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Enumeration of L-convex polyominoes by rows and columns

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Abstract

In this paper, we consider the class of L-convex polyominoes, i.e. the convex polyominoes in which any two cells can be connected by a path of cells in the polyomino that switches direction between the vertical and the horizontal at most once.

Using the ECO method, we prove that the number f_n of L-convex polyominoes with perimeter $2(n+2)$ satisfies the rational recurrence relation $f_n = 4f_{n-1} - 2f_{n-2}$, with $f_0 = 1$, $f_1 = 2$, $f_2 = 7$. Moreover, we give a combinatorial interpretation of this statement. In the last section, we present some open problems.
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1. Introduction

One of the most famous problems in combinatorics is the enumeration of self-avoiding walks, or self-avoiding polygons, on a regular lattice. A closely related problem is that of enumerating polyominoes: a *polyomino* is a finite union of elementary cells of the lattice $Z \times Z$, whose interior is connected.

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A series of problems is related to these objects such as, for example, the decidability problems concerning the tiling of the plane, or of a rectangle, using polyominoes [5,13] and, on the other hand, the covering problems of a polyomino by rectangles [11].

The general enumeration problem of polyominoes is difficult to solve and still open. In order to simplify enumeration problems of polyominoes, several subclasses were defined by combining the two simple notions of convexity and directed growth.

A significant result in the enumeration of convex polyominoes was first obtained by Delest and Viennot in [14], where the authors determined the number of convex polyominoes with semi-perimeter equal to $n + 2$. Substantially, they encode convex polyominoes by means of words of a context-free language generated by a non-ambiguous grammar, and successively translate the grammar productions into an algebraic system of equations, thus obtaining the generating function of the desired class.

In the present paper, we deal with the family of *L-convex polyominoes*, recently introduced in [9] as the first level in a classification of convex polyominoes. In that work the authors observed that convex polyominoes have the property that every pair of cells is connected by a monotone path (see Section 2). In this way each convex polyomino is characterized by a parameter k that represents the minimal number of changes of direction in these paths. More precisely, a convex polyomino is called k-convex if, for every pair of its cells, there is at least a monotone path with at most k changes of direction that connects them. When the value of k is 1 we have the so-called L-convex polyominoes, where this terminology is motivated by the L-shape of the path that connects any two cells.

Successively this class of polyominoes has been considered by several points of view: for example, in [10] it is shown that L-convex polyominoes are a well-ordering according to the sub-picture order. Moreover, the authors have investigated some tomographical aspects of this family, and have discovered that L-convex polyominoes are uniquely determined by their horizontal and vertical projections [8].

Here, we face the problem of the enumeration of L-convex polyominoes according to various parameters, through the application of the ECO method, where ECO stands for enumeration of combinatorial objects. Such a method, introduced by Pinzani and his collaborators, is a constructive method to produce all the objects of a given class, according to the growth of a certain parameter (the *size*) of the objects. Basically, the idea is to perform local expansions on each object of size n, thus constructing a set of objects of the successive size (see [3] for more details).

The application of the ECO method often leads to an easy solution for problems that are commonly believed "hard" to solve. For example, in [15] the authors give an ECO construction for the classes of convex polyominoes and column-convex polyominoes according to the semi-perimeter. A simple algebraic computation leads then to the determination of generating functions for the two classes.

In [1] it is also shown that an ECO construction easily leads to an efficient algorithm for the exhaustive generation of the examined class. Moreover, an ECO construction can often produce interesting combinatorial information about the class of objects studied, as shown in [3] using analytic methods, or in [4], using bijective techniques.

In [2], Banderier et al. reintroduced the *kernel method* in order to determine the generating function of various types of ECO systems.

As main result we prove that the number f_n of L-convex polyominoes having semi-perimeter equal to $n + 2$ satisfies the recurrence relation:

$$f_n = 4 f_{n-1} - 2 f_{n-2}, \quad n \geqslant 3, \tag{1}$$

with $f_0 = 1$, $f_1 = 2$, $f_2 = 7$.

In the last part of the paper we give a simple bijective proof of this relation, providing also some statistics on the class of L-convex polyominoes and many open questions.

In a successive paper [7], the authors have studied the problem of enumerating L-convex polyominoes by the area, and provided a coding of L-convex polyominoes in terms of words of a regular language.

2. Basic definitions

A *polyomino* is a finite union of elementary cells of the lattice $Z \times Z$, whose interior is connected (see Fig. 1(a)). A polyomino is said to be *h-convex* (resp., *v-convex*) if every its row (resp., column) is connected. A polyomino is said to be *hv-convex*, or simply *convex*, if it is both *h*-convex and *v*-convex (see Fig. 1(b)).

In a convex polyomino P we will consider the following parameters:

- $h(P)$ (the *height*), the number of rows of P;
- $w(P)$ (the *width*), the number of columns of P;
- $p(P)$ (the *semi-perimeter*), the sum of $h(P)$ and $w(P)$;
- $a(P)$ (the *area*), the number of cells of P.

