb{R}^N},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The coefficient $\beta(\cdot)$ is nonnegative. We can have that $\beta = 0$ and this case corresponds to the Neumann problem.

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We are looking for positive solutions and our goal is to determine the nonexistence, existence and multiplicity of positive solutions as the parameter $\lambda > 0$ varies. Also, we show the existence of a smallest positive solution $\tilde{u}_\lambda$ and investigate the continuity and monotonicity properties of the map $\lambda \to \tilde{u}_\lambda$.

More precisely, first we prove a bifurcation-type result, producing a critical parameter value $\lambda_* > 0$ such that

- for all $\lambda > \lambda_*$ problem (1.1) has at least two positive solutions;
- for $\lambda = \lambda_*$ problem (1.1) has at least one positive solution;
- for $\lambda \in (0, \lambda_*)$ problem (1.1) has no positive solutions.

Moreover, we show that for all $\lambda \geq \lambda_*$ problem (1.1) admits a smallest positive solution $\tilde{u}_\lambda$ and we determine the continuity and monotonicity properties of the map $\lambda \to \tilde{u}_\lambda$.

The starting point of our work here is the recent paper of Averna-Papageorgiou-Tornatore [3]. Our work here extends and complements that paper. In [3] $\xi = 0$, $\beta = 0$ (Neumann problem) and the reaction term $f(z, \cdot)$ is asymptotically $(p-1)$-linear as $x \to +\infty$. Similar bifurcation-type results for different classes of nonlinear parametric elliptic equations, were proved by Brock-Itturiaga-Ubilla [4], Filippakis-O'Regan-Papageorgiou [7], Garcia Azorero- Manfredi-Peral Alonso [10], Gasinski-Papageorgiou [12], Guo-Zhang [14], Takeuchi [24, 25] (Dirichlet problems), Cardinali-Papageorgiou-Rubbioni [5], Papageorgiou-Smyrlis [22] (Neumann problems) and Papageorgiou-Radulescu [18] (Robin problems).

Our approach uses variational methods based on the critical point theory together with truncation and comparison techniques. For easy reference, in the next section we recall the main notions and results which we will use in the sequel.

2. Mathematical background-hypotheses

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair (X, X^*). Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds:

> Every sequence $\{u_n\}_{n\geq 1} \subset X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subset \mathbb{R}$ is bounded and
> $$(1 + \|u_n\|)\varphi'(u_n) \to 0 \qquad \text{in } X^* \text{ as } n \to \infty,$$
> admits a strongly convergent subsequence.

This is a compactness-type condition on the functional φ. It leads to a deformation theorem from which one can deduce the mimimax theory for the critical values of φ. A basic result in that theory, is the "mountain pass theorem" of Ambrosetti-Rabinowitz [2]. Here we state the result in a slightly more general form (see Gasinski-Papageorgiou [11]).

Theorem 2.1. *If X is a Banach space, $\varphi \in C^1(X; \mathbb{R})$ satisfies the C-condition, u_0, $u_1 \in X$ and $\varrho > 0$ are such that, $\|u_1 - u_0\| > \varrho$,*
$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\left[\varphi(u) : \|u - u_0\| = \varrho\right] = m_\varrho$$
and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$ where $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\}$, then $c \geq m_\varrho$ and c is a critical value of φ (that is, there exists $u \in X$ such that $\varphi'(u) = 0$ and $\varphi(u) = c$).

The following spaces will be used in the analysis of problem (1.1):

$$W^{1,p}(\Omega) \quad C^1(\overline{\Omega}) \quad \text{and} \quad L^\eta(\partial\Omega) 1 \le \eta \le \infty.$$

The Sobolev space $W^{1,p}(\Omega)$ $(1 < p < \infty)$ is a reflexive Banach space with the norm

$$\|u\| = \left(\|u\|_p^p + \|Du\|_p^p\right)^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

The space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) \ge 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior containing the set

$$D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the "boundary" Lebesgue spaces $L^\eta(\partial\Omega)$ $1 \le \eta \le \infty$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u_{|\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

Therefore we understand $\gamma_0(u)$ as representing the "boundary values" of an arbitrary Sobolev function u. The trace map γ_0 is compact into $L^\eta(\partial\Omega)$ for all $\eta \in [1, \frac{(N-1)p}{N-p})$ if $p < N$ and into $L^\eta(\partial\Omega)$ for all $1 \le \eta < \infty$ if $p \ge N$. Also, we have

$$\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \quad (\frac{1}{p} + \frac{1}{p'} = 1), \quad \ker\gamma_0 = W_0^{1,p}(\Omega).$$

In the sequel, for the sake of notational simplicity, we drop the use of the trace map γ_0. All restrictions of Sobolev functions u on $\partial\Omega$ are defined in the sense of traces. Given a measurable function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example a Carathéodory function), by $N_g(\cdot)$ we denote the Nemytskii (superposition) map corresponding to g defined by

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)), \quad \text{for all } u \in W^{1,p}(\Omega).$$

Evidently $z \to N_g(u)(z)$ is measurable.

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz, \quad \text{for all } u, h \in W^{1,p}(\Omega). \tag{2.1}$$

The next proposition summarizes the properties of this map (see [11]).

Proposition 2.2. *The map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by (2.1) is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is,*

$u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0 \Rightarrow u_n \to u$ in $W^{1,p}(\Omega)$.

In what follows

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } N > p \\ +\infty & \text{if } N \le p, \end{cases}$$

which is the critical Sobolev exponent. Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$|f_0(z,x)| \le a_0(z)(1 + |x|^{\eta-1}) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)$, $1 \le \eta \le p^*$. Also, let $k_0 \in C^{0,\alpha}(\partial\Omega \times \mathbb{R})$ with $\alpha \in (0,1)$ and assume that

$$|k_0(z,x)| \le c_0 |x|^q \text{ for all } (z,x) \in \partial\Omega \times \mathbb{R}, \text{ with } c_0 > 0,\ q \in (1,p].$$

We set $F_0(z,x) = \int_0^x f_0(z,s)ds$, $K_0(z,x) = \int_0^x k_0(z,s)ds$ and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega \xi(z)|u|^p\,dz + \int_{\partial\Omega} K_0(z,u)d\sigma - \int_\Omega F_0(z,u)\,dz$$

for all $u \in W^{1,p}(\Omega)$. From Papageorgiou-Radulescu [20], we have the following result.

Proposition 2.3. *If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\varrho_0 > 0$ such that*

$$\varphi_0(u_0) \le \varphi_0(u_0 + h) \quad \text{for all } h \in C^1(\overline{\Omega}),\ \|h\|_{C^1(\overline{\Omega})} \le \varrho_0,$$

then $u_0 \in C_0^{1,\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$ and it is a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\varrho_1 > 0$ such that

$$\varphi_0(u_0) \le \varphi_0(u_0 + h) \quad \text{for all } h \in W^{1,p}(\Omega),\ \|h\| \le \varrho_1.$$

It is well known that for nonlinear problems it is difficult to have strong comparison results. Additional conditions on the data are needed. So, suppose that $h_1, h_2 \in L^\infty(\Omega)$. We write that $h_1 \prec h_2$ if for every $K \subset \Omega$ compact, we can find $\varepsilon = \varepsilon(K) > 0$ such that

$$h_1(z) + \varepsilon \le h_2(z) \quad \text{for a.a. } z \in K.$$

Evidently, if h_1, $h_2 \in C(\Omega)$ and $h_1(z) < h_2(z)$ for all $z \in \Omega$, then $h_1 \prec h_2$.

In Fragnelli-Mugnai-Papageorgiou [9] we can find the following strong comparison principle.

Proposition 2.4. *If ξ, h_1, $h_2 \in L^\infty(\Omega)$, $h_1 \prec h_2$, $v \in D_+$, $u \le v$*

$$\begin{gathered} -\Delta_p u + \xi(z)|u|^{p-2}u = h_1 \quad \text{in } \Omega \\ -\Delta_p v + \xi(z)v^{p-1} = h_2 \quad \text{in } \Omega \\ \frac{\partial v}{\partial n}|_{\partial\Omega} < 0, \end{gathered}$$

then $(v-u)(z) > 0$ for all $z \in \Omega$ and $\frac{\partial(v-u)}{\partial n}|_{\Sigma_0} < 0$, where $\Sigma_0 = \{z \in \partial\Omega : v(z) = u(z)\}$.

Remark 2.5. We consider the following open cone in $C^1(\overline{\Omega})$

$$\begin{aligned} \hat{D}_+ = \big\{ &u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}\big|_{\partial\Omega \cap u^{-1}(0)} < 0 \\ &\text{if } \partial\Omega \cap u^{-1}(0) \ne \emptyset \} \end{aligned}$$

then $\hat{D}_+ \supseteq D_+$, and the conclusion of Proposition 2.4 says that

$$v - u \in \hat{D}_+.$$

Consider the nonlinear eigenvalue problem

$$
\begin{gathered}
-\Delta_p u + \xi(z)|u|^{p-2}u = \hat{\lambda}|u|^{p-2}u \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega.
\end{gathered}
$$

We know (see Papageorgiou-Radulescu [19]), that this problem has a smallest eigenvalue $\hat{\lambda}_1 \in \mathbb{R}$ which is isolated, simple and

$$
\hat{\lambda}_1 = \inf\big[\frac{\tau(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega),\ u \neq 0\big], \tag{2.2}
$$

where $\tau : W^{1,p}(\Omega) \to \mathbb{R}$ is the C^1-functional defined by

$$
\tau(u) = \|Du\|_p^p + \int_\Omega \xi(z)|u|^p\,dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).
$$

The infimum in (2.2) is realized on the corresponding one-dimensional eigenspace. The eigenfunctions corresponding to $\hat{\lambda}_1$ do not change sign. By $\hat{u}_1$ we denote the positive, L^p-normalized (that is $\|\hat{u}_1\|_p = 1$) eigenfunction. The nonlinear regularity theory (see [16, 20]) and the nonlinear maximum principle (see [11, 23]), imply that $\hat{u}_1 \in D_+$. Note that if $\xi \geq 0$, $\xi \neq 0$ or if $\xi = 0$, $\beta \neq 0$, then $\hat{\lambda}_1 > 0$.

Now let us introduce our hypotheses on the data of problem (1.1).

