

# BIWEIGHTS AND $\ast$ -HOMOMORPHISMS OF PARTIAL $\ast$ -ALGEBRAS

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Consider two partial  $\ast$ -algebras,  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$ , and an  $\ast$ -homomorphism  $\Phi$  from  $\mathfrak{A}_1$  into  $\mathfrak{A}_2$ . Given a biweight  $\varphi$  on  $\mathfrak{A}_2$ , we discuss conditions under which the natural composition  $\varphi \circ \Phi$  of  $\varphi$  and  $\Phi$  is a biweight on  $\mathfrak{A}_1$ . In particular, we examine whether the restriction of a biweight to a partial  $\ast$ -subalgebra is again a biweight.

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## 1. Introduction

In the representation theory of  $\ast$ -algebras, one of the cornerstones is the well-known Gel'fand-Naïmark-Segal (GNS) construction, which provides a way of building a concrete representation of an abstract  $\ast$ -algebra. This is often crucial for physical applications, for instance, in statistical mechanics or in quantum field theory.

It has been realized some time ago that  $\ast$ -algebras of bounded operators are not sufficient for such applications, one needs also  $\ast$ -algebras of unbounded operators and even *partial  $\ast$ -algebras* of operators. A full-fledged theory of partial  $\ast$ -algebras has been developed, covering both abstract partial  $\ast$ -algebras and their operator realizations, for which we refer to our recent monograph [4]. There the reader will find how the GNS construction can be performed in the new extended framework.

The crucial point is to replace positive linear forms by special sesquilinear forms (sometimes called invariant), in such a way that one can bypass many of the difficulties due to the lack of a noneverywhere defined multiplication and the nonassociativity of the partial multiplication. Moreover, for technical reasons, it has been found necessary to specialize further the sesquilinear forms in question, thus introducing the concept of *biweight* [3]. Biweights on partial  $\ast$ -algebras indeed allow a GNS construction and thus permit to develop from there a representation theory for partial  $\ast$ -algebras. As a matter of fact, they turn out to be the right objects for this purpose, since they are sufficiently flexible to avoid many difficulties inherent to the “partial” character of the multiplication.

Nevertheless, biweights exhibit unfamiliar features (in particular when compared with the sesquilinear forms defined by positive linear functionals on an  $\ast$ -algebra  $\mathfrak{A}$ ): the restriction of a biweight to a partial  $\ast$ -subalgebra is not necessarily a biweight; if  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$

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are partial  $*$ -algebras,  $\varphi$  is a biweight on  $\mathfrak{A}_2$ , and  $\Phi$  is an  $*$ -homomorphism from  $\mathfrak{A}_1$  to  $\mathfrak{A}_2$ , then the natural *composition*  $\varphi \circ \Phi$  of  $\varphi$  and  $\Phi$  may fail to be a biweight. This is the subject of the Section 3, where we give sufficient conditions for a biweight to pass safely through these two operations.

On the other hand, families of biweights having a common *core* can be used to define unbounded  $C^*$ -seminorms on a given partial  $*$ -algebra  $\mathfrak{A}$  [7, 9, 10]; and these seminorms provide a relevant tool for the study of the structure of a partial  $*$ -algebra, in particular when the latter carries a locally convex topology, and comparisons are possible. In Section 4, we make use of these seminorms to study the continuity properties of an  $*$ -homomorphism  $\Phi$  from a normed partial  $*$ -algebra into another one. A particular family of seminorms is used to define a locally convex topology called  *$C^*$ -like*, and sufficient conditions for  $\Phi$  to be continuous with respect to these topologies are given.

In Section 5, finally, we propose a notion of  $*$ -radical for a partial  $*$ -algebra, once more in terms of the unbounded  $C^*$ -seminorms defined by families of biweights. We do not undertake a systematic exploration of the consequences of this definition, we simply show that it behaves as expected in some particular situation.

Clearly, the theory is far from finished. Even in the relatively simple case of Banach partial  $*$ -algebras, many definitions are tentative and simply meant to open new research avenues. We hope to make further progress on these topics and to report on it in future papers.

### 2. Preliminaries

In order to keep the paper reasonably self-contained, we summarize in this section the basic facts on partial  $*$ -algebras and on their topological structure. Further details and proofs may be found in [1] or in the monograph [4].