In a polyomino we will define a *path* as a self-avoiding sequence of unitary steps of four types: north $N = (0, 1)$, south $S = (0, -1)$, east $E = (1, 0)$, and west $W = (-1, 0)$. A path $\Pi_{(A,B)}$, connecting two distinct cells A and B, starts from the center of A, and ends in the center of B (see Fig. 2(a)).

We say that a path is *monotone* if it is made with steps of only two types (see Fig. 2(b)).

In [9] it is proved that the cells of convex polyominoes satisfy a particular connection property that involves the shape of the paths connecting any pair of cells.

Proposition 2.1. *A polyomino P is convex iff every pair of cells is connected by a monotone path.*
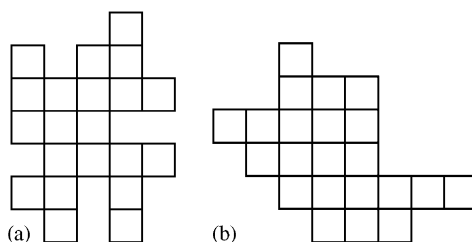

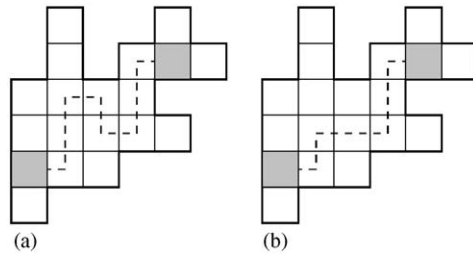Fig. 1. (a) A polyomino; (b) a convex polyomino.

Fig. 2. (a) A path between two highlighted cells in a polyomino; (b) a monotone path between the same cells, made only of north and east steps.
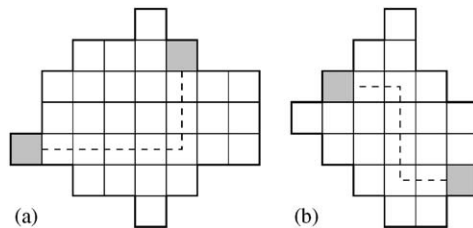
Fig. 3. (a) A L-convex polyomino, and a monotone path with a single change of direction joining two of its cells; (b) a convex but not L-convex polyomino: the two highlighted cells cannot be connected by a path with only one change of direction.

2.1. The class of L-convex polyominoes

Given a path $w = u_1 \ldots u_k$, with $u_i \in \{N, S, E, W\}$, each pair of steps $u_i u_{i+1}$ such that $u_i \neq u_{i+1}$, $0 < i < k$, is called a *change of direction*. For instance, the monotone path depicted in Fig. 2(b) has four changes of direction.

In [9] it is proposed a classification of convex polyominoes based on the number of changes of direction in the paths connecting any two cells of a polyomino. More precisely, we call *k-convex* a convex polyomino such that every pair of cells can be connected by a monotone path with at most k changes of direction. The convex polyominoes in the first level of this classification, called *L-convex polyominoes*, are such that every pair of their cells can be connected by a path with at most one change of direction (see Fig. 3(a)). So, for example, the reader can easily check that the polyomino in Fig. 2(b) is not L-convex.

Let us denote by $\mathcal{L}$ the class of L-convex polyominoes. In what follows we will often use an alternative characterization of $\mathcal{L}$, which involves the notion of *maximal rectangle*.

By abuse of notation, for any two polyominoes P and P' we will write $P \subseteq P'$ to mean that P is geometrically included in P'.

A *rectangle*, that we denote by $[x, y]$, with $x, y \in \mathbb{N} \setminus \{0\}$, is a rectangular polyomino with x columns and y rows. We say $[x, y]$ to be *maximal* in P if

$$\forall [x', y'], \quad [x, y] \subseteq [x', y'] \subseteq P \Rightarrow [x, y] = [x', y'].$$

Two rectangles $[x, y]$ and $[x' y']$ contained in P have a *crossing intersection* if their intersection is a rectangle whose basis is the smallest of the two bases and whose height is
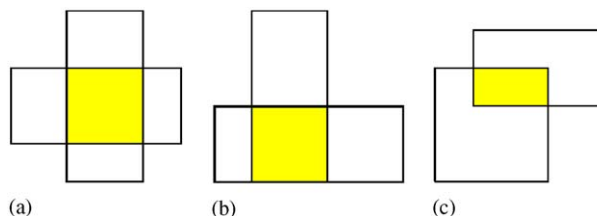
Fig. 4. (a), (b) Two rectangles having a crossing intersection; (c) two rectangles having a non-crossing intersection.
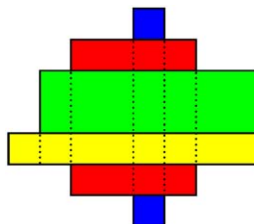

Fig. 5. The L-convex polyomino in Fig. 3(a) defined by the overlapping of four non-comparable rectangles having crossing intersection.

the smallest of the two heights, i.e.

$$[x, y] \cap [x', y'] = \big[\min(x, x'), \min(y, y')\big],$$

see Fig. 4, for an example. The following theorem gives a characterization of L-convex polyominoes in terms of maximal rectangles [9].

