EXISTENCE AND ASYMPTOTIC PROPERTIES FOR QUASILINEAR ELLIPTIC EQUATIONS WITH GRADIENT DEPENDENCE

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Abstract. The paper focuses on a Dirichlet problem driven by the (p,q)-Laplacian containing a parameter $\mu > 0$ in the principal part of the elliptic equation and a (convection) term fully depending on the solution and its gradient. Existence of solutions, uniqueness, a priori estimates, and asymptotic properties as $\mu \to 0$ and $\mu \to \infty$ are established under suitable conditions.

1. Introduction

In this paper we focus on the following nonlinear Dirichlet problem driven the (p,q)-Laplacian operator

$$(P_\mu) \qquad \begin{cases} -\Delta_p u - \mu\Delta_q u = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^N$ is a nonempty bounded open set with the boundary $\partial\Omega$, and μ is a positive real parameter. In the statement of problem (P_μ), with given numbers $1 < q < p$, Δ_p and Δ_q stand for the p-Laplacian and q-Laplacian, respectively, that is $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $\Delta_q u = \operatorname{div}(|\nabla u|^{q-2}\nabla u)$. The right-hand side of the equation in (P_μ) is expressed through $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$, which is a Carathéodory function, i.e $f(\cdot,s,\xi)$ is measurable for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$ and $f(x,\cdot,\cdot)$ is continuous for a.e. $x \in \Omega$.

We also examine the limiting case of problem (P_μ), namely if $\mu = 0$. In this case (P_μ) becomes the problem driven by the p-Laplacian operator

$$(P_0) \qquad \begin{cases} -\Delta_p u = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The main point in our study is the fact that the right-hand side of problems (P_μ) and (P_0) depends on the solution u and on its gradient ∇u. The expression $f(x,u,\nabla u)$ is often called convection term. Due to the presence of the gradient ∇u in the term $f(x,u,\nabla u)$, problems (P_μ) and (P_0) do not have generally variational structure, so the variational methods are not applicable. In view of this difficulty, problem (P_μ) in its general form is rarely studied in the literature. It is more investigated problem (P_0) (see [2], [3], [4], [5], [10], [11], and the references therein) and the variational case in problem (P_μ) where the right-hand side does not depend

2010 *Mathematics Subject Classification.* 35J92; 35J25.

Key words and phrases. Quasilinear elliptic equation; (p,q)-Laplacian; gradient dependence; asymptotic properties.

The authors were partially supported by INdAM - GNAMPA Project 2015.

on the gradient ∇u, i.e., $f(x,s,\xi) = f(x,s)$ (see [6], [7], [9], and the references therein).

Under only two hypotheses on the function $f(x,s,\xi)$, we show that we have existence of solutions for all problems (P_μ), with $\mu > 0$, and (P_0). Our approach relies on the theory of pseudomonotone operators for which we refer to the monographs [1], [7], [12]. Adding a further condition, a uniqueness result is also produced. Under the same hypotheses as for the existence part, we establish a priori estimates for the solutions of (P_μ). Based on them, we look at asymptotic properties of the solution sets of (P_μ) regarding μ as parameter. In this respect, a principal objective of the present paper is to show that in the limit as $\mu \to 0$ we obtain a solution of (P_0) that is approached in the space $W_0^{1,p}(\Omega)$ through a sequence of solutions of problems (P_μ), whereas letting $\mu \to +\infty$ along the solutions of problems (P_μ) we reach zero in the space $W_0^{1,q}(\Omega)$.

The rest of the paper is organized as follows. Section 2 deals with existence and uniqueness of solution to problem (P_μ). Section 3 is devoted to the asymptotic properties related to problem (P_μ) when $\mu \to 0$ and $\mu \to +\infty$.

2. Existence and uniqueness of solution to problem (P_μ)

In the sequel, for every $r \in [1,+\infty]$ we denote by r' its Hölder conjugate, i.e., r' satisfies $\frac{1}{r} + \frac{1}{r'} = 1$. In particular, this applies to the Sobolev critical exponent p^* with its conjugate $(p^*)'$. Recall that $p^* = \frac{pN}{N-p}$ if $N > p$ and $p^* = +\infty$ if $N \le p$. The strong convergence and the weak convergence are denoted by $\to$ and $\rightharpoonup$, respectively.