- (H1) $\xi \in L^\infty(\Omega)$.
- (H2) $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$.
- (H3) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and
  - (i) for every $\varrho > 0$, there exists $a_\varrho \in L^\infty(\Omega)$ such that
$$
\begin{gathered}
|f(z,x)| \leq a_\varrho(z) \quad \text{for a.a. } z \in \Omega,\ \text{all } |x| \leq \varrho, \\
\lim_{x \to +\infty} \frac{f(z,x)}{x^{p^*-1}} = 0 \quad \text{uniformly for a.a. } z \in \Omega;
\end{gathered}
$$
  - (ii) if $F(z,x) = \int_0^x f(z,s)ds$, then $\lim_{x\to+\infty} \frac{F(z,x)}{x^p} = +\infty$ uniformly for a.a. $z \in \Omega$;
  - (iii) if $e(z,x) = f(z,x)x - pF(z,x)$, then there exists $d \in L^1(\Omega)$ such that
$$
e(z,x) \leq e(z,y) + d(z) \quad \text{for a.a. } z \in \Omega, \text{all } 0 \leq x \leq y;
$$
  - (iv) there exist $\delta_0 > 0$, $q \in (1,p)$ and a function $\hat{\eta} \in L^\infty(\Omega)$ such that
$$
\begin{gathered}
f(z,x) \geq c_1 x^{q-1} \quad \text{for a.a. } z \in \Omega,\ \text{all } 0 \leq x \leq \delta_0,\ \text{some } c_1 > 0, \\
\hat{\lambda}_1 \leq \hat{\eta}(z) \quad \text{for a.a. } z \in \Omega \text{ and } \hat{\eta} \neq \hat{\lambda}_1, \\
\hat{\eta}(z)x^{p-1} \leq f(z,x) \text{ for a.a. } z \in \Omega,\ \text{all } x \geq 0.
\end{gathered}
$$

Note that when $\beta = 0$, we have the usual Neumann problem.

Remark 2.6. Since we are looking for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality, we may assume that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$. Note that $f(z,\cdot)$ does not satisfy the usual subcritical polynomial growth. Indeed, according to hypothesis (H3)(i), given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$
|f(z,x)| \leq c_\varepsilon + \varepsilon x^{p^*-1} \quad \text{for a.a. } z \in \Omega,\ \text{all } x \geq 0. \tag{2.3}
$$

This is a kind of almost critical growth for $f(z,\cdot)$ and it is the source of technical difficulties since $W^{1,p}(\Omega)$ is embedded only continuously and not compactly into $L^{p^*}(\Omega)$. Nevertheless we overcome this difficulty with the use Vitali's theorm (the extended dominated convergence theorem). Hypotheses (H3)(ii), (iii) imply that

$$\lim_{x\to+\infty}\frac{f(z,x)}{x^{p-1}}=+\infty \quad \text{uniformly for a.a. } z\in\Omega.$$

So, the reaction term $f(z,\cdot)$ is $(p-1)$-superlinear near $+\infty$. However, we do not employ the usual for superlinear problems AR-condition. Recall that the AR-condition (unilateral version, since $f(z,x)|_{(-\infty,0]}=0$), says that there exist $r>p$ and $M>0$ such that

$$0<rF(z,x)\le f(z,x)x \quad \text{for a.a. } z\in\Omega,\ \text{all } x\ge M, \tag{2.4}$$

$$0<\operatorname{ess\,inf}_\Omega F(\cdot,M). \tag{4b}$$

Integrating (2.4) and using (4b), we obtain the weaker condition

$$c_2x^r\le F(z,x) \quad \text{for a.a. } z\in\Omega,\ \text{all } x\ge M,\ \text{some } c_2>0. \tag{2.5}$$

From (2.5) and hypothesis (H3)(iii), we see that $f(z,\cdot)$ has at least $(r-1)$-polynomial growth near $+\infty$. This excludes from consideration $(p-1)$-superlinear functions with slower growth near $+\infty$, which fail to satisfy the AR-condition. For example consider the following function (for the sake of simplicity we drop the z-dependence):

$$f(x)=\begin{cases} cx^{q-1} & \text{if } x\in[0,1]\\ cp(\log x+\frac{1}{p})x^p & \text{if } 1<x\end{cases}$$

with $c>\max\{0,\hat\lambda_1\}$. The function $f(\cdot)$ satisfies hypotheses (H3), but fails to satisfy the AR-condition (see (2.4)).

We introduce the following two sets:

$$\begin{gathered}\mathcal{L}=\big\{\lambda>0:\text{problem (1.1) admits a positive solution}\big\}\\ \text{(this is the set of admissible parameters)},\\ S(\lambda)=\text{set of positive solutions for problem (1.1)}\\ (\text{if } \lambda\notin\mathcal{L},\text{ then } S(\lambda)=\emptyset).\end{gathered}$$

We use the following notation. If $x\in\mathbb{R}$, then $x^\pm=\max\{\pm x,0\}$. Then for all $u\in W^{1,p}(\Omega)$ we define $u^\pm(\cdot)=u(\cdot)^\pm$.

We know that $u^\pm\in W^{1,p}(\Omega)$, $u=u^+-u^-$, $|u|=u^++u^-$. Moreover, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Finally if $1\le p<\infty$, then $1<p'\le\infty$ and satisfies $\frac{1}{p}+\frac{1}{p'}=1$.

3. Bifurcation-type theorem

In this section we prove a bifurcation-type theorem for problem (1.1) as described in the Introduction. First we determine the nature of the solutions.

Proposition 3.1. *If hypotheses* (H1)–(H3) *hold, then for every* $\lambda>0$ *we have* $S(\lambda)\subseteq D_+$.

Proof. Let $u \in S(\lambda)$. Then

$$\langle A(u), h\rangle + \int_\Omega (\xi(z)+\lambda)u^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)u^{p-1}hd\sigma = \int_\Omega f(z,u)h\,dz$$

for all $h \in W^{1,p}(\Omega)$, hence we have

$$\begin{gathered} -\Delta_p u(z) + (\xi(z)+\lambda)u(z)^{p-1} = f(z,u(z)) \quad \text{for a.a. } z \in \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

(see Papageorgiou-Radulescu [19]).

From Papageorgiou-Radulescu [20], we have $u \in L^\infty(\Omega)$. So, we can apply [16, Theorem 2] and conclude that $u \in C_+ \setminus \{0\}$.

Let $\varrho = \|u\|_\infty$. Hypotheses (H3)(i),(iv) imply that we can find $\hat{\xi}_\varrho > 0$ such that

$$f(z,x) + \hat{\xi}_\varrho x^{p-1} \geq 0 \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \varrho. \tag{3.2}$$

Using (3.1) and (3.2), we obtain

$$\Delta_p u(z) \leq [\xi(z) + \lambda + \hat{\xi}_\varrho]u(z)^{p-1} \quad \text{for a.a. } z \in \Omega,$$

which implies $u \in D_+$ by the nonlinear maximum principle (see, for example Gasinski-Papageorgiou [11, p. 736]). □

Next we show that $\mathcal{L} \neq \emptyset$ and prove a structural property of $\mathcal{L}$

Proposition 3.2. *If hypotheses* (H1)–(H3) *hold, then* $\mathcal{L} \neq \emptyset$ *and if* $\lambda \in \mathcal{L}$*, then* $[\lambda, +\infty) \subseteq \mathcal{L}$.

Proof. Let $\eta > \|\xi\|_\infty$ (see hypothesis (H1)). We consider the nonlinear Robin problem

$$\begin{gathered} -\Delta_p u(z) + (\xi(z)+\eta)|u(z)|^{p-2}u(z) = 1 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.3}$$

We claim that problem (3.3) admits a unique solution $\bar{u} \in D_+$. To see this consider the C^1-functional $\Psi : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\Psi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega (\xi(z)+\eta)|u|^p\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega u\,dz.$$

Since $\eta > \|\xi\|_\infty$, we have

$$\Psi(u) \geq c_3\|u\|^p - c_4\|u\| \quad \text{for some } c_3,\ c_4 > 0,$$

hence Ψ is coercive.

Also, using the Sobolev embedding theorem and the compactness of the trace map, we see that Ψ is sequentially weakly lower semicontinuous. Therefore, by the Weierstrass-Tonelli theorem, we can find $\bar{u} \in W^{1,p}(\Omega)$ such that

$$\Psi(\bar{u}) = \inf[\Psi(u) : u \in W^{1,p}(\Omega)]. \tag{3.4}$$

Hypotheses (H1), (H2) imply that

$$\Psi(u) \leq c_5\|u\|^p - \int_\Omega u\,dz \quad \text{for some } c_5 > 0, \text{ and all } u \in W^{1,p}(\Omega).$$

Let $u \in D_+$ and $t > 0$, we have

$$\Psi(tu) \le c_5 t^p \|u\|^p - t \int_\Omega u\,dz.$$

Since $p > 1$, choosing $t \in (0,1)$ small we see that $\Psi(tu) < 0 = \Psi(0)$, which implies $\Psi(\bar{u}) < 0 = \Psi(0)$ (see (3.4)), then $\bar{u} \neq 0$.

From (3.4) we have $\Psi'(\bar{u}) = 0$, so we obtain

$$\langle A(\bar{u}), h\rangle + \int_\Omega (\xi(z) + \eta)|\bar{u}|^{p-2}\bar{u}h\,dz + \int_{\partial\Omega} \beta(z)|\bar{u}|^{p-2}\bar{u}h d\sigma = \int_\Omega h\,dz, \tag{3.5}$$

for all $h \in W^{1,p}(\Omega)$.

In (3.5) we choose $h = -\bar{u}^- \in W^{1,p}(\Omega)$. Since $\eta > \|\xi\|_\infty$ and using (H2), we have

$$c_6\|\bar{u}^-\| \le 0 \quad \text{for some } c_6 > 0,$$

hence $\bar{u} \ge 0$, $\bar{u} \neq 0$.

Therefore (3.3) has a positive solution. Moreover, as before the nonlinear regularity theory and the nonlinear maximum principle, imply that

$$\bar{u} \in D_+.$$

Next we show the uniqueness of this solution. So, suppose that $\tilde{u}$ is another solution. Again we can show that $\tilde{u} \in D_+$. Evidently we can find $t > 0$ such that $t\tilde{u} \le \bar{u}$. Suppose that $t_0 > 0$ is the biggest such positive real. Assume that $t_0 < 1$. Then

$$-\Delta_p(t_0\tilde{u}) + (\xi(z) + \eta)(t_0\tilde{u})^{p-1} = t_0^{p-1} < 1 = -\Delta_p\bar{u} + (\xi(z) + \eta)\bar{u}^{p-1}$$

for a.a. $z \in \Omega$. Invoking Proposition 2.4 we have

$$\bar{u} - t_0\tilde{u} \in \hat{D}_+$$

which contradicts the maximality of t_0. Therefore $t_0 \ge 1$ and we have

$$\tilde{u} \le \bar{u}.$$

Interchanging the roles of $\tilde{u}$ and $\bar{u}$ in the above argument, we also have that

$$\bar{u} \le \tilde{u},$$

hence $\tilde{u} = \bar{u}$. This proves the uniqueness of the solution $\bar{u} \in D_+$ of (3.3).