Theorem 2.1. *A convex polyomino P is L-convex iff every pair of its maximal rectangles has crossing intersection.*

Thus, a L-convex polyomino can be seen as the overlapping of maximal rectangles. On the other hand, each finite overlapping of non-comparable rectangles such that any pair of them has a crossing intersection determines a L-convex polyomino (see Fig. 5 for an example). As a consequence we have that, given a L-convex polyomino P, there is a unique maximal rectangle $[x, y]$, such that $x = w(P)$. In the sequel, we denote such a rectangle by $R(P)$, and its height (i.e. y) by $\eta(P)$ (briefly, η); similarly, there is a maximal rectangle $[x, y]$, such that $y = h(P)$. In Fig. 6 a L-convex polyomino with $\eta(P) = 3$ is depicted.

The characterization of L-convex polyominoes given in Theorem 2.1 will be useful in order to solve the problem of their enumeration.

2.2. ECO method

In this section, we will recall some basics about the ECO method; it is a method for the enumeration and the recursive construction of a class of combinatorial objects, $\mathcal{O}$, by means of an operator ϑ which performs "local expansions" on the objects of $\mathcal{O}$. More precisely,
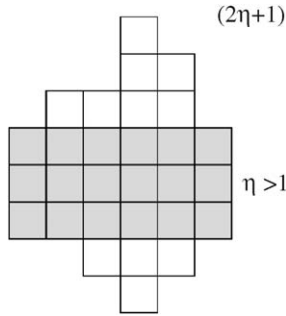
Fig. 6. A L-convex polyomino of class A. The maximal rectangle $R(P)$ is emphasized.

let p be a parameter on $\mathcal{O}$, such that $|\mathcal{O}_n| = |\{O \in \mathcal{O} : p(O) = n\}|$ is finite. An operator ϑ on the class $\mathcal{O}$ is a function from $\mathcal{O}_n$ to $2^{\mathcal{O}_{n+1}}$, where $2^{\mathcal{O}_{n+1}}$ is the power set of $\mathcal{O}_{n+1}$.

Proposition 2.2. *Let ϑ be an operator on $\mathcal{O}$. If ϑ satisfies the following conditions*:
1. *for each $O' \in \mathcal{O}_{n+1}$, there exists $O \in \mathcal{O}_n$ such that $O' \in \vartheta(O)$,*
2. *for each $O, O' \in \mathcal{O}_n$ such that $O \neq O'$, then $\vartheta(O) \cap \vartheta(O') = \emptyset$,*
then the family of sets $\mathcal{F}_{n+1} = \{\vartheta(O) : O \in \mathcal{O}_n\}$ is a partition of $\mathcal{O}_{n+1}$.

This method was successfully applied to the enumeration of various classes of walks, permutations, and polyominoes. We refer to [3,16] for further details and results.

The recursive construction determined by ϑ can be suitably described through a *generating tree*, i.e. a rooted tree whose vertices are objects of $\mathcal{O}$. The objects having the same value of the parameter p lie at the same level, and the sons of an object are the objects it produces through ϑ.

If the construction determined by the ECO operator ϑ is regular enough it is then possible to describe it by means of a *succession rule* of the form:

$$\begin{cases} (b), \\ (h) \rightsquigarrow (c_1)(c_2) \dots (c_h), \end{cases}$$

where $b, h, c_i \in \mathbb{N}$, meaning that the root object has b sons, and the h objects $O'_1, \dots, O'_h$, produced by an object O are such that $\left|\vartheta(O'_i)\right| = c_i$, $1 \leqslant i \leqslant h$. A succession rule describes a sequence $(f_n)_{n \geqslant 0}$ of positive integers, where f_n is the number of nodes at level n of the generating tree.

3. Enumeration of the class of L-convex polyominoes

Let $\mathcal{L}$ be the class of L-convex polyominoes, and $\mathcal{L}_n$ be the set of L-convex polyominoes having semi-perimeter equal to $n \geqslant 2$. Moreover, let $f_n = |\mathcal{L}_{n+2}|$. For any $n \geqslant 2$, we partition the set $\mathcal{L}_n$ into three mutually disjoint subsets, namely A, B, and C:
  (i) the class A is made of the polyominoes for which $\eta > 1$ (see Fig. 6);
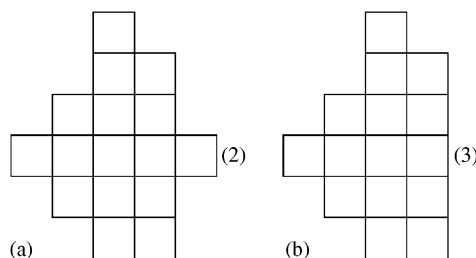
Fig. 7. (a) A L-convex polyomino of class B; (b) a L-convex polyomino of class C.