Consider the Sobolev space $W_0^{1,p}(\Omega)$ endowed with the norm $\|u\| := \|\nabla u\|_{L^p(\Omega)}$ for all $u \in W_0^{1,p}(\Omega)$. In studying problem (P_μ) we rely on the negative p-Laplacian $-\Delta_p : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^* = W^{-1,p'}(\Omega)$. It is well known that the operator $-\Delta_p$ is continuous, bounded, pseudomonotone and has the S_+-property (see [1], [7]). We denote by $\lambda_{1,p}$ the first eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$. It has the variational characterization

$$\lambda_{1,p} = \inf_{u \in W_0^{1,p}(\Omega),\, u \neq 0} \frac{\|\nabla u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\Omega)}}.$$

Throughout the paper we assume that the nonlinearity $f(x,s,\xi)$ satisfies the hypotheses:

(H1) There exist constants $a_1 \ge 0$, $a_2 \ge 0$, $\alpha \in [0, p^*-1[$, $\beta \in [0, \frac{p}{(p^*)'}[$ and a function $\sigma \in L^{\gamma'}(\Omega)$, with $\gamma \in [1, p^*[$, such that

$$|f(x,s,\xi)| \le \sigma(x) + a_1|s|^\alpha + a_2|\xi|^\beta \text{ a.e. } x \in \Omega,\ \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N;$$

(H2) there exist constants $d_1 \ge 0$, $d_2 \ge 0$ with $\lambda_{1,p}^{-1} d_1 + d_2 < 1$, and a function $\omega \in L^1(\Omega)$ such that

$$f(x,s,\xi)s \le \omega(x) + d_1|s|^p + d_2|\xi|^p \text{ a.e. } x \in \Omega,\ \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N.$$

A (weak) solution of problem (P_μ) for $\mu \ge 0$ is any $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx + \mu \int_\Omega |\nabla u|^{q-2} \nabla u \nabla v \, dx = \int_\Omega f(x,u,\nabla u) v \, dx \tag{2.1}$$

for all $v \in W_0^{1,p}(\Omega)$. According to hypothesis (H1) and Hölder's inequality, the integrals exist in the definition of weak solution as given in (2.1). Indeed, let us note that

$$f(x,u,\nabla u) \in L^{r'}(\Omega),\ \forall u \in W_0^{1,p}(\Omega), \tag{2.2}$$

with some $r \in [1, p^*[$, as can be easily checked by using the growth condition in (H1) and Sobolev embedding theorem.

Theorem 1. *Assume that conditions (H1) and (H2) hold. Then problem* (P_μ)*, with* $\mu \geq 0$*, admits at least one weak solution* $u_\mu \in W_0^{1,p}(\Omega)$*.*

Proof. We are going to prove the existence of weak solutions to problem (P_μ) by means of the theory for pseudomonotone operators. Specifically, corresponding to (P_μ) we introduce the nonlinear operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by

$$A(u) = -\Delta_p u - \mu\Delta_q u - N(u), \tag{2.3}$$

where $N : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ denotes the Nemytskii operator associated to f, that is $N(u) = f(x,u,\nabla u)$. It is known from (2.2), that $N(u) \in W^{-1,p'}(\Omega)$ for all $u \in W_0^{1,p}(\Omega)$.

It is clear from the growth condition in (H1) that $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is bounded, which means that it maps bounded sets onto bounded sets.