A *partial  $*$ -algebra* is a complex vector space  $\mathfrak{A}$ , endowed with an involution  $x \mapsto x^*$  (i.e., a bijection such that  $x^{**} = x$ ) and a partial multiplication defined by a set  $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$  (a binary relation) such that

- (i)  $(x, y) \in \Gamma$  implies  $(y^*, x^*) \in \Gamma$ ;
- (ii)  $(x, y_1), (x, y_2) \in \Gamma$  implies  $(x, \lambda y_1 + \mu y_2) \in \Gamma$ , for all  $\lambda, \mu \in \mathbb{C}$ ;
- (iii) for any  $(x, y) \in \Gamma$ , there is defined a product  $x \cdot y \in \mathfrak{A}$ , which is distributive with respect to the addition and satisfies the relation  $(x \cdot y)^* = y^* \cdot x^*$ .

We will assume the partial  $*$ -algebra  $\mathfrak{A}$  contains a unit  $e$ , that is,  $e^* = e$ ,  $(e, x) \in \Gamma$ , for all  $x \in \mathfrak{A}$ , and  $e \cdot x = x \cdot e = x$ , for all  $x \in \mathfrak{A}$ . (If  $\mathfrak{A}$  has no unit, it may always be embedded into a larger partial  $*$ -algebra with unit, in the standard fashion [5].)

Given the defining set  $\Gamma$ , spaces of multipliers are defined in the obvious way:

$$\begin{aligned} (x, y) \in \Gamma &\iff x \in L(y) \quad \text{or} \quad x \text{ is a left multiplier of } y \\ &\iff y \in R(x) \quad \text{or} \quad y \text{ is a right multiplier of } x. \end{aligned} \tag{2.1}$$

For any subset  $\mathfrak{N} \subset \mathfrak{A}$ , we write

$$L\mathfrak{N} = \bigcap_{x \in \mathfrak{N}} L(x), \quad R\mathfrak{N} = \bigcap_{x \in \mathfrak{N}} R(x), \tag{2.2}$$

and, of course, the involution exchanges the two:

$$(L\mathfrak{N})^* = R\mathfrak{N}^*, \quad (R\mathfrak{N})^* = L\mathfrak{N}^*. \quad (2.3)$$

Clearly, all these multiplier spaces are vector subspaces of  $\mathfrak{A}$ , containing  $e$ .

The partial  $*$ -algebra is *abelian* if  $L(x) = R(x)$ , for all  $x \in \mathfrak{A}$ , and then  $x \cdot y = y \cdot x$ , for all  $x \in L(y)$ . In that case, we write simply for the multiplier spaces  $L(x) = R(x) \equiv M(x)$ ,  $L\mathfrak{N} = R\mathfrak{N} \equiv M\mathfrak{N}$  ( $\mathfrak{N} \subset \mathfrak{A}$ ).

Notice that the partial multiplication is *not* required to be associative (and often it is not). A partial  $*$ -algebra  $\mathfrak{A}$  is said to be *associative* if the following condition holds for any  $x, y, z \in \mathfrak{A}$ : whenever  $x \in L(y)$ ,  $y \in L(z)$ , and  $x \cdot y \in L(z)$ , then  $y \cdot z \in R(x)$  and

$$(x \cdot y) \cdot z = x \cdot (y \cdot z). \quad (2.4)$$

This condition is rather strong and rarely realized in practice. A weaker notion is sometimes useful, however. A partial  $*$ -algebra  $\mathfrak{A}$  is said to be *semi-associative* if  $y \in R(x)$  implies  $y \cdot z \in R(x)$  for every  $z \in R\mathfrak{A}$  and then (2.4) holds. Of course, if the partial  $*$ -algebra  $\mathfrak{A}$  is semi-associative, both  $R\mathfrak{A}$  and  $L\mathfrak{A}$  are algebras. From here on, we will write simply  $xy$  for the product  $x \cdot y$ .

We recall some basic definitions on  $*$ -representations of partial  $*$ -algebras. We refer to [2, 4] for details.

Let  $\mathcal{H}$  be a complex Hilbert space and  $\mathcal{D}$  a dense subspace of  $\mathcal{H}$ . We denote by  $\mathcal{L}^+(\mathcal{D}, \mathcal{H})$  the set of all (closable) linear operators  $X$  such that  $\mathcal{D}(X) = \mathcal{D}$ ,  $\mathcal{D}(X^*) \supseteq \mathcal{D}$ . The set  $\mathcal{L}^+(\mathcal{D}, \mathcal{H})$  is a partial  $*$ -algebra with respect to the following operations: the usual sum  $X_1 + X_2$ , the scalar multiplication  $\lambda X$ , the involution  $X \mapsto X^\dagger = X^* \upharpoonright \mathcal{D}$ , and the (weak) partial multiplication  $X_1 \square X_2 = X_1^\dagger X_2$ , defined whenever  $X_2$  is a weak right multiplier of  $X_1$ ,  $X_2 \in R^w(X_1)$  (or  $X_1 \in L^w(X_2)$ ), that is, if and only if  $X_2\mathcal{D} \subset \mathcal{D}(X_1^\dagger)$  and  $X_1^* \mathcal{D} \subset \mathcal{D}(X_2^*)$ .