(ii) the class B is made of the polyominoes for which $\eta = 1$, and for which the last column is made of only one cell (see Fig. 7(a)). This class contains the one cell polyomino;

(iii) the class C is made of the polyominoes for which $\eta = 1$, and do not belong to the class B (see Fig. 7(b)).

We now define an ECO operator ϑ on the class $\mathcal{L}$,

$$\vartheta : \mathcal{L}_n \to 2^{\mathcal{L}_{n+1}},$$

working differently on polyominoes belonging to different classes. More precisely, let $P \in \mathcal{L}_n$:

- if P belongs to the class A or to the class C, then the operator ϑ produces:
  ○ 2η polyominoes by appending a cell onto each cell of the left and right sides of $R(P)$,
  ○ one polyomino by adding an entire row to $R(P)$.
  In this case $|\vartheta(P)| = 2\eta + 1$ and all the produced polyominoes have size $n + 1$ (see Fig. 8);

- if P belongs to the class B, then the operator ϑ produces:
  ○ one polyomino by appending a cell onto the right side of $R(P)$,
  ○ one polyomino by adding an entire row to $R(P)$.
  In this case $|\vartheta(P)| = 2$ and all the produced polyominoes have size $n + 1$ (see Fig. 9).

We would like to point out that, for any $P \in \mathcal{L}$, all the polyominoes in $\vartheta(P)$ preserve the L-convexity. Moreover, it is easy to verify that the operator ϑ satisfies conditions 1 and 2 of Proposition 2.2.

The succession rule. Translating the construction of the operator ϑ onto the framework of succession rules means to label with (k), $k \in \mathbb{N}^+$ each polyomino that produces exactly k objects through ϑ, and then represent the performance of the operator with a set of *productions*.

Actually, it is easy to recognize that each polyomino of class A or class C has label $(2\eta + 1)$ (in particular, the polyominoes in class C have label (3)), while the polyominoes of class B have label (2). The succession rule associated with the ECO operator ϑ is then:

$$\Omega : \begin{cases} (2), \\ (2) \rightsquigarrow (2)(5), \\ (2h + 1) \rightsquigarrow (2)^h (3)^h (2h + 3) \quad h > 0, \end{cases} \tag{2}$$
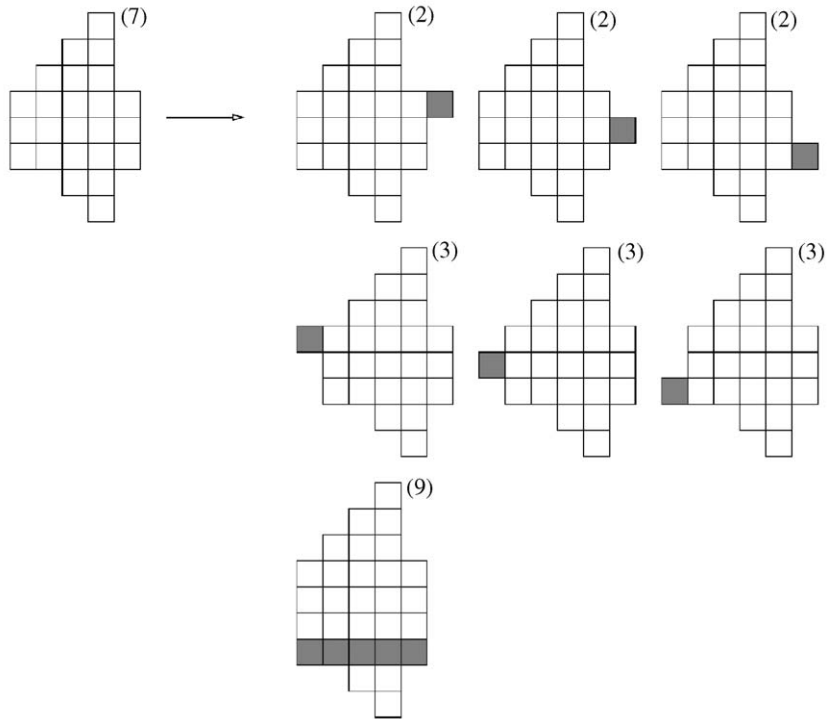
Fig. 8. The operator ϑ applied to a polyomino of class A (the performance of ϑ is the same for polyominoes of class C).

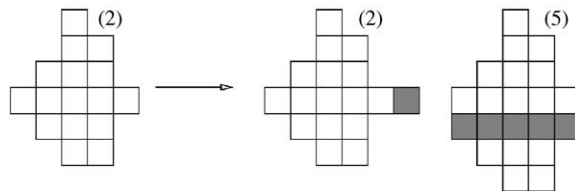Fig. 9. The operator ϑ applied to a polyomino of class B.

where the power notation stands for repetitions. Fig. 10 depicts the first three levels of the generating tree of Ω.