We claim that the operator A in (2.3) is pseudomonotone. To this end let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be such that $u_n \rightharpoonup u$ and $\limsup_{n\to+\infty}\langle Au_n, u_n - u\rangle \leq 0$. We provide the proof in the case where $N > p$. The case $N \leq p$ is easier and thus we omit it. It is seen from hypothesis (H1) that $\gamma, \frac{p^*}{p^*-\alpha}, \frac{p}{p-\beta} < p^*$. Then Rellich's compact embedding theorem implies that $u_n \to u$ in $L^\gamma(\Omega)$, $L^{\frac{p^*}{p^*-\alpha}}(\Omega)$, and $L^{\frac{p}{p-\beta}}(\Omega)$. This, in conjunction with hypothesis (H1) and applying Hölder inequality, leads to

$$\lim_{n\to\infty}\int_\Omega f(x,u_n,\nabla u_n)(u_n - u)\,dx = 0. \tag{2.4}$$

Taking into account (2.3) and (2.4), we infer that

$$\limsup_{n\to+\infty}\langle -\Delta_p u_n - \mu\Delta_q u_n, u_n - u\rangle = \limsup_{n\to+\infty}\langle Au_n, u_n - u\rangle \leq 0.$$

At this point, the S_+-property of the operator $-\Delta_p - \mu\Delta_q$ on the space $W_0^{1,p}(\Omega)$ can be used (see, e.g., [7, Proposition 2.70]) to derive the strong convergence $u_n \to u$ in $W_0^{1,p}(\Omega)$. Now it is straightforward to get that $A(u_n) \to A(u)$ in $W^{-1,p'}(\Omega)$, which ensures in particular that the operator A is pseudomonotone.

Let us prove that A is coercive, which means to have

$$\lim_{\|u\|\to\infty}\frac{\langle Au, u\rangle}{\|u\|} = +\infty.$$

On the basis of hypothesis (H2), it turns out that

$$\langle Au, u\rangle = \|\nabla u\|^p_{L^p(\Omega)} + \mu\|\nabla u\|^q_{L^q(\Omega)} - \int_\Omega f(x,u,\nabla u)u\,dx \geq (1 - d_1\lambda_{1,p}^{-1} - d_2)\|\nabla u\|_p^p - \|\omega\|_{L^1(\Omega)}.$$

It follows that A is coercive because $p > 1$ and $\lambda_{1,p}^{-1}d_1 + d_2 < 1$.

Since $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is pseudomonotone, bounded and coercive, we can apply the main theorem on pseudomonotone operators (see [1, Theorem 2.99], [7, Theorem 2.63]). Therefore there is at least one element $u_\mu \in W_0^{1,p}(\Omega)$ such that $Au_\mu = 0$, so u_μ is a weak solution of problem (P_μ), which completes the proof. □

The final part of the section deals with the uniqueness of solution to problem (P_μ), which can hold only under strong hypotheses (see [8] for the case where f in (P_μ) does not depend on the gradient ∇u). We illustrate this topic by presenting a uniqueness result in the case where $p = 2$ or $q = 2$. Our assumption is as follows:

(U)(a) there exists a constant $b_1 \geq 0$ such that

$$(f(x,s,\xi) - f(x,t,\xi))(s-t) \leq b_1|s-t|^2 \text{ a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^N,\ \forall s,t \in \mathbb{R};$$

(U)(b) there exist a function $\tau \in L^\delta(\Omega)$, with some $\delta \in [1, p^*[$, and a constant $b_2 \geq 0$ such that the function $f(x,s,\cdot) - \tau(x)$ is linear and

$$|f(x,s,\xi) - \tau(x)| \leq b_2|\xi| \text{ a.e. } x \in \Omega,\ \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N.$$

Theorem 2. *Assume that conditions (H1), (H2), (U)(a) and (U)(b) hold.*

(i) *If $p = 2 > q > 1$ and $b_1\lambda_{1,2}^{-1} + b_2\lambda_{1,2}^{-\frac{1}{2}} < 1$, then the solution of problem (P_μ) is unique for every $\mu > 0$.*

(ii) *If $p > q = 2$, then the solution of problem (P_μ) is unique for every $\mu > b_1\lambda_{1,2}^{-1} + b_2\lambda_{1,2}^{-\frac{1}{2}}$.*