A  $*$ -homomorphism of a partial  $*$ -algebra  $\mathfrak{A}$  into another one  $\mathfrak{B}$  is a linear map  $\rho : \mathfrak{A} \rightarrow \mathfrak{B}$  such that (i)  $\rho(x^*) = \rho(x)^*$  for every  $x \in \mathfrak{A}$ ; and (ii) whenever  $x \in L(y)$  in  $\mathfrak{A}$ , then  $\rho(x) \in L(\rho(y))$  in  $\mathfrak{B}$  and  $\rho(x)\rho(y) = \rho(xy)$ . The map  $\rho$  is an  $*$ -isomorphism if it is a bijection and  $\rho^{-1} : \mathfrak{B} \rightarrow \mathfrak{A}$  is also a  $*$ -homomorphism.

We remark that if the  $*$ -homomorphism  $\rho$  is not a  $*$ -isomorphism, the image  $\rho(\mathfrak{A})$  need not be a partial  $*$ -subalgebra of  $\mathfrak{B}$ . Indeed, there could be pairs  $x, y \in \mathfrak{A}$  such that  $x \notin L(y)$ , but  $\rho(x) \in L(\rho(y))$ ; then the product  $\rho(x)\rho(y)$  is well defined, but does not belong to  $\rho(\mathfrak{A})$ .

A  $*$ -representation of a partial  $*$ -algebra  $\mathfrak{A}$  in the Hilbert space  $\mathcal{H}$  is an  $*$ -homomorphism of  $\mathfrak{A}$  into  $\mathcal{L}^+(\mathcal{D}, \mathcal{H})$ , for some  $\mathcal{D} \subset \mathcal{H}$ , that is, a linear map  $\pi : \mathfrak{A} \rightarrow \mathcal{L}^+(\mathcal{D}, \mathcal{H})$  such that: (i)  $\pi(x^*) = \pi(x)^\dagger$  for every  $x \in \mathfrak{A}$ ; (ii)  $x \in L(y)$  in  $\mathfrak{A}$  implies  $\pi(x) \in L^w(\pi(y))$  and  $\pi(x)\pi(y) = \pi(xy)$ . The  $*$ -representation  $\pi$  is said to be *bounded* if  $\overline{\pi(x)} \in \mathcal{B}(\mathcal{H})$  for every  $x \in \mathfrak{A}$ .

A partial  $*$ -algebra  $\mathfrak{A}$  is said to be a *normed partial  $*$ -algebra* [6] if it carries a norm  $\|\cdot\|$  such that

- (i) the involution  $x \mapsto x^*$  is isometric:  $\|x\| = \|x^*\|$ , for all  $x \in \mathfrak{A}$ ;

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(ii) for every  $a \in L\mathfrak{A}$ , there exists a constant  $\gamma_a > 0$  such that

$$\|ax\| \leq \gamma_a \|x\|, \quad \forall x \in \mathfrak{A}. \quad (2.5)$$

$\mathfrak{A}[\|\cdot\|]$  is called a *Banach partial  $*$ -algebra* if, in addition,  $\mathfrak{A}[\|\cdot\|]$  is a Banach space.

### 3. Representable forms, biweights, and $*$ -homomorphisms

**3.1. Representable forms.** The possibility of performing a GNS construction starting from certain noneverywhere defined sesquilinear forms, called *biweights*, on a given partial  $*$ -algebra  $\mathfrak{A}$ , has been extensively studied in [3, 4]. Here we introduce the class of *representable forms*. They constitute, in a sense, the largest family of positive sesquilinear forms for which a GNS construction is possible. Of course, every biweight is representable.

Let  $\varphi$  be a positive sesquilinear form on  $\mathcal{D}(\varphi) \times \mathcal{D}(\varphi)$ , where  $\mathcal{D}(\varphi)$  is a subspace of  $\mathfrak{A}$ . Then, we have

$$\varphi(x, y) = \overline{\varphi(y, x)}, \quad \forall x, y \in \mathcal{D}(\varphi), \quad (3.1)$$

$$|\varphi(x, y)|^2 \leq \varphi(x, x)\varphi(y, y), \quad \forall x, y \in \mathcal{D}(\varphi). \quad (3.2)$$

We put

$$N_\varphi = \{x \in \mathcal{D}(\varphi) : \varphi(x, x) = 0\}. \quad (3.3)$$

By (3.2), we have

$$N_\varphi = \{x \in \mathcal{D}(\varphi) : \varphi(x, y) = 0, \forall y \in \mathcal{D}(\varphi)\}, \quad (3.4)$$

and so  $N_\varphi$  is a subspace of  $\mathcal{D}(\varphi)$  and the quotient space  $\mathcal{D}(\varphi)/N_\varphi \equiv \{\lambda_\varphi(x) \equiv x + N_\varphi; x \in \mathcal{D}(\varphi)\}$  is a pre-Hilbert space with respect to the inner product  $\langle \lambda_\varphi(x) | \lambda_\varphi(y) \rangle = \varphi(x, y)$ ,  $x, y \in \mathcal{D}(\varphi)$ . We denote by  $\mathcal{H}_\varphi$  the Hilbert space obtained by completion of  $\mathcal{D}(\varphi)/N_\varphi$ .