For instance, the production $(7) \rightsquigarrow (2)(2)(2)(3)(3)(3)(9)$ means that a L-convex polyomino with label (7) (that is, belonging to class B) produces seven different polyominoes having immediately greater size, and labels (2), (2), (2), (3), (3), (3), and (9), respectively (see also Fig. 8). The root of the tree is (2), which is the label of the one cell polyomino.
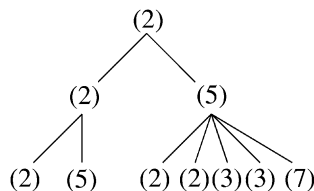
Fig. 10. The first levels of the generating tree of Ω.

3.1. The calculus of the generating function of Ω

In this section, we will determine the generating function of L-convex polyominoes according to various parameters, using the simple remark that the number f_n of L-convex polyominoes having semi-perimeter equal to $n + 2$ is given by the number of objects at level n of the generating tree of Ω.

Throughout this section, we will use the following notation:

- L is the set of labels of the generating tree of Ω;
- D is the set of labels (2) in the generating tree of Ω;
- E is the set of labels $(2\eta + 1)$, $\eta \geqslant 1$ in the generating tree of Ω.

Moreover, for any polyomino P, let $l(P)$ (briefly, l) be the label of the polyomino. We will deal with the generating function:

$$L(s, x, y) = \sum_{P \in \mathrm{L}} s^{l(P)} x^{w(P)} y^{h(P)} = s^2 xy + s^2 x^2 y + s^5 xy^2 + \cdots.$$

We will also consider:

- $D(s, x, y) = \sum_{P \in \mathrm{D}} s^2 x^w y^h$, i.e. the generating function of all polyominoes having label (2), and
- $E(s, x, y) = \sum_{P \in \mathrm{E}} s^l x^w y^h$, i.e. the generating function of all polyominoes having labels with odd numbers.

Clearly, that $L(s, x, y) = D(s, x, y) + E(s, x, y)$. The productions of Ω determine the following relations:

$$D(s, x, y) = s^2 xy + x D(s, x, y) + \frac{s^2 x}{2} \left[\frac{\partial}{\partial s} E(s, x, y)\big|_{s=1} - E(1, x, y) \right], \qquad (3)$$

$$E(s, x, y) = s^3 y D(s, x, y) + \frac{s^3 x}{2} \left[\frac{\partial}{\partial s} E(s, x, y)\big|_{s=1} - E(1, x, y) \right]$$
$$+ s^2 y E(s, x, y). \qquad (4)$$

The system is constituted by two equations and six unknown functions, but it is possible to obtain four more equations deriving (3) and (4) with respect to s, and letting $s = 1$ in (3) and (4). The system of six equations leads to the desired solution, and in particular:

$$D(s, x, y) = \frac{s^2 xy \left(1 - x - 2y + y^2\right)}{1 - 2x - 2y + x^2 + y^2}, \qquad (5)$$

$$E(s, x, y) = \frac{s^3 x y^2 \left(2 s^2 y + s^2 x - s^2 - s^2 y^2 - 2 x + x y\right)}{\left(s^2 y - 1\right) \left(1 - 2x - 2y + x^2 + y^2\right)}. \tag{6}$$

Summing (5) and (6) we obtain the generating function for the class of L-convex polyominoes. Letting $s = 1$ we determine the generating function according to the number of rows and columns:

$$L(x, y) = \frac{(1 - x)(1 - y) \, xy}{1 - 2\,x - 2\,y + x^2 + y^2}. \tag{7}$$

Finally, letting $x = y$ in (7) one obtains the generating function according to the semi-perimeter:

$$L(x) = \frac{x^2 \left(1 - 2\,x + x^2\right)}{1 - 4\,x + 2\,x^2}. \tag{8}$$

As a consequence, we have that the number f_n of L-convex polyominoes having semi-perimeter $n + 2$ satisfies the recurrence relation:

$$\begin{aligned} &f_0 = 1, \quad f_1 = 2, \quad f_2 = 7, \\ &f_n = 4 f_{n-1} - 2 f_{n-2}, \quad n \geqslant 3 \end{aligned} \tag{9}$$

(sequence A003480 in [19]). The closed expression for f_n is then straightforward:

$$f_n = \frac{\sqrt{2} + 1}{4} \left(2 + \sqrt{2}\right)^n + \frac{1 - \sqrt{2}}{4} \left(2 - \sqrt{2}\right)^n.$$

3.2. A bijective proof of (9)

In this section, we will determine, in a purely bijective fashion, recurrence relations for the classes A, B, and C of L-convex polyominoes. As a consequence, we will obtain a combinatorial proof of (9).