Proof. Since conditions (H1) and (H2) are supposed to be fulfilled, we may apply Theorem 1, which asserts that there exists a solution $u_\mu \in W_0^{1,p}(\Omega)$ of problem (P_μ) for every $\mu > 0$. Suppose that $v_\mu \in W_0^{1,p}(\Omega)$ is a second solution of (P_μ). Acting with $u_\mu - v_\mu$ on the equation in (P_μ) gives

$$\begin{aligned} &\langle -\Delta_p u_\mu + \Delta_p v_\mu, u_\mu - v_\mu\rangle + \mu\langle -\Delta_q u_\mu + \Delta_q v_\mu, u_\mu - v_\mu\rangle \\ &= \int_\Omega (f(x,u_\mu,\nabla u_\mu) - f(x,v_\mu,\nabla u_\mu))(u_\mu - v_\mu)\,dx \\ &+ \int_\Omega (f(x,v_\mu,\nabla u_\mu) - f(x,v_\mu,\nabla v_\mu))(u_\mu - v_\mu)\,dx. \end{aligned} \tag{2.5}$$

(i) For $p = 2$, hypotheses (U)(a) and (U)(b), in conjunction with (2.5), the monotonicity of $-\Delta_q$ and Hölder inequality imply

$$\begin{aligned} &\|\nabla(u_\mu - v_\mu)\|^2_{L^2(\Omega)} \leq b_1\|u_\mu - v_\mu\|^2_{L^2(\Omega)} + \int_\Omega (f(x,v_\mu,\nabla(\tfrac{1}{2}(u_\mu - v_\mu)^2)) - \tau(x))\,dx \\ &\leq (b_1\lambda_{1,2}^{-1} + b_2\lambda_{1,2}^{-\frac{1}{2}})\,\|\nabla(u_\mu - v_\mu)\|^2_{L^2(\Omega)}. \end{aligned}$$

Using that $b_1\lambda_{1,2}^{-1} + b_2\lambda_{1,2}^{-\frac{1}{2}} < 1$, the equality $u_\mu = v_\mu$ follows.

(ii) For $p > q = 2$, arguing as in the case of part (i), we find the estimate

$$\mu\|\nabla(u_\mu - v_\mu)\|^2_{L^2(\Omega)} \leq (b_1\lambda_{1,2}^{-1} + b_2\lambda_{1,2}^{-\frac{1}{2}})\,\|\nabla(u_\mu - v_\mu)\|^2_{L^2(\Omega)}.$$

The conclusion that $u_\mu = v_\mu$ ensues provided that $b_1\lambda_{1,2}^{-1} + b_2\lambda_{1,2}^{-\frac{1}{2}} < \mu$, which completes the proof. □

3. Asymptotic properties as $\mu \to 0$ and $\mu \to +\infty$

It is shown in Theorem 1 that problem (P_μ) possesses a solution $u_\mu \in W_0^{1,p}(\Omega)$ for every $\mu > 0$. We establish the following a priori estimate.

Lemma 1. *Assume that conditions (H1) and (H2) hold. Then there exists a constant $b > 0$ independent of $\mu > 0$ such that*

$$\|\nabla u_\mu\|_{L^p(\Omega)} \leq b,\ \forall \mu > 0. \tag{3.1}$$

Proof. Fix $\mu > 0$. Since $u_\mu \in W_0^{1,p}(\Omega)$ is a solution of (P_μ), we can insert $v = u = u_\mu$ in (2.1). Thanks to assumption (H2), for every $\mu > 0$ we get the estimate

$$\|\nabla u_\mu\|_{L^p(\Omega)}^p \leq \int_\Omega f(x, u_\mu, \nabla u_\mu) u_\mu dx \leq (d_1 \lambda_{1,p}^{-1} + d_2) \|\nabla u_\mu\|_p^p + \|\omega\|_{L^1(\Omega)}.$$

We have by hypothesis (H2) that $\lambda_{1,p}^{-1} d_1 + d_2 < 1$. Consequently, (3.1) is obtained by choosing $b = \left(\frac{\|\omega\|_{L^1(\Omega)}}{1 - d_1 \lambda_{1p}^{-1} - d_2} \right)^{\frac{1}{p}}$. ☐

Next, taking advantage that μ is a parameter, we consider the limit points of the net (u_μ) as $\mu \to 0$ and $\mu \to +\infty$. We start by letting $\mu \to 0$ in problem (P_μ).