Now, let $\overline{m} = \min_{\overline{\Omega}} \bar{u} > 0$ (since $\bar{u} \in D_+$) and let $\lambda_0 = \eta + \frac{\|N_f(\bar{u})\|_\infty}{\overline{m}^{p-1}}$ (see hypothesis (H3)(i)). For all $h \in W^{1,p}(\Omega)$ with $h \ge 0$, and since $\bar{u} \in D_+$ is the unique solution of (3.3), we have

$$\begin{aligned} &\langle A(\bar{u}), h\rangle + \int_\Omega (\xi(z) + \lambda_0)\bar{u}^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)\bar{u}^{p-1}hd\sigma \\ &= \langle A(\bar{u}), h\rangle + \int_\Omega \big[\xi(z) + \eta + \frac{\|N_f(\bar{u})\|_\infty}{\overline{m}^{p-1}}\big]h\,dz + \int_{\partial\Omega} \beta(z)\bar{u}^{p-1}hd\sigma \\ &\ge \int_\Omega [1 + f(z,\bar{u})]h\,dz \ge \int_\Omega f(z,\bar{u})h\,dz. \end{aligned} \tag{3.6}$$

We consider the following truncation of $f(z,\cdot)$,

$$\bar{f}(z,x) = \begin{cases} f(z,x) & \text{if } x \le \bar{u}(z) \\ f(z,\bar{u}(z)) & \text{if } \bar{u}(z) < x \end{cases}. \tag{3.7}$$

This is a Caratheodory function. We set $\bar{F}(z,x) = \int_0^x \bar{f}(z,s)ds$ and consider the C^1-functional $\bar{\varphi} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\bar{\varphi}(u) = \frac{1}{p}\tau(u) + \frac{\lambda_0}{p}\|u\|_p^p - \int_\Omega \bar{F}(z,u)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Since $\lambda_0 \ge \eta > \|\xi\|_\infty$, and using (3.7), we see that $\bar{\varphi}$ is coercive.

Also, it is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\bar{\varphi}(u_0) = \inf[\bar{\varphi}(u) : u \in W^{1,p}(\Omega)]. \tag{3.8}$$

Use (3.7), hypothesis (H3)(iv) and recall that $\overline{m} = \min_{\overline{\Omega}} \bar{u}$, for $u \in D_+$ and let $t \in (0,1)$ be small such that $tu = \min\{\delta_0, \overline{m}\}$. Then we have

$$\bar{\varphi}(tu) = \frac{t^p}{p}\tau(u) + \frac{\lambda_0 t^p}{p}\|u\|^p - \frac{c_1}{q}t^q\|u\|_q^q.$$

Since $q < p$, choosing $t \in (0,1)$ even smaller if necessary, we see that $\bar{\varphi}(tu) < 0 = \bar{\varphi}(0)$, which implies $\bar{\varphi}(u_0) < 0 = \bar{\varphi}(0)$ (see (3.8)), then $u_0 \neq 0$.

From (3.8) we have $\bar{\varphi}'(u_0) = 0$, so we obtain

$$\begin{aligned}
&\langle A(u_0), h\rangle + \int_\Omega (\xi(z) + \lambda_0)|u_0|^{p-2}u_0 h\,dz + \int_{\partial\Omega} \beta(z)|u_0|^{p-2}u_0 h d\sigma \\
&= \int_\Omega \bar{f}(z,u_0)h\,dz
\end{aligned} \tag{3.9}$$

for all $h \in W^{1,p}(\Omega)$.

In (3.9) first we act with $h = -u_0^- \in W^{1,p}(\Omega)$. Since $\lambda_0 \ge \eta > \|\xi\|_\infty$ and $f(z,\cdot)|_{(-\infty,0]} = 0$ for a.a. $z \in \Omega$, we obtain

$$c_7\|u_0^-\|^p \le 0 \quad \text{for some } c_7 > 0$$

which implies $u_0 \ge 0$, $u_0 \neq 0$.

Next in (3.9) we choose $h = (u_0 - \overline{u})^+ \in W^{1,p}(\Omega)$. From (3.6), (3.7) and hypothesis (H2) we have

$$\begin{aligned}
&\langle A(u_0), (u_0 - \overline{u})^+\rangle + \int_\Omega (\xi(z) + \lambda_0)u_0^{p-1}(u_0 - \overline{u})^+\,dz + \int_{\partial\Omega} \beta(z)u_0^{p-1}(u_0 - \overline{u})^+ d\sigma \\
&= \int_\Omega f(z,\overline{u})(u_0 - \overline{u})^+\,dz \\
&\le \langle A(\overline{u}), (u_0 - \overline{u})^+\rangle + \int_\Omega (\xi(z) + \lambda_0)\bar{u}^{p-1}(u_0 - \overline{u})^+\,dz + \int_{\partial\Omega} \beta(z)\overline{u}^{p-1}(u_0 - \overline{u})^+ d\sigma\,.
\end{aligned}$$

Then

$$\langle A(u_0) - A(\overline{u}), (u_0 - \overline{u})^+\rangle + \int_\Omega (\xi(z) + \lambda_0)(u_0^{p-1} - \overline{u}^{p-1})(u_0 - \overline{u})^+\,dz \le 0,$$

then, we have $u_0 \le \overline{u}$. So, we have proved that

$$\begin{gathered}
u_0 \in [0, \overline{u}] = \{u \in W^{1,p}(\Omega) : 0 \le u(z) \le \overline{u}(z) \\
\text{for a.a. } z \in \Omega\}, \quad u_0 \neq 0.
\end{gathered} \tag{3.10}$$

Then from (3.7), (3.9) and (3.10), we have

$$\langle A(u_0), h\rangle + \int_\Omega (\xi(z) + \lambda_0)u_0^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)u_0^{p-1}h d\sigma = \int_\Omega f(z,u_0)h\,dz$$

for all $h \in W^{1,p}(\Omega)$. Then, from Proposition 3.1 we have $u_0 \in S(\lambda_0) \subseteq D_+$. Therefore $\lambda_0 \in \mathcal{L}$ and so $\mathcal{L} \neq \emptyset$.

Next let $\lambda \in \mathcal{L}$ and let $\mu > \lambda$. Consider $u_\lambda \in S(\lambda) \subseteq D_+$. We have

$$-\Delta_p u_\lambda(z) + (\xi(z)+\lambda)u_\lambda(z)^{p-1} \leq -\Delta_p u_\lambda(z) + (\xi(z)+\mu)u_\lambda(z)^{p-1} \tag{3.11}$$

for a.a. $z \in \Omega$. As before we introduce

$$\tilde{f}(z,x) = \begin{cases} f(z,x) & \text{if } x \leq u_\lambda(z) \\ f(z,u_\lambda(z)) & \text{if } u_\lambda(z) < x. \end{cases} \tag{3.12}$$

This is a Caratheodory function. We set $\tilde{F}(z,x) = \int_0^x \tilde{f}(z,s)ds$ and consider the C^1-functional $\tilde{\varphi}_\mu : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\bar{\varphi}_\mu(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega (\xi(z)+\mu)|u|^p\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega \tilde{F}(z,u)\,dz$$

for all $u \in W^{1,p}(\Omega)$.

Reasoning as in the first part of the proof and using (3.11), (3.12), via the direct method of the calculus of variations, we produce $u_\mu \in W^{1,p}(\Omega)$ such that

$$\tilde{\varphi}_\mu'(u_\mu) = 0 \quad \text{and } u_\mu \in [0, u_\lambda].$$

Then from (3.12), we have

$$\langle A(u_\mu), h\rangle + \int_\Omega (\xi(z)+\mu)u_\mu^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)u_\mu^{p-1}h d\sigma = \int_\Omega f(z,u_\mu)h\,dz$$

for all $h \in W^{1,p}(\Omega)$. Then, $u_\mu \in S(\mu) \subseteq D_+$. So, we have $\mu \in \mathcal{L}$ and $[\lambda, +\infty) \subseteq \mathcal{L}$. □

Remark 3.3. Proposition 3.2 implies that $\mathcal{L}$ is a half-line. Moreover, from the proof we have that, if $\lambda \in \mathcal{L}$ and $\mu > \lambda$, $u_\lambda \in S(\lambda) \subseteq D_+$, then $\mu \in \mathcal{L}$ and we can find $u_\mu \in S(\mu) \subseteq D_+$ such that $u_\lambda - u_\mu \in C_+ \setminus \{0\}$. We can improve this monotonicity property, if we strengthen the conditions on $f(z,\cdot)$.

The new conditions on $f(z,\cdot)$, are the following:

(H4) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, hypotheses (H4)(i),(ii),(iii),(iv) are the same as the corresponding hyphoteses (H3)(i),(ii),(iii),(iv) and
- (v) for every $0 < \eta < \theta$, we can find $\hat{\xi}_{\theta\eta} > 0$ such that for a.a. $z \in \Omega$, the function
$$x \to f(z,x) + \hat{\xi}_{\theta\eta}x^{p-1}$$
is nondecreasing on $[\eta, \theta]$.

Remark 3.4. This new condition on $f(z,\cdot)$ is satisfied if, for example, for a.a. $z \in \Omega$, $f(z,\cdot)$ is differentiable and for $0 < \eta < \theta$, there exists $a_{\theta\eta} \in L^\infty(\Omega)$ such that

$$|f_x'(z,x)| \leq a_{\theta\eta}(z) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [\eta,\theta]$$

(just use the mean value theorem to check this).

We have the following stronger monotonicity property.

Proposition 3.5. *If* (H1), (H2), (H4) *hold,* $\lambda \in \mathcal{L}$, $\mu > \lambda$ *and* $u_\lambda \in S(\lambda) \subseteq D_+$, *then we can find* $u_\mu \in S(\mu) \subseteq D_+$ *such that* $u_\lambda - u_\mu \in \hat{D}_+$.

Proof. From Proposition 3.2 and its proof, we know that we can find $u_\mu \in S(\mu) \subseteq D_+$ such that

$$u_\mu \le u_\lambda, \quad u_\mu \ne u_\lambda. \tag{3.13}$$

Let $\eta = \min_{\overline{\Omega}} u_\mu$ and $\theta = \|u_\lambda\|_\infty$. Consider $\hat{\xi}_{\theta\mu} > 0$ as postulated by hypothesis (H4)(v). We have

$$\begin{aligned}
&-\Delta_p u_\mu(z) + (\xi(z) + \lambda + \hat{\xi}_{\theta\mu}) u_\mu(z)^{p-1} \\
&-\Delta_p u_\mu(z) + (\xi(z) + \mu + \hat{\xi}_{\theta\mu}) u_\mu(z)^{p-1} - (\mu - \lambda) u_\mu(z)^{p-1} \\
&= f(z, u_\mu(z)) + \hat{\xi}_{\theta\mu} u_\mu(z)^{p-1} - (\mu - \lambda) u_\mu(z)^{p-1} \\
&< f(z, u_\mu(z)) + \hat{\xi}_{\theta\mu} u_\mu(z)^{p-1} \\
&= -\Delta_p u_\lambda(z) + (\xi(z) + \lambda + \hat{\xi}_{\theta\eta}) u_\lambda(z)^{p-1} \quad \text{for a.a. } z \in \Omega,
\end{aligned}$$

which implies $u_\lambda - u_\mu \in \hat{D}_+$ (see Proposition 2.4). $\square$

For $\lambda > 0$, let $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-energy (Euler) functional for problem (1.1) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\tau(u) + \frac{\lambda}{p}\|u\|_p^p - \int_\Omega F(z,u)\,dz \quad \text{for all} \quad u \in W^{1,p}(\Omega).$$

Recall that

$$\tau(u) = \|Du\|_p^p + \int_\Omega \xi(z)|u|^p\,dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

We set $\lambda_* = \inf \mathcal{L} \ge 0$.