Let $a_{n,i}$ be the number of L-convex polyominoes of the class A having semi-perimeter equal to $n + 2$ and height of $R(P)$ equal to $i > 1$, b_n (resp., c_n) the number of L-convex polyominoes of the class B (resp., C) having semi-perimeter equal to $n + 2$, and let

$$A_i(x) = \sum_{n \geqslant 0} a_{n,i} x^n, \ i \geqslant 2, \quad B(x) = \sum_{n \geqslant 0} b_n x^n, \quad C(x) = \sum_{n \geqslant 0} c_n x^n.$$

The first few terms of these sequences are presented in the table in Fig. 11. Naturally, as one can check in the table, for any $n \geqslant 0$, $f_n = b_n + c_n + \sum_{i \geqslant 2} a_{n,i}$.

f_n	n	b_n	c_n	$a_{n,2}$	$a_{n,3}$	$a_{n,4}$	$a_{n,5}$	...
1	0	1						
2	1	1	0	1				
7	2	3	2	1	1			
24	3	10	7	5	1	1		
82	4	34	24	17	5	1	1	
280	5	116	82	58	17	5	1	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Fig. 11. The numbers b_n, c_n, and $a_{n,i}$, for $n = 0, \ldots, 5$.
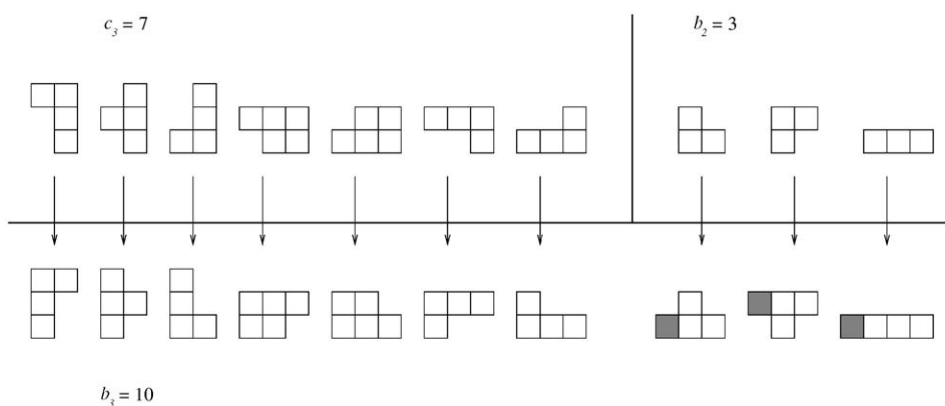
Fig. 12. A graphical proof that $b_3 = b_2 + c_3$.

Looking at the table one can verify the following statements:

(i) for any $n \geqslant 1$, $c_n = b_n - b_{n-1}$;

(ii) for any $n \geqslant 0$, $i \geqslant 2$, $a_{n,i} = b_{n-i+1} + c_{n-i+1}$; in particular, $a_{n,2} = b_{n-1} + c_{n-1}$;

(iii) for any $i \geqslant 3$, $n \geqslant 0$, $a_{n+1,i+1} = a_{n,i}$.

For brevity, we will only prove (i), leaving the other simple proofs to the reader. So let $n \geqslant 1$ be fixed; each polyomino of class B having size $n + 2$ can uniquely be obtained in one of the two ways:

• By reflecting a polyomino of class C, with the same size, with respect to the y-axis.

• By adding a cell to the left of $R(P)$, P being a polyomino of class B having size equal to $n + 1$.

Therefore, the number b_n of polyominoes in B with size $n + 2$ is equal to $c_n + b_{n-1}$ (see Fig. 12 for an example, with $n = 3$).

By manipulating (i), (ii), and (iii), and then using induction we have that, for all $n > 1$

$$\sum_{i \geqslant 2} a_{n,i} = c_n. \tag{10}$$

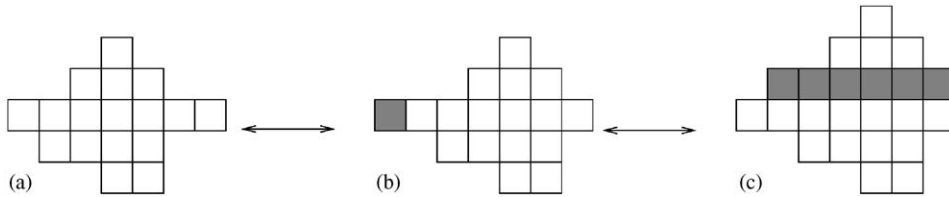Fig. 13. (a) A polyomino in $\mathcal{L}_7$ of class A; (b) the corresponding polyomino in C_8.

Fig. 14. (a) A polyomino P in $\mathcal{L}_{12}$ of class B; (b) the polyomino P' obtained in step 1; (c) the polyomino in C_{13} corresponding to P.

From (i) and (10) we obtain that, for $n > 1$:

$$f_n = b_n + 2c_n = 3b_n - 2b_{n-1}.$$ (11)

One more result deserves to be examined carefully.