Theorem 3. *For any sequence $\mu_n \to 0^+$, there exists a relabeled subsequence of solutions (u_{μ_n}) of the corresponding problems (P_{μ_n}) such that $u_{\mu_n} \to u$ in $W_0^{1,p}(\Omega)$, with $u \in W_0^{1,p}(\Omega)$ weak solution of problem (P_0).*

Proof. Set, for simplicity, $u_n := u_{\mu_n}$. Since u_n is a weak solution of problem (P_{μ_n}), we can apply Lemma 1 and deduce that the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$. Then along a relabeled subsequence one has that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ for some $u \in W_0^{1,p}(\Omega)$.

Following the same reasoning based on hypothesis (H1) as in the proof of Theorem 1, we can show the validity of relation (2.4). Through the equation in (P_{μ_n}), the fact that $\mu_n \to 0^+$ and using (2.4) we are led to

$$\lim_{n \to +\infty} \langle -\Delta_p u_n, u_n - u \rangle = 0.$$

Recalling that the operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ satisfies the S_+-property, we conclude that $u_n \to u$ in $W_0^{1,p}(\Omega)$. As arrived at the strong convergence $u_n \to u$ in $W_0^{1,p}(\Omega)$, we can pass to the limit in the equation in problem (P_{μ_n}) as $n \to \infty$. Specifically, $u_n \to u$ in $W_0^{1,p}(\Omega)$ implies that $\nabla u_n \to \nabla u$ in $L^p(\Omega)^N$, so the growth condition in assumption (H1) and Krasnoselskii's theorem ensure

$$N(u_n) = f(\cdot, u_n(\cdot), \nabla u_n(\cdot)) \to N(u) = f(\cdot, u(\cdot), \nabla u(\cdot)) \tag{3.2}$$

in $L^{r'}(\Omega)$ as $n \to \infty$, for some $r \in [1, p^*[$. Bearing in mind that $-\Delta_p u_n \to -\Delta_p u$ in $W^{-1,p'}(\Omega)$, $\mu_n \to 0^+$, and (3.2), letting $n \to \infty$ in the equation of (P_{μ_n}) allows us to see that u is a weak solution of problem (P_0), which completes the proof. ☐

We turn to the asymptotic property as $\mu \to +\infty$.

Theorem 4. *For any sequence $\mu_n \to +\infty$, the sequence of solutions (u_{μ_n}) of the corresponding problems (P_{μ_n}) satisfies $u_{\mu_n} \to 0$ in $W_0^{1,q}(\Omega)$.*

Proof. Proceeding as in the proof of Theorem 3, we set $u_n := u_{\mu_n}$ and apply Lemma 1 to derive that the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$, so up to a relabeled subsequence we have $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ for some $u \in W_0^{1,p}(\Omega)$.

We note that u_n satisfies

$$\begin{cases} -\frac{1}{\mu_n} \Delta_p u_n - \Delta_q u_n = \frac{1}{\mu_n} f(x, u_n, \nabla u_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

If we act with $u_n - u$ in (3.3), we find that

$$\lim_{n \to +\infty} \langle -\Delta_q u_n, u_n - u \rangle = 0.$$

This follows from (3.3) because $\mu_n \to +\infty$, the sequence $(\Delta_p u_n)$ is bounded in $W^{-1,p'}(\Omega)$, and the sequence $(f(\cdot, u_n(\cdot), \nabla u_n(\cdot)))$ is bounded in $L^{r'}(\Omega)$, for some $r \in [1, p^*[$ (arguing as for (2.4) in the proof of Theorem 1). Then the S_+-property of the operator $-\Delta_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ guarantees that $u_n \to u$ in $W_0^{1,q}(\Omega)$. Letting $n \to \infty$ in (3.3) entails $\Delta_q u = 0$, so $u = 0$. Taking into account that the preceding argument applies for every convergent subsequence of (u_n), we conclude that for the whole sequence we have that $u_n \to 0$ in $W_0^{1,q}(\Omega)$. The proof is thus complete. □

Acknowledgement. The authors are grateful to L. Fresse and V.V. Motreanu for their useful comments.

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