Proposition 3.6. *If* (H1)–(H3) *hold, then* $\lambda_* > 0$.

Proof. We argue indirectly. So, suppose that $\lambda_* = 0$ and consider a sequence $\{\lambda_n\}_{n\ge1} \subset \mathcal{L}$ such that $\lambda_n \downarrow 0$. From the proof of Proposition 3.2, we know that we can find $u_n \in S(\lambda_n) \subseteq D_+$ such that $\tilde{u} \le u_n$ for some $\tilde{u} \in D_+$, all $n \in \mathbb{N}$ and

$$\varphi_{\lambda_n}(u_n) < 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.14}$$

From (3.14) we have

$$-\int_\Omega pF(z,u_n)\,dz \le -\|Du_n\|_p^p - \int_\Omega [\xi(z) + \lambda_n] u_n^p\,dz - \int_{\partial\Omega} \beta(z) u_n^p d\sigma \tag{3.15}$$

for all $n \in \mathbb{N}$. Since $u_n \in S(\lambda_n)$ for all $n \in \mathbb{N}$, we have

$$\int_\Omega f(z,u_n) u_n\,dz = \|Du_n\|_p^p + \int_\Omega [\xi(z) + \lambda_n] u_n^p\,dz + \int_{\partial\Omega} \beta(z) u_n^p d\sigma \tag{3.16}$$

for all $n \in \mathbb{N}$. Adding (3.15) and (3.16), we obtain

$$\int_\Omega e(z,u_n)\,dz \le 0 \text{ for all } n \in \mathbb{N}. \tag{3.17}$$

Claim: The sequence $\{u_n\}_{n\ge1} \subset W^{1,p}(\Omega)$ is bounded. We argue by contradiction. Suppose that, by passing to a subsequence if necessary $\|u_n\| \to \infty$ as $n \to +\infty$.

Let $y_n = u_n/\|u_n\|$ for all $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \ge 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \rightharpoonup y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ and } L^p(\partial\Omega). \tag{3.18}$$

First assume that $y \neq 0$. Let $\Omega_+ = \{z \in \Omega : y(z) > 0\}$. We have $|\Omega_+|_N > 0$ (recall that $y \geq 0$) and

$$u_n(z) \to +\infty \text{ for a.a. } z \in \Omega_+. \tag{3.19}$$

On account of (3.19) and hypothesis (H3)(ii), we have

$$\frac{F(z, u_n(z))}{\|u_n\|^p} = \frac{F(z, u_n(z))}{u_n(z)^p} y_n(z)^p \to +\infty \quad \text{for a.a. } z \in \Omega_+. \tag{3.20}$$

Using Fatou's lemma and (3.20), we see that

$$\int_{\Omega_+} \frac{F(z, u_n(z))}{\|u_n\|^p}\, dz \to +\infty. \tag{3.21}$$

Hypothesis (H3)(i)(iii) imply that we can find $c_8 > 0$ such that

$$F(z, x) \geq x^p - c_8 \quad \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.22}$$

Taking into account (3.22), we have

$$\int_{\Omega\setminus\Omega_+} \frac{F(z, u_n)}{\|u_n\|^p}\, dz \geq \int_{\Omega\setminus\Omega_+} y_n^p\, dz - \frac{c_8}{\|u_n\|^p}|\Omega|_N \quad \text{for all } n \in \mathbb{N}. \tag{3.23}$$

From (3.23) follows that

$$\begin{aligned} \int_{\Omega} \frac{F(z, u_n)}{\|u_n\|^p}\, dz &= \int_{\Omega_+} \frac{F(z, u_n)}{\|u_n\|^p}\, dz + \int_{\Omega\setminus\Omega_+} \frac{F(z, u_n)}{\|u_n\|^p}\, dz \\ &\geq \int_{\Omega_+} \frac{F(z, u_n)}{\|u_n\|^p}\, dz - \frac{c_8}{\|u_n\|^p}|\Omega|_N \quad \text{for all } n \in \mathbb{N} \end{aligned}$$

which implies, by using (3.21),

$$\int_{\Omega} \frac{F(z, u_n)}{\|u_n\|^p}\, dz \to +\infty \quad \text{as } n \to \infty. \tag{3.24}$$

Hypothesis (H3)(iii) implies that

$$0 \leq e(z, x) + d(z) \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0,$$

hence

$$pF(z, x) - d(z) \leq f(z, x)x \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.25}$$

From (3.16) and (3.25) we have

$$\int_{\Omega} \frac{pF(z, u_n)}{\|u_n\|^p}\, dz - \frac{\|d\|_1}{\|u_n\|^p} \leq \tau(y_n) \leq c_9 \quad \text{for some } c_9 > 0, \text{ all } n \in \mathbb{N}. \tag{3.26}$$

Comparing (3.24) and (3.26) we reach a contradiction. This takes care of the case $y \neq 0$.

Now we assume that $y = 0$. For $k > 0$ let $v_n = (kp)^{1/p} y_n \in W^{1,p}(\Omega)$ for all $n \in \mathbb{N}$. Taking into account (3.18) and recalling $y \geq 0$, we have

$$v_n \rightharpoonup 0 \text{ in } W^{1,p}(\Omega) \text{ and } v_n \to 0 \text{ in } L^p(\Omega) \text{ and } L^p(\partial\Omega). \tag{3.27}$$

Let $c_{10} = \sup_{n\geq 1} \|v_n\|_{p^*}^{p^*} < \infty$ (see (3.27)). From hypothesis (H3)(i) we see that given $\varepsilon > 0$, we can find $c_{11} = c_{11}(\varepsilon)$ such that

$$|F(z, x)| \leq \frac{\varepsilon}{2c_{10}} x^{p^*} + c_{11} \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.28}$$

Let $E \subseteq \Omega$ be a measurable set with $|E|_N \le \frac{\varepsilon}{2c_{11}}$. We have

$$\int_E |F(z, v_n)\,dz \le \frac{\varepsilon}{2c_{10}} \|v_n\|_{p^*}^{p^*} + c_{11}|E|_N \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

It is evident from (3.28) that $\{F(\cdot, v_n(\cdot))\}_{n\ge1} \subseteq L^1(\Omega)$ is bounded. It follows that

$$\{F(\cdot, v_n(\cdot))\}_{n\ge1} \subset L^1(\Omega) \text{ is uniformly integrable} \tag{3.29}$$

(see, for example, Gasinski-Papageorgiou [13, p. 36]). Moreover, at least for a subsequence, we have

$$F(z, v_n(z)) \to 0 \quad \text{for a.a. } z \in \Omega. \tag{3.30}$$

Then (3.29), (3.30) and Vitali's theorem (see, for example Gasinski-Papgeorgiou [13, p. 5]), we have

$$\int_\Omega F(z, v_n(z))\,dz \to 0 \quad \text{as } n \to +\infty. \tag{3.31}$$

Recall that $\|u_n\| \to \infty$. So, we can find $n_0 \in \mathbb{N}$ such that

$$0 < (kp)^{1/p} \frac{1}{\|u_n\|} \le 1 \quad \text{for all } n \ge n_0. \tag{3.32}$$

We choose $t_n \in [0, 1]$ such that

$$\varphi_{\lambda_n}(t_n u_n) = \max[\varphi_{\lambda_n}(t u_n) : 0 \le t \le 1]. \tag{3.33}$$

For $n \ge n_0$ and taking into account (3.32) and (3.33), we have

$$\begin{aligned} &\varphi_{\lambda_n}(t_n u_n) \\ &\ge \varphi_{\lambda_n}(v_n) \\ &= k\|Dy_n\|_p^p + k\int_\Omega [\xi(z) + \lambda_n] y_n^p\,dz + k\int_{\partial\Omega} \beta(z) y_n^p d\sigma - \int_\Omega F(z, v_n)\,dz \\ &= k[\tau(y_n) + \eta\|y_n\|_p^p] + k\lambda_n\|y_n\|_p^p - \int_\Omega F(z, v_n)\,dz - k\eta\|y_n\|_p^p \\ &\ge k[c_{12} - \eta\|y_n\|_p^p] + k\lambda_n\|y_n\|_p^p - \int_\Omega F(z, v_n)\,dz \ge kc_{13} \end{aligned} \tag{3.34}$$

for some $c_{12} > 0$ (since $\eta > \|\xi\|_\infty$) and for some $c_{13} > 0$ all $n \ge n_1 \ge n_0$ (see (3.18), (3.31) and recall that $y = 0$). But $k > 0$ is arbitrary. So, from (3.34) we infer that

$$\varphi_{\lambda_n}(t_n u_n) \to 0 \text{ as } n \to \infty. \tag{3.35}$$

Recall that

$$\varphi_{\lambda_n}(u_n) < 0 = \varphi_{\lambda_n}(0) \quad \text{for all } n \in \mathbb{N}.$$

This and (3.35) imply that

$$t_n \in (0, 1) \quad \text{for all } n \ge n_2. \tag{3.36}$$

Then from (3.33) and (3.36) it follows that for $n \ge n_2$ we have

$$\frac{d}{dt}\varphi_{\lambda_n}(t u_n)|_{t=t_n} = 0,$$

which implies $\langle \varphi'_{\lambda_n}(t_n u_n), u_n\rangle = 0$ (by the chain rule), then

$$\tau(t_n u_n) = \int_\Omega f(z, t_n u_n)(t_n u_n)\,dz \quad \text{for all } n \ge n_2. \tag{3.37}$$

From hypothesis (H3)(iii) and (3.36), we have

$$\int_\Omega e(z,t_nu_n)\,dz \le \int_\Omega e(z,u_n)\,dz + \|d\|_1 \quad \text{for all } n \ge n_2,$$

hence

$$\int_\Omega f(z,t_nu_n)(t_nu_n)\,dz \le c_{14} + \int_\Omega pF(z,t_nu_n)\,dz \tag{3.38}$$

for some $c_{14} > 0$, all $n \ge n_2$.

Using (3.37) in (3.38), we obtain

$$p\varphi_{\lambda_n}(t_nu_n) \le c_{14} \quad \text{for all } n \ge n_2. \tag{3.39}$$

Comparing (3.35) and (3.39) we have a contradiction. This proves the Claim.