Lemma 3.1. *For any $n \geqslant 1$, $c_{n+1} = f_n$. In particular, this yields to*

$$c_{n+1} = b_n + c_n + \sum_{i \geqslant 2} a_{n,i}$$

i.e. each c_n is obtained as the sum of all the terms in the previous row.

To prove this statement we will determine a bijection between the set of L-convex polyominoes having semi-perimeter $n + 2$, counted by f_n, and the set of polyominoes of class C having semi-perimeter $n + 3$, denoted by C_{n+3}, $n \geqslant 1$, counted by c_{n+1}. To explain the bijection, we must distinguish three cases:
  (i) P is a polyomino in $\mathcal{L}_{n+2}$ of class A. In this case, the polyomino of C_{n+3} corresponding to P is obtained simply adding one cell to the left of the first row of $R(P)$ (see Fig. 13).
 (ii) P is a polyomino in $\mathcal{L}_{n+2}$ of class B. In this case, the polyomino of C_{n+3} corresponding to P is obtained by performing the following sequence of operations on P:
     1. We take the rightmost cell of $R(P)$, and transpose it on the left of the first cell of $R(P)$, thus obtaining a polyomino P' with the same size as P (see Fig. 14(b)).
     2. We add one cell above each cell of $R(P')$ except the leftmost (see Fig. 14(c)).
(iii) P is a polyomino in C_{n+2}. In this case, the polyomino of C_{n+3} corresponding to P is obtained simply by adding one cell to the leftmost cell of $R(P)$ (see Fig. 15).
   An example of the bijection for $n = 2$ is given in Fig. 16. The inverse bijection is left to the reader. $\square$

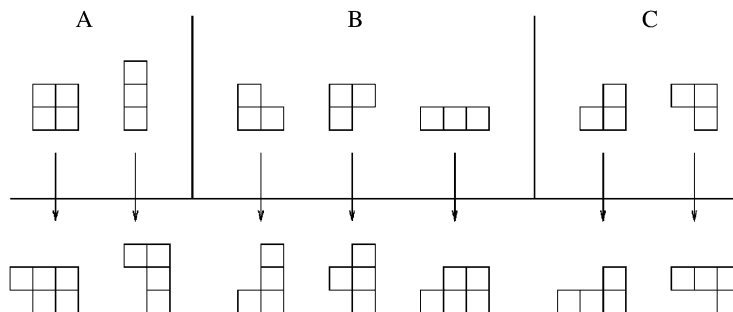Fig. 15. (a) A polyomino in C_7; (b) the corresponding polyomino in C_8.

Fig. 16. The bijection between the polyominoes in $\mathcal{L}_4$ and those in C_5.

Using (11), and Lemma 3.1, we have that for $n \geqslant 2$:

$$f_n = a_n + 2 f_{n-1}.$$

Finally, substituting $a_n = f_n - 2 f_{n-1}$ into (11) we get the desired recurrence relation, (9), for $n \geqslant 2$:

$$f_n = 4 f_{n-1} - 2 f_{n-2}.$$

Generating functions. Using the recurrence relations previously determined we obtain the generating functions of the classes B, and C:

$$B(x) = \frac{1 - 3x + x^2}{1 - 4x + 2x^2}, \quad C(x) = \frac{x(2 - x^2)}{1 - 4x + 2x^2}, \tag{12}$$

and A,

$$A(x) = \sum_{i \geqslant 2} x^{i-1} A_i(x) = \sum_{i \geqslant 2} \frac{x^{i-1}(1 - x)^3}{1 - 4x + 2x^2} = \frac{x(1 - x)^2}{1 - 4x + 2x^2}. \tag{13}$$

3.3. The number of L-convex polyominoes according to rows and columns

Starting from (7), or using simple bijective arguments, we can now refine the result in (9), considering the numbers $\left(f_{r,c} \right)_{r,c \geqslant 1}$, of L-convex polyominoes having r rows and c columns. Table in Fig. 17 depicts the first few terms.

$r \setminus c$	1	2	3	4	5	...
1	1	1	1	1	1	...
2	1	5	11	19	29	...
3	1	11	42	110	235	...
4	1	19	110	402	1135	...
5	1	29	235	1135	4070	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Fig. 17. The first terms of the sequences $(f_{r,c})_{r,c \geqslant 1}$, for $r, c = 1, \ldots, 5$.
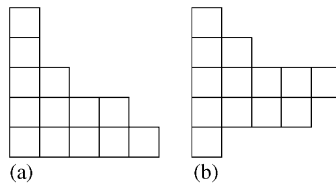
Fig. 18. (a) A Ferrer diagram representing the partition (5, 3, 2, 2, 1); (b) a stack polyomino.