The construction of an  $*$ -representation starting from a positive sesquilinear form  $\varphi$  on  $\mathfrak{A}$  makes use of certain subspaces of  $R\mathfrak{A}$ , called *pre-cores* and *cores*. The notion of core for  $\varphi$  was introduced in [3].

*Definition 3.1.* Let  $\varphi$  be a positive sesquilinear form on  $\mathcal{D}(\varphi) \times \mathcal{D}(\varphi)$ . A subspace  $B(\varphi)$  of  $\mathcal{D}(\varphi)$  is said to be a *precore* for  $\varphi$  if

- (i)  $B(\varphi) \subset R\mathfrak{A}$ ;
- (ii)  $\{ax : a \in \mathfrak{A}, x \in B(\varphi)\} \subset \mathcal{D}(\varphi)$ ;
- (iii)  $\varphi(ax, y) = \varphi(x, a^*y)$ , for all  $a \in \mathfrak{A}$ , for all  $x, y \in B(\varphi)$ ;
- (iv)  $\varphi(a^*x, by) = \varphi(x, (ab)y)$ , for all  $a \in L(b)$ , for all  $x, y \in B(\varphi)$ .

The subspace  $B(\varphi)$  is called a *core* if, in addition,

- (v)  $\lambda_\varphi(B(\varphi))$  is dense in  $\mathcal{H}_\varphi$ .

We denote by  $\mathcal{D}_\varphi$  the set of all pre-cores for  $\varphi$  and with  $\mathcal{B}_\varphi$  the set of all cores  $B(\varphi)$  for  $\varphi$ .

*Definition 3.2.* A positive sesquilinear form  $\varphi$  on  $\mathcal{D}(\varphi) \times \mathcal{D}(\varphi)$  such that  $\mathcal{B}_\varphi \neq \emptyset$  is called a *biweight* on  $\mathfrak{A}$ .

*Remark 3.3.* If  $\varphi$  is a biweight, then condition (v) implies, in particular, that if  $x \in \mathcal{D}(\varphi)$ , there exists a sequence  $(z_n)$ ,  $z_n \in B(\varphi)$ , such that  $\varphi(x - z_n, x - z_n) \rightarrow 0$ . This, in turn, implies that  $|\varphi(y, x) - \varphi(y, z_n)| \rightarrow 0$ , for every  $z \in \mathcal{D}(\varphi)$ . Moreover, if  $\varphi(x, x) = 1$ , then we can choose  $(z_n)$  such that  $\varphi(z_n, z_n) = 1$  for every  $n \in \mathbb{N}$ .

*Definition 3.4.* A positive sesquilinear form  $\varphi$  on  $\mathcal{D}(\varphi) \times \mathcal{D}(\varphi)$  is called *representable* if there exists a precore  $B(\varphi)$  for  $\varphi$  and an  $*$ -representation  $\pi_\varphi$  defined on a dense subspace  $\mathcal{D}(\pi_\varphi)$  of  $\mathcal{H}_\varphi$  with  $\lambda_\varphi(B(\varphi)) \subseteq \mathcal{D}(\pi_\varphi)$  and such that

$$\varphi(ax, by) = \langle \pi_\varphi(a)\lambda_\varphi(x) \mid \pi_\varphi(b)\lambda_\varphi(y) \rangle, \quad \forall a, b \in \mathfrak{A}, x, y \in B(\varphi). \quad (3.5)$$

*Remark 3.5.* The following question is natural: if  $B(\varphi)$  is a precore and  $\varphi$  is representable, is  $B(\varphi)$  necessarily a core for  $\varphi$ ? The answer is, in general, negative. Indeed, let  $X$  be a bounded selfadjoint operator in Hilbert space  $\mathcal{H}$  such that  $X^2$  is not a complex multiple of  $X$ . Then, the subspace  $\mathcal{M}$  of  $\mathcal{B}(\mathcal{H})$  generated by  $X$  may be viewed as a partial  $*$ -algebra: two elements  $\alpha X, \beta X$ ,  $\alpha, \