From the Claim it follows that at least for a subsequence, we have

$$u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega) \text{ and } L^p(\partial\Omega). \tag{3.40}$$

Again let $c_{15} = \sup_{n\ge1}\|u_n\|_{p^*} < \infty$ (see (3.40)). Hypothesis (H3)(i) implies that given $\varepsilon > 0$, we can find $c_{16} = c_{16}(\varepsilon)$ such that

$$|f(z,x)| \le \frac{\varepsilon}{3c_{15}^{p^*}}x^{p^*-1} + c_{16} \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0. \tag{3.41}$$

Consider $E \subset \Omega$ a measurable set. From (3.41), we obtain

$$\begin{aligned} &|\int_E f(z,u_n)(u_n-u)\,dz| \\ &\le \int_E |f(z,u_n)|\,|(u_n-u)|\,dz \\ &\le \frac{\varepsilon}{3c_{15}^{p^*}}\int_\Omega |u_n|^{p^*-1}|u_n-u|\,dz + c_{16}\int_E |u_n-u|\,dz. \end{aligned} \tag{3.42}$$

Using Holder's inequality, we have

$$c_{16}\int_\Omega |u_n-u|\,dz \le c_{16}|E|_N^{1/(p^*)'}\|u_n-u\|_{p^*} \le 2c_{16}c_{15}|E|_N^{1/(p^*)'}. \tag{3.43}$$

Also, we have

$$\frac{\varepsilon}{3c_{15}^{p^*}}\int_\Omega |u_n|^{p^*-1}|u_n-u|\,dz \le \frac{\varepsilon}{3c_{15}^{p^*}}\|u_n\|_{p^*}^{p^*-1}\|u_n-u\|_{p^*} \le \frac{2\varepsilon}{3}. \tag{3.44}$$

We assume that

$$|E|_N \le \Big(\frac{\varepsilon}{6c_{15}c_{16}}\Big)^{1/(p^*)'}.$$

Then from (3.43) we have

$$c_{16}\int_\Omega |u_n-u|\,dz \le \frac{\varepsilon}{3}. \tag{3.45}$$

Using (3.44), (3.45) in (3.42), we obtain

$$\int_E |f(z,u_n)|\,|(u_n-u)|\,dz \le \varepsilon \quad \text{for all } n \in \mathbb{N},$$

hence it follows that

$$\{f(\cdot,u_n(\cdot))\,(u_n-u)(\cdot)\}_{n\ge1} \subset L^1(\Omega) \text{ is uniformly integrable.} \tag{3.46}$$

For at least a subsequence (see (3.40)), we have

$$f(z,u_n(z))(u_n-u)(z) \to 0 \quad \text{for a.a. } z \in \Omega \text{ as } n \to \infty. \tag{3.47}$$

Then (3.46), (3.47) and Vitali's theorem imply that

$$\int_\Omega f(z,u_n(z))(u_n-u)\,dz \to 0 \quad \text{as } n\to\infty. \tag{3.48}$$

Recall that

$$\langle A(u_n),h\rangle + \int_\Omega [\xi(z)+\lambda_n]u_n^{p-1}h\,dz + \int_{\partial\Omega}\beta(z)u_n^{p-1}h d\sigma = \int_\Omega f(z,u_n)h\,dz \tag{3.49}$$

for all $h\in W^{1,p}(\Omega)$, all $n\in\mathbb{N}$.

In (3.49) we choose $h=u_n-u\in W^{1,p}(\Omega)$, pass to the limit as $n\to\infty$ and use (3.40) and (3.48). Then

$$\lim_{n\to\infty}\langle A(u_n),u_n-u\rangle = 0,$$

which implies (see also Proposition 2.2 and (3.40))

$$u_n\to u \quad \text{in } W^{1,p}(\Omega). \tag{3.50}$$

As before, using Vitali's theorem, we have

$$\int_\Omega f(z,u_n)h\,dz \to \int_\Omega f(z,u)h\,dz \quad \text{for all } h\in W^{1,p}(\Omega). \tag{3.51}$$

So, if in (3.49) we pass to the limit as $n\to\infty$ and use (3.50) and (3.51), then

$$\langle A(u),h\rangle + \int_\Omega \xi(z)u^{p-1}h\,dz + \int_{\partial\Omega}\beta(z)u^{p-1}h d\sigma = \int_\Omega f(z,u)h\,dz \tag{3.52}$$

for all $h\in W^{1,p}(\Omega)$ (recall $\lambda_n\downarrow 0$).

Recall that $\tilde u\le u_n$ for all $n\in\mathbb{N}$, with $\tilde u\in D_+$. Hence $\tilde u\le u$ and so $u\ne 0$. From (3.52) it follows that

$$\begin{gathered} -\Delta_p u(z)+\xi(z)u^{p-1}(z) = f(z,u(z) \quad \text{for a.a. } z\in\Omega,\\ \frac{\partial u}{\partial n_p}+\beta(z)u^{p-1}=0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.53}$$

(see Papageorgiou-Radulescu [19]), then $u\in D_+$ (by the nonlinear regularity theory [16, 20] and the nonlinear maximum principle [23]).

Consider the function

$$R(\hat u_1,u)(z) = |D\hat u_1(z)|^p - |Du(z)|^{p-2}\Big(Du(z), D\big(\frac{\hat u_1^p}{u^{p-1}}\big)(z)\Big)_{\mathbb{R}^N}.$$

From the nonlinear Picone's identity (see Motreanu-Motreanu-Papageorgiou [17], p.255), we have

$$0\le R(\hat u_1,u)(z) \text{ for a.a. } z\in\Omega,$$

which implies using also Green's identity, (see Gasinski-Papageorgiou [11]) and hypothesis (H3)(iv)

$$\begin{aligned} 0 &\le \|D\hat u_1\|_p^p - \int_\Omega(-\Delta_p u)\frac{\hat u_1^p}{u^{p-1}}\,dz + \int_{\partial\Omega}\beta(z)u^{p-1}\frac{\hat u_1^p}{u^{p-1}}d\sigma\\ &= \|D\hat u_1\|_p^p + \int_\Omega \xi(z)\hat u_1^p\,dz + \int_{\partial\Omega}\beta(z)\hat u_1^p d\sigma - \int_\Omega f(z,u)\frac{\hat u_1^p}{u^{p-1}}\,dz\\ &\le \tau(\hat u_1) - \int_\Omega \hat\eta(z)\hat u_1^p\,dz = \hat\lambda_1\|\hat u_1\|_p^p - \int_\Omega \hat\eta(z)\hat u_1^p\,dz\\ &= \int_\Omega[\hat\lambda_1-\hat\eta(z)]\hat u_1^p\,dz < 0 \end{aligned}$$

which is a contradiction. Therefore $\lambda_* > 0$. $\square$

Now we show that for $\lambda \in (\lambda_*, +\infty)$ we have at least two positive solutions.

Proposition 3.7. *If* (H1), (H2), (H4) *hold and* $\lambda \in (\lambda_*, +\infty)$ *then problem* (1.1) *admits at least two positive solutions* $u_\lambda,\ \hat{u}_\lambda \in D_+$, $u_\lambda \neq \hat{u}_\lambda$.

Proof. Let $\mu_1,\ \mu_2 \in \mathcal{L}$ such that $\mu_1 < \lambda < \mu_2$. From Proposition 3.5, we know that there exist $u_{\mu_1} \in S(\mu_1) \subseteq D_+$ and $u_{\mu_2} \in S(\mu_2) \subseteq D_+$ such that

$$u_{\mu_1} - u_{\mu_2} \in \hat{D}_+.$$

We consider the Caratheodory function

$$g_0(z,x) = \begin{cases} f(z, u_{\mu_2}(z)) + \eta u_{\mu_2}(z)^{p-1} & \text{if } x < u_{\mu_2}(z) \\ f(z,x) + \eta x^{p-1} & \text{if } u_{\mu_2}(z) \le x \le u_{\mu_1}(z) \\ f(z, u_{\mu_1}(z)) + \eta u_{\mu_1}(z)^{p-1} & \text{if } u_{\mu_1}(z) < x \end{cases} \tag{3.54}$$

(recall that $\eta > \|\xi\|_\infty$, see the proof of Proposition 3.2).

We set $G_0(z,x) = \int_0^x g_0(z,s)ds$ and consider the C^1-functional $k_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$k_\lambda(u) = \frac{1}{p}\tau(u) + \frac{\lambda+\eta}{p}\|u\|_p^p - \int_\Omega G_0(z,u)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (3.54) and since $\eta > \|\xi\|_\infty$, it follows that $k_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$k_\lambda(u_\lambda) = \inf[k_\lambda(u) : u \in W^{1,p}(\Omega)],$$

then $k'_\lambda(u_\lambda) = 0$, so we obtain

$$\begin{aligned} &\langle A(u_\lambda), h\rangle + \int_\Omega (\xi(z) + \lambda + \eta)|u_\lambda|^{p-2}u_\lambda h\,dz + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda h d\sigma \\ &= \int_\Omega g_0(z,u_\lambda)h\,dz \end{aligned} \tag{3.55}$$

for all $h \in W^{1,p}(\Omega)$.

In (3.55) first we act with $h = (u_\lambda - u_{\mu_1})^+ \in W^{1,p}(\Omega)$, since $\mu_1 < \lambda$ and $u_{\mu_1} \in S(\mu_1)$, we obtain

$$\begin{aligned} &\langle A(u_\lambda), (u_\lambda - u_{\mu_1})^+\rangle + \int_\Omega (\xi(z) + \lambda + \eta)u_\lambda^{p-1}(u_\lambda - u_{\mu_1})^+\,dz \\ &+ \int_{\partial\Omega} \beta(z)u_\lambda^{p-1}(u_\lambda - u_{\mu_1})^+ d\sigma \\ &= \int_\Omega [f(z,u_{\mu_1}) + \eta u_{\mu_1}^{p-1}](u_\lambda - u_{\mu_1})^+\,dz \\ &= \langle A(u_{\mu_1}), (u_\lambda - u_{\mu_1})^+\rangle + \int_\Omega (\xi(z) + \mu_1 + \eta)u_{\mu_1}^{p-1}(u_\lambda - u_{\mu_1})^+\,dz \\ &\quad + \int_{\partial\Omega} \beta(z)u_{\mu_1}^{p-1}(u_\lambda - u_{\mu_1})^+ d\sigma \\ &\le \langle A(u_{\mu_1}), (u_\lambda - u_{\mu_1})^+\rangle + \int_\Omega (\xi(z) + \lambda + \eta)u_{\mu_1}^{p-1}(u_\lambda - u_{\mu_1})^+\,dz \end{aligned}$$

$$+\int_{\partial\Omega}\beta(z)u_{\mu_1}^{p-1}(u_\lambda-u_{\mu_1})^+d\sigma\,.$$

Then

$$\begin{aligned}&\langle A(u_\lambda)-A(u_{\mu_1}),(u_\lambda-u_{\mu_1})^+\rangle\\&+\int_\Omega(\xi(z)+\lambda+\eta)(u_\lambda^{p-1}-u_{\mu_1}^{p-1})(u_\lambda-u_{\mu_1})^+\,dz\\&+\int_{\partial\Omega}\beta(z)(u_\lambda^{p-1}-u_{\mu_1}^{p-1})(u_\lambda-u_{\mu_1})^+\,dz\le 0,\end{aligned}$$

hence, we have $u_\lambda\le u_{\mu_1}$.

Similarly, using $h=(u_{\mu_2}-u_\lambda)^+\in W^{1,p}(\Omega)$ in (3.55), we obtain $u_{\mu_1}\le u_\lambda$.