The reader can easily check that the numbers $(f_{r,c})_{r,c \geqslant 1}$ satisfy the simple recurrence relation:

$$f_{r,c} = 2(f_{r-1,c} + f_{r,c-1}) - f_{r-2,c} - f_{r,c-2}, \quad r, c \geqslant 3. \tag{14}$$

4. Further work

In this paper, we have considered the enumeration problem for the class of L-convex polyominoes the semi-perimeter, and the number of rows and columns. In this final section, we underline some interesting connections between L-convex polyominoes and the well-known classes of *integer partitions*, and of *stack polyominoes*.

Integer partitions, stack polyominoes, and L-convex polyominoes. Every integer partition $\lambda = (p_1, \ldots, p_l)$, with $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_l$, has a simple graphical representation in terms of *Ferrer diagrams* made of l columns such that the ith column is made of p_i cells (see Fig. 18(a)).

The number p_n of Ferrer diagrams having semi-perimeter equal to $n + 2$ is easily proved to satisfy the recurrence relation:

$$p_n = 2p_{n-1}, \quad n \geqslant 1, \tag{15}$$

with $p_0 = 1$.

A *stack polyomino* is a convex polyomino such that, for each column:

- The ordinate of the top-cell is higher than or equal to the ordinate of the top-cells of the columns on its right.
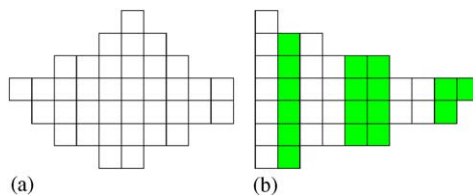
Fig. 19. (a) A L-convex polyomino; (b) the corresponding two-colored stack polyomino.

• The ordinate of the bottom-cell is lower than or equal to the ordinate of the bottom-cells of the columns on its right (see Fig. 18(b)).

The number s_n of stack polyominoes having semi-perimeter equal to $n + 2$ satisfies the recurrence relation:

$$s_n = 3s_{n-1} - s_{n-2}, \quad n \geqslant 2, \tag{16}$$

with $s_0 = 1$, $s_1 = 2$ [18]. The sequence $(s_n)_{n \geqslant 0}$ is well-known as the *bisection of the Fibonacci sequence* (sequence A001519 in [19]).

There is a strict connection between the classes of stack and L-convex polyominoes. In fact, stack polyominoes are characterized, among L-convex polyominoes, by the property that any two cells are connected by a path down and across, or up and across. That is, by just two orientations of the L.

On the other side, a L-convex polyomino can be represented as a stack polyomino whose columns can have two colors, say black and white, and such that:
1. The columns having maximal length are white.
2. For any fixed length, the white columns precede the black columns having the same length.

Without going further into details, we present in Fig. 19 a pictorial explanation of this simple correspondence.

In this paper, we have proved that the number f_n of convex polyominoes having size $n + 2$ satisfies recurrence (9). The reader will easily realize that recurrence relations (9), (15), and (16) have the form:

$$x_n = (u + 2)x_{n-1} - ux_{n-2}, \quad u = 0, 1, 2. \tag{17}$$

The authors would like to point out the problem of determining some class of polyominoes extending the class of L-convex, and such that the number q_n of elements having size $n + 2$ satisfies the recurrence $q_n = (u + 2)q_{n-1} - uq_{n-2}$, with $u \geqslant 3$.

A renewal property of the sequence $(f_n)_{n \geqslant 1}$. In [6], Cameron considers the number sequences coming out of the *quasi-inversion* of the generating function

$$h_k(x) = \sum_{n \geqslant k} nx^{n-k+1},$$

of the sequence $k, k+1, k+2, \ldots,$ of positive integers starting from a given $k \geqslant 1$. The quasi-inverse $f_k(x)$ of $h_k(x)$ is defined as

$$f_k(x) = \frac{1}{1 - h_k(x)} = 1 + h_k(x) + h_k(x)^2 + \cdots . \tag{18}$$

For any fixed $k \geqslant 1$ we have the explicit formula

$$f_k(x) = \frac{(1-x)^2}{1 - (k+2)x + x^2}, \tag{19}$$

We are especially interested in the case $k = 2$, where we get

$$h_2(x) = \frac{x(2-x)}{(1-x)^2},$$

and successively we obtain the generating function of L-convex polyominoes (divided by x^2):

$$f_2(x) = \frac{(1-x)^2}{1 - 4x + 2x^2} = 1 + 2x + 7x^2 + 24x^3 + 82x^4 + \cdots ,$$

and then we have

$$f_2(x) = 1 + h_2(x) f_2(x). \tag{20}$$

Eq. (20) suggests that the coefficients f_n satisfy the *renewal recurrence relation*:

$$f_n = n f_1 + (n-1) f_2 + \cdots + 2 f_{n-1}, \quad n \geqslant 1, \tag{21}$$

with $f_1 = 1$. We would like to point out the problem of giving a combinatorial proof that L-convex polyominoes satisfy relation (21).

Acknowledgements

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