So, we have proved that

$$u_\lambda\in[u_{\mu_2},u_{\mu_1}]=\{u\in W^{1,p}(\Omega):u_{\mu_2}(z)\le u(z)\le u_{\mu_1}(z)\text{ for a.a. }z\in\Omega\}.\tag{3.56}$$

Then (3.54) and (3.56) imply that (3.55) become

$$\langle A(u_\lambda),h\rangle+\int_\Omega(\xi(z)+\lambda)u_\lambda^{p-1}h\,dz+\int_{\partial\Omega}\beta(z)u_\lambda^{p-1}hd\sigma=\int_\Omega f(z,u_\lambda)h\,dz.$$

for all $h\in W^{1,p}(\Omega)$. Hence $u_\lambda\in S(\lambda)\subset D_+$.

Moreover, arguing as in the proof of Proposition 3.5, we show that

$$u_\lambda-u_{\mu_2}\in\hat D_+\quad\text{and}\quad u_{\mu_1}-u_\lambda\in\hat D_+\,.$$

Then

$$u_\lambda\in\operatorname{int}_{C^1(\overline\Omega)}[u_{\mu_2},u_{\mu_1}].\tag{3.57}$$

Next we consider the Caratheodory function

$$\vartheta(z,x)=\begin{cases}f(z,u_{\mu_2}(z))+\eta u_{\mu_2}(z)^{p-1}&\text{if }x<u_{\mu_2}(z)\\ f(z,x)+\eta x^{p-1}&\text{if }u_{\mu_2}(z)\le x.\end{cases}\tag{3.58}$$

We set $\Theta(z,x)=\int_0^x\vartheta(z,s)ds$ and consider the C^1-functional $\Psi_\lambda:W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\Psi_\lambda(u)=\frac1p\tau(u)+\frac{\lambda+\eta}{p}\|u\|_p^p-\int_\Omega\Theta(z,u)\,dz\quad\text{for all }u\in W^{1,p}(\Omega).$$

We introduce the sets

$$\begin{gathered}[u_{\mu_2})=\{u\in W^{1,p}(\Omega):u_{\mu_2}(z)\le u(z)\text{ for a.a. }z\in\Omega\},\\ K_{\Psi_\lambda}=\{u\in W^{1,p}(\Omega):\Psi'(u)=0\}.\end{gathered}$$

Claim: $K_{\Psi_\lambda}\subseteq[u_{\mu_2})\cap C^1(\overline\Omega)$ Let $u\in K_{\Psi_\lambda}$. Then $\Psi_\lambda'(u)=0$ and we obtain

$$\begin{aligned}&\langle A(u),h\rangle+\int_\Omega(\xi(z)+\lambda+\eta)|u|^{p-2}uh\,dz+\int_{\partial\Omega}\beta(z)|u|^{p-2}uhd\sigma\\&=\int_\Omega\vartheta(z,u)h\,dz\end{aligned}\tag{3.59}$$

for all $h\in W^{1,p}(\Omega)$. In (3.59) we choose $h=(u_{\mu_2}-u)^+\in W^{1,p}(\Omega)$. Then, since $\mu_2>\lambda$ and $u_{\mu_2}\in S(\mu_2)$, we obtain

$$\langle A(u),(u_{\mu_2}-u)^+\rangle+\int_\Omega(\xi(z)+\lambda+\eta)|u|^{p-2}u(u_{\mu_2}-u)^+\,dz$$

$$
\begin{aligned}
&+\int_{\partial\Omega}\beta(z)|u|^{p-2}u(u_{\mu_2}-u)^+d\sigma\\
&=\int_\Omega[f(z,u_{\mu_2})+\eta u_{\mu_2}^{p-1}](u_{\mu_2}-u)^+\,dz\\
&=\langle A(u_{\mu_2}),(u_{\mu_2}-u)^+\rangle+\int_\Omega(\xi(z)+\mu_2+\eta)u_{\mu_2}^{p-1}(u_{\mu_2}-u)^+\,dz\\
&\quad+\int_{\partial\Omega}\beta(z)u_{\mu_2}^{p-1}(u_{\mu_2}-u)^+d\sigma\\
&\geq\langle A(u_{\mu_2}),(u_{\mu_2}-u)^+\rangle\\
&\quad+\int_\Omega(\xi(z)+\lambda+\eta)u_{\mu_2}^{p-1}(u_{\mu_2}-u)^+\,dz+\int_{\partial\Omega}\beta(z)u_{\mu_2}^{p-1}(u_{\mu_2}-u)^+d\sigma\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\langle A(u_{\mu_2})-A(u),(u_{\mu_2}-u)^+\rangle+\int_\Omega(\xi(z)+\lambda+\eta)(u_{\mu_2}^{p-1}-|u|^{p-2}u)(u_{\mu_2}-u)^+\,dz\\
&+\int_{\partial\Omega}\beta(z)(u_{\mu_1}^{p-1}-|u|^{p-2}u)(u_{\mu_2}-u)^+\,dz\leq 0,
\end{aligned}
$$

then, we have $u_{\mu_2}\leq u$ and $u\in C^1(\overline{\Omega})$ (by the nonlinear regularity theory), hence $K_{\Psi_\lambda}\subseteq[u_{\mu_2})\cap C^1(\overline{\Omega})$. This proves the Claim.

From (3.58) it is clear that $\vartheta(z,\cdot)$ has the same asymptotic behavior as $x\to+\infty$ as the reaction $f(z,\cdot)$. Hence reasoning as in the proof of Proposition 3.6 we show that

$$\Psi_\lambda \quad \text{satisfies the C-condition,} \tag{3.60}$$

(we show that every C-condition is bounded, see the Claim in the proof of Proposition 8 and then use Proposition 2.2).

Moreover, if $u\in D_+$, then taking into account hypotesis (H4)(i) we have

$$\Psi_\lambda(tu)\to-\infty \quad \text{as } t\to+\infty. \tag{3.61}$$

Note that

$$\Psi_\lambda|_{[u_{\mu_2},u_{\mu_1}]}=k_\lambda|_{[u_{\mu_2},u_{\mu_1}]}$$

where $[u_{\mu_2},u_{\mu_1}]=\{u\in W^{1,p}(\Omega):u_{\mu_2}(z)\leq u(z)\leq u_{\mu_1}(z)\text{ for a.a. }z\in\Omega\}$ Then from (3.57) it follows that u_λ is a local $C^1(\overline{\Omega})$-minimizer of Ψ_λ hence u_λ is a local $W^{1,p}(\Omega)$-minimizer of Ψ_λ (see Proposition 2.3).

We assume that K_{Ψ_λ} is finite or otherwise on account of (3.58) and the Claim, we see that we already have an infinity of positive solutions for problem (1.1), $\lambda>\lambda_*$. Thus we have finished. Since K_{Ψ_λ} is finite and u_λ is a local minimizer of Ψ_λ, we can find $\varrho_\lambda\in(0,1)$ small such that

$$\Psi_\lambda(u_\lambda)<\inf[\Psi_\lambda(u):\|u-u_\lambda\|=\varrho_\lambda]=m_\lambda \tag{3.62}$$

(see Aizicovici-Papageorgiou-Staicu [1, Proposition 29]). Then (3.60), (3.61), (3.62) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $\hat{u}_\lambda\in W^{1,p}(\Omega)$ such that

$$\hat{u}_\lambda\in K_{\Psi_\lambda}\subseteq[u_{\mu_2})\cap C^1(\overline{\Omega}) \quad\text{and}\quad m_\lambda\leq\Psi_\lambda(\hat{u}_\lambda). \tag{3.63}$$

From (3.58) and (3.63) it follows that $\hat{u}_\lambda\in S(\lambda)\subseteq D_+$ and from (3.62) and (3.63) we see that $\hat{u}_\lambda\neq u_\lambda$. □

Proposition 3.8. *If* (H1)–(H3) *hold, then* $\lambda_*\in\mathcal{L}$.

Proof. Let $\{\lambda_n\}_{n\geq 1} \subset \mathcal{L}$ such that $\lambda_n \downarrow \lambda_*$. As before (see the proof of Proposition 3.6) we can find $u_n \in S(\lambda_n) \subseteq D_+$, $n \in \mathbb{N}$ such that $\tilde{u} \leq u_n$ for some $\tilde{u} \in D_+$, all $n \in \mathbb{N}$ and

$$\varphi_{\lambda_n}(u_n) < 0 \text{ all } n \in \mathbb{N}.$$

Reasoning as in the proof of Proposition 3.6, we show that $\{u_n\} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \rightharpoonup u_* \ \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_* \ \text{ in } L^p(\Omega) \text{ and } L^p(\partial\Omega).$$

Moreover, in the limit as $n \to +\infty$, we obtain

$$\langle A(u_*), h\rangle + \int_\Omega (\xi(z) + \lambda_*) u_*^{p-1} h\,dz + \int_{\partial\Omega} \beta(z) u_*^{p-1} h d\sigma = \int_\Omega f(z, u_*) h\,dz \tag{3.64}$$

for all $h \in W^{1,p}(\Omega)$ (see the proof of Proposition 3.6). Since $\tilde{u} \leq u_*$, we have $u_* \neq 0$ and so from (3.64) we infer that $u_* \in S(\lambda_*) \subseteq D_+$ hence $\lambda_* \in \mathcal{L}$. ☐

Corollary 3.9. *If* (H1)–(H3) *hold, then* $\mathcal{L} = [\lambda_*, +\infty)$.

We can now formulate our nonexistence, existence and multiplicity theorem for problem (1.1) (bifurcation-type theorem).

Theorem 3.10. (a) *If* (H1)–(H3) *hold, then there exists* $\lambda_* > 0$ *such that* $\mathcal{L} = [\lambda_*, +\infty)$.

(b) *If* (H1), (H2), (H4) *hold, then with* $\lambda_* > 0$ *as in (a) we have*
  (1) *for* $\lambda \in (0, \lambda_*)$ *problem* (1.1) *has no positive solutions;*
  (2) *for* $\lambda = \lambda_*$ *problem* (1.1) *has at least one positive solution* $u_{\lambda_*} \in D_+$*;*
  (3) *for* $\lambda \in (\lambda_*, +\infty)$ *problem* (1.1) *has at least two positive solutions* $u_\lambda, \hat{u}_\lambda \in D_+$ *and* $u_\lambda \neq \hat{u}_\lambda$.

4. Minimal positive solutions

In this section we show that for every $\lambda \in \mathcal{L}$ problem (1.1) has a smallest positive solution $\tilde{u}_\lambda \in D_+$ and then we examine the continuity and monotonicity properties of the map $\lambda \to \tilde{u}_\lambda$.

Hypotheses (H3)(i), (iv) imply that

$$f(z, x) \geq c_1 x^{q-1} - c_{17} x^{p^*-1} \tag{4.1}$$

for a.a. $z \in \Omega$, all $x \geq 0$ and some $c_{17} > 0$.

The unilateral growth condition on $f(z, \cdot)$ leads to the following auxiliary parametric Robin problem with parameter $\tilde{\lambda} \geq 0$:

$$\begin{gathered}
-\Delta_p u(z) + (\xi(z) + \tilde{\lambda}) u(z)^{p-1} = c_1 u(z)^{q-1} - c_{17} u(z)^{p^*-1} \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n_p} + \beta(z) u^{p-1} = 0, \quad \text{on } \partial\Omega, \\
u > 0.
\end{gathered} \tag{4.2}$$

Proposition 4.1. *If* (H1), (H2) *hold and* $\tilde{\lambda} \geq 0$*, then problem* (4.2) *admits a unique positive solution* $\bar{u}_{\tilde{\lambda}} \in D_+$

Proof. First we show the existence of a positive solution for problem (4.2). To this end, we consider the C^1-functional $\hat{\gamma}_{\tilde{\lambda}} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\gamma}_{\tilde{\lambda}}(u) = \frac{1}{p}\tau(u) + \frac{\tilde{\lambda}}{p}\|u\|_p^p + \frac{\eta}{p}\|u^-\|_p^p - \frac{c_1}{q}\|u^+\|_p^p + \frac{c_{17}}{p^*}\|u^+\|_{p^*}^{p^*}$$

for all $u \in W^{1,p}(\Omega)$.

Here as before $\eta > \|\xi\|_\infty$ (see the proof of Proposition 3.2). Since $\eta > \|\xi\|_\infty$ and $q < p < p^*$, we see that $\hat{\gamma}_{\tilde{\lambda}}$ is coercive. Also, it is sequentially weakly lower semicontinuos (recall that $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ continuously). So, by the Weierstrass-Tonelli theorem, we can find $\bar{u}_{\tilde{\lambda}} \in W^{1,p}(\Omega)$ such that

$$\hat{\gamma}_{\tilde{\lambda}}(\bar{u}_{\tilde{\lambda}}) = \inf \big[\hat{\gamma}_{\tilde{\lambda}}(u) : u \in W^{1,p}(\Omega)\big]. \tag{4.3}$$

For $u \in D_+$ and $t > 0$, since $q < p < p^*$ we see that if we take $t \in (0,1)$ small, we have $\hat{\gamma}_{\tilde{\lambda}}(tu) < 0$, which implies

$$\hat{\gamma}_{\tilde{\lambda}}(\bar{u}_{\tilde{\lambda}}) < 0 = \hat{\gamma}_{\tilde{\lambda}}(0)$$

hence $\bar{u}_{\tilde{\lambda}} \neq 0$.

From (4.3) we have $\hat{\gamma}'_{\tilde{\lambda}}(\bar{u}_{\tilde{\lambda}}) = 0$ so we obtain

$$\begin{aligned} &\langle A(\bar{u}_{\tilde{\lambda}}), h\rangle + \int_\Omega (\xi(z) + \tilde{\lambda})|\bar{u}_{\tilde{\lambda}}|^{p-2}\bar{u}_{\tilde{\lambda}} h\,dz + \int_{\partial\Omega} \beta(z)|\bar{u}_{\tilde{\lambda}}|^{p-2}\bar{u}_{\tilde{\lambda}} h d\sigma \\ &- \int_\Omega \eta(\bar{u}_{\tilde{\lambda}}^-)^{p-1} h\,dz \\ &= \int_\Omega [c_1(\bar{u}_{\tilde{\lambda}}^+)^{q-1} - c_{17}(\bar{u}_{\tilde{\lambda}}^+)^{p^*-1}]) h\,dz \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{4.4}$$

In (4.4) we choose $h = -\bar{u}_{\tilde{\lambda}}^- \in W^{1,p}(\Omega)$. Then

$$\tau(\bar{u}_{\tilde{\lambda}}^-) + \eta\|\bar{u}_{\tilde{\lambda}}^-\|_p^p = 0,$$

since $\eta > \|\xi\|_\infty$, we obtain

$$c_{18}\|\bar{u}_{\tilde{\lambda}}^-\|^p \le 0$$

for some constant $c_{18} > 0$, hence $\bar{u}_{\tilde{\lambda}} \ge 0$, $\bar{u}_{\tilde{\lambda}} \neq 0$.

From (4.4) it follows that $\bar{u}_{\tilde{\lambda}}$ is a positive solution of (4.2). From Papageorgiou-Radulescu [19], we have $\bar{u}_{\tilde{\lambda}} \in L^\infty(\Omega)$.

Then Lieberman [16, Theorem 2] implies that $\bar{u}_{\tilde{\lambda}} \in C_+ \setminus \{0\}$. Moreover, we have

$$\Delta_p \bar{u}_{\tilde{\lambda}}(z) \le [\|\xi\|_\infty + \tilde{\lambda} + c_{17}\|\bar{u}_{\tilde{\lambda}}\|_\infty^{p^*-p}]\bar{u}_{\tilde{\lambda}}(z)^{p-1} \quad \text{for a.a. } z \in \Omega,$$

then $\bar{u}_{\tilde{\lambda}} \in D_+$ (from the nonlinear maximum principle, see Pucci-Serrin [23, pp. 111,120]).

Next we show the uniqueness of this positive solution. Consider the integral functional $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\|Du^{1/p}\|_p^p & \text{if } u \ge 0,\ u^{1/p} \in W^{1,p}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

From Diaz-Saa [6, Lemma 1], we have that $j(\cdot)$ is convex. Suppose that $\bar{u}_{\tilde{\lambda}}^*$ is another positive solution of (4.2). Again we have $\bar{u}_{\tilde{\lambda}}^* \in D_+$. Note that

$$\bar{u}_{\tilde{\lambda}}^*,\ (\bar{u}_{\tilde{\lambda}}^*)^p \in \operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < +\infty\}$$

(the effective domain of $j(\cdot)$).

Then for every $h \in C^1(\overline{\Omega})$, $j(\cdot)$ is Gateaux differentiable at $\bar{u}_{\tilde{\lambda}}^p$, $(\bar{u}_{\tilde{\lambda}}^*)^p$ in the direction h. By the chain rule and the nonlinear Green's identity, we have

$$j'(\bar{u}_{\tilde{\lambda}}^p)(h) = \int_\Omega \frac{-\Delta_p \bar{u}_{\tilde{\lambda}}}{\bar{u}_{\tilde{\lambda}}^{p-1}} h\,dz - \int_{\partial\Omega} \beta(z) h d\sigma$$

$$j'((\bar{u}^*_{\tilde{\lambda}})^p)(h) = \int_\Omega \frac{-\Delta_p \bar{u}^*_{\tilde{\lambda}}}{(\bar{u}^*_{\tilde{\lambda}})^{p-1}} h\, dz - \int_{\partial\Omega} \beta(z) h d\sigma$$

for all $h \in W^{1,p}(\Omega)$. The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. So, since $q < p < p^*$ we have

$$\begin{aligned}
0 &\le \int_\Omega \Big[\frac{-\Delta_p \bar{u}_{\tilde{\lambda}}}{\bar{u}^{p-1}_{\tilde{\lambda}}} - \frac{-\Delta_p \bar{u}^*_{\tilde{\lambda}}}{(\bar{u}^*_{\tilde{\lambda}})^{p-1}}\Big] \Big(\bar{u}^p_{\tilde{\lambda}} - (\bar{u}^*_{\tilde{\lambda}})^p\Big)\, dz \\
&= \int_\Omega \Big(c_1\Big[\frac{1}{\bar{u}^{p-q}_{\tilde{\lambda}}} - \frac{1}{(\bar{u}^*_{\tilde{\lambda}})^{p-q}}\Big] - c_{17}\Big[\bar{u}^{p^*-p}_{\tilde{\lambda}} - (\bar{u}^*_{\tilde{\lambda}})^{p^*-p}\Big]\Big) \Big(\bar{u}^p_{\tilde{\lambda}} - (\bar{u}^*_{\tilde{\lambda}})^p\Big)\, dz \\
&\le 0
\end{aligned}$$

hence $\bar{u}_{\tilde{\lambda}} = \bar{u}^*_{\tilde{\lambda}}$ which implies that $\bar{u}_{\tilde{\lambda}} \in D_+$ is the unique positive solution of (4.2). ☐

Proposition 4.2. *If* (H1), (H2), (H3) *hold,* $\lambda \in \mathcal{L}$ *and* $u \in S(\lambda) \subseteq D_+$*, then* $\bar{u}_{\tilde{\lambda}} \le u$ *with* $\tilde{\lambda} \ge \lambda$*.*

Proof. We consider the Caratheodory function $k_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$k_0(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ c_1 x^{q-1} - c_{17} x^{p^*-1} + \eta x^{p-1} & \text{if } 0 \le x \le u(z) \\ c_1 u(z)^{q-1} - c_{17} u(z)^{p^*-1} + \eta u_{\mu_1}(z)^{p-1} & \text{if } u(z) < x \end{cases} \tag{4.5}$$

(recall that $\eta > \|\xi\|_\infty$). We set $K_0(z,x) = \int_0^x k_0(z,s) ds$ and consider the C^1-functional $\Psi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\Psi_0(u) = \frac{1}{p}\tau(u) + \frac{\tilde{\lambda} + \eta}{p}\|u\|_p^p - \int_\Omega K_0(z,u)\, dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (4.5) and since $\eta > \|\xi\|_\infty$, we see that $\Psi_0(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\tilde{u} \in W^{1,p}(\Omega)$ such that

$$\Psi_0(\tilde{u}) = \inf[\Psi_0(u) \ : \ u \in W^{1,p}(\Omega)]. \tag{4.6}$$

Using hypothesis (H3)(iv) as in the proof of Proposition 3.2, we have $\Psi_0(\tilde{u}) < 0 = \Psi_0(0)$, which implies that $\tilde{u} \neq 0$.

From (4.6) we have $\Psi_0'(\tilde{u}) = 0$, so we obtain

$$\begin{aligned}
&\langle A(\tilde{u}), h\rangle + \int_\Omega (\xi(z) + \tilde{\lambda} + \eta)|\tilde{u}|^{p-2}\tilde{u} h\, dz + \int_{\partial\Omega} \beta(z)|\tilde{u}|^{p-2}\tilde{u} h d\sigma \\
&= \int_\Omega k_0(z,\tilde{u}) h\, dz \quad \text{for all } h \in W^{1,p}(\Omega).
\end{aligned} \tag{4.7}$$

In (4.7) first we choose $h = -\tilde{u}^- \in W^{1,p}(\Omega)$. Then

$$\tau(\tilde{u}^-) + [\tilde{\lambda} + \eta]\|\tilde{u}^-\|_p^p = 0,$$

hence, $c_{19}\|\tilde{u}^-\|^p \le 0$ for some positive constant c_{19} and since $\tilde{\lambda} + \eta > \|\xi\|_\infty$, we obtain $\tilde{u} \ge 0$ and $\tilde{u} \neq 0$.

Next in (4.7) we choose $h = (\tilde{u} - u)^+ \in W^{1,p}(\Omega)$, since $u \in S(\lambda)$ and $\tilde{\lambda} \ge \lambda$ we obtain

$$\langle A(\tilde{u}), (\tilde{u} - u)^+\rangle + \int_\Omega (\xi(z) + \tilde{\lambda} + \eta)\tilde{u}^{p-1}(\tilde{u} - u)^+\, dz$$

$$
\begin{aligned}
&+\int_{\partial\Omega}\beta(z)\tilde{u}^{p-1}(\tilde{u}-u)^{+}d\sigma\\
&=\int_{\Omega}[c_1u^{q-1}-c_{17}u^{p^*-1}+\eta u^{p-1}](\tilde{u}-u)^{+}\,dz\\
&\le\int_{\Omega}[f(z,u)+\eta u^{p-1}](\tilde{u}-u)^{+}\,dz\\
&=\langle A(u),(\tilde{u}-u)^{+}\rangle+\int_{\Omega}(\xi(z)+\tilde{\lambda}+\eta)u^{p-1}(\tilde{u}-u)^{+}\,dz\\
&\quad+\int_{\partial\Omega}\beta(z)u^{p-1}(\tilde{u}-u)^{+}d\sigma;
\end{aligned}
$$

then

$$
\begin{aligned}
&\langle A(\tilde{u})-A(u),(\tilde{u}-u)^{+}\rangle+\int_{\Omega}(\xi(z)+\tilde{\lambda}+\eta)(\tilde{u}^{p-1}-u^{p-1})(\tilde{u}-u)^{+}\,dz\\
&+\int_{\partial\Omega}\beta(z)(\tilde{u}^{p-1}-u^{p-1})(\tilde{u}-u)^{+}\,dz\le 0,
\end{aligned}
$$

so, we have $\tilde{u}\le u$. We have proved that

$$
\tilde{u}\in[0,u]=\{v\in W^{1,p}(\Omega):0\le v(z)\le u(z)\text{ for a.a. }z\in\Omega\},\quad \tilde{u}\ne 0. \tag{4.8}
$$

From (4.5), (4.7) and (4.8) it follows that $\tilde{u}$ is a nontrivial positive solution of (4.2), hence $\tilde{u}=\bar{u}_{\tilde{\lambda}}\in D_+$.

Therefore we conclude that $\bar{u}_{\tilde{\lambda}}\le u$ for all $u\in S(\lambda)$ (see Proposition 4.1). □

Now we are ready to produce the smallest positive solution for problem (1.1), $\lambda\in\mathcal{L}$.

Proposition 4.3. *If* (H1)–(H3) *hold and* $\lambda\in\mathcal{L}$ *then problem* (1.1) *has a smallest positive solution* $\tilde{u}_\lambda\in S(\lambda)\subseteq D_+$.

Proof. From Papageorgiou-Radulescu-Repovs [21], we know that $S(\lambda)$ is downward directed (that is, if u, $v\in S(\lambda)$, then we can find $y\in S(\lambda)$ such that $y\le u$, $y\le v$, see also Filippakis-Papageorgiou [8]). Invoking Hu-Papageorgiou [15, Lemma 3.10, p 178], we can find $\{u_n\}_{n\ge1}\subseteq S(\lambda)$ decreasing such that

$$
\inf S(\lambda)=\inf_{n\ge1}u_n.
$$

For every $n\in\mathbb{N}$ we have

$$
\langle A(u_n),h\rangle+\int_{\Omega}(\xi(z)+\lambda)u_n^{p-1}h\,dz+\int_{\partial\Omega}\beta(z)u_n^{p-1}h d\sigma=\int_{\Omega}f(z,u_n)h\,dz \tag{4.9}
$$

for all $h\in W^{1,p}(\Omega)$.

$$
0\le u_n\le u_1 \tag{4.10}
$$

From (4.9) and (4.10) it follows that $\{u_n\}_{n\ge1}\subset W^{1,p}(\Omega)$ is bounded. We may assume that

$$
u_n\rightharpoonup\hat{u}_\lambda\quad\text{in }W^{1,p}(\Omega)\text{ and }u_n\to\hat{u}_\lambda\ \text{ in }L^p(\Omega)\text{ and }L^p(\partial\Omega). \tag{4.11}
$$

In (4.7) we choose $h=u_n-\tilde{u}_\lambda\in W^{1,p}(\Omega)$, pass to the limit as $n\to\infty$ and use (4.11). Then

$$
\lim_{n\to\infty}\langle A(u_n),u_n-\tilde{u}_\lambda\rangle=0,
$$

which implies

$$
u_n\to\tilde{u}_\lambda\quad\text{in }W^{1,p}(\Omega)\quad\text{as }n\to\infty. \tag{4.12}
$$

Passing to the limit as $n \to \infty$ in (4.7) and using (3.55), we obtain

$$\langle A(\tilde{u}_\lambda), h\rangle + \int_\Omega (\xi(z)+\lambda)\tilde{u}_\lambda^{p-1} h\,dz + \int_{\partial\Omega} \beta(z)\tilde{u}_\lambda^{p-1} h d\sigma = \int_\Omega f(z,\tilde{u}_\lambda) h\,dz$$

for all $h \in W^{1,p}(\Omega)$ and $\bar{u}_\lambda \le \tilde{u}_\lambda$ (see Proposition 4.2). From these facts we conclude that $\tilde{u}_\lambda \in S(\lambda)$ and $\tilde{u}_\lambda = \inf S(\lambda)$. □

We examine the monotonicity and continuity properties of the map

$$\chi(\lambda) = \tilde{u}_\lambda \quad \text{for } \lambda \in \mathcal{L} = [\lambda_*, +\infty).$$

We have the following monotonicity and continuity result.

Proposition 4.4. *If* (H1)–(H3) *hold, then* $\chi : \mathcal{L} \to C^1(\overline{\Omega})$ *is decreasing in the sense that*

$$\lambda, \theta \in \mathcal{L},\ \lambda < \theta \Rightarrow \tilde{u}_\lambda - \tilde{u}_\theta \in C_+ \setminus \{0\}.$$

and it is also right continuous.

Proof. From the proof of Proposition 3.2, we know that we can find $u_\theta \in S(\theta)$ such that

$$\tilde{u}_\lambda - u_\theta \in C_+ \setminus \{0\},$$

which implies

$$\tilde{u}_\lambda - \tilde{u}_\theta \in C_+ \setminus \{0\}$$

(since $\tilde{u}_\theta \le u_\theta$). Hence $\chi(\cdot)$ is decreasing.

Next suppose that $\{\lambda_n\}_{n\ge1} \subseteq \mathcal{L}$ and $\lambda_n \to \lambda^+$ ($\lambda^+ \in \mathcal{L}$). We know that $\{\tilde{u}_{\lambda_n}\}_{n\ge1} \subseteq D_+$ is increasing and

$$0 \le \tilde{u}_{\lambda_n} \le \tilde{u}_\lambda \quad \text{for all } n \in \mathbb{N}. \tag{4.13}$$

Then Lieberman [16, Theorem 2] implies that we can find $\alpha \in (0,1)$ and $c_{20} > 0$ such that

$$\tilde{u}_{\lambda_n} \in C^{1,\alpha}(\overline{\Omega}),\ \|\tilde{u}_{\lambda_n}\|_{C^{1,\alpha}(\overline{\Omega})} \le c_{20} \quad \text{for all } n \in \mathbb{N}. \tag{4.14}$$

The compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and (4.14), imply that at least for a subsequence, we have

$$\tilde{u}_{\lambda_n} \to u_\lambda^* \quad \text{in } C^1(\overline{\Omega}) \text{ as } n \to \infty. \tag{4.15}$$

If $u_\lambda^* \ne \tilde{u}_\lambda$, then we can find $z_0 \in \overline{\Omega}$ such that $\tilde{u}_\lambda(z_0) < u^*(z_0)$ and from (4.15) we obtain $\tilde{u}_\lambda(z_0) < \tilde{u}_{\lambda_n}(z_0)$ for all $n \ge n_0$, which contradicts (4.13). Therefore $\tilde{u}_\lambda = u_\lambda^*$ and this proves the right continuity of $\chi(\cdot)$ □

Under stronger conditions on $f(z,\cdot)$, we can improve the monotonicity of $\chi(\cdot)$.

Proposition 4.5. *If* (H1), (H2), (H4) *hold, then* $\chi : \mathcal{L} \to C^1(\overline{\Omega})$ *is strictly decreasing in the sense that*

$$\lambda, \theta \in \mathcal{L},\ \lambda < \theta \Rightarrow \tilde{u}_\lambda - \tilde{u}_\theta \in \hat{D}_+.$$

and it is also right continuous.

Proof. As in the proof of Proposition 4.4, using this time Proposition 3.5, we have

$$\lambda, \theta \in \mathcal{L},\ \lambda < \theta \Rightarrow \tilde{u}_\lambda - \tilde{u}_\theta \in \hat{D}_+.$$

Hence $\chi(\cdot)$ is strictly increasing. The right continuity of $\chi(\cdot)$ follows from Proposition 4.4. □

Summarizing our findings in this section on the minimal positive solution of problem (1.1) for $\lambda \in \mathcal{L} = [\lambda_*, +\infty)$, we can state the following theorem.

Theorem 4.6. (a) *If hypotheses* (H1)–(H3) *hold, then for every* $\lambda \in \mathcal{L} = [\lambda_*, +\infty)$ *problem* (1.1) *has a smallest positive solution* $\tilde{u}_\lambda \in D_+$ *and the map* $\lambda \to \tilde{u}_\lambda$ *from* $\mathcal{L} = [\lambda_*, +\infty)$ *into* $C^1(\overline{\Omega})$ *is decreasing (that is* $\lambda, \theta \in \mathcal{L}$, $\lambda < \theta \Rightarrow \tilde{u}_\lambda - \tilde{u}_\theta \in C_+ \setminus \{0\}$*) and right continuous.*

(b) *If hypotheses* (H1), (H2), (H4) *hold, then for every* $\lambda \in \mathcal{L} = [\lambda_*, +\infty)$ *problem* (1.1) *has a smallest positive solution* $\tilde{u}_\lambda \in D_+$ *and the map* $\lambda \to \tilde{u}_\lambda$ *from* $\mathcal{L} = [\lambda_*, +\infty)$ *into* $C^1(\overline{\Omega})$ *is strictly decreasing (that is* $\lambda, \theta \in \mathcal{L}$, $\lambda < \theta \Rightarrow \tilde{u}_\lambda - \tilde{u}_\theta \in \hat{D}_+$*) and right continuous.*

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Diego Averna
Dipartimento di Matematica e Informatica, Università degli studi di Palermo, Via Archirafi, 90123 - Palermo, Italy
E-mail address: `diego.averna@unipa.it`

Nikolaos S. Papageorgiou
Department of Mathematics, National Technical University, Zografou Campus, Athens 15780, Greece
E-mail address: `npapg@math.ntua.gr`

Elisabetta Tornatore
Dipartimento di Matematica e Informatica, Università degli studi di Palermo, Via Archirafi, 90123 Palermo, Italy
E-mail address: `elisa.tornatore@unipa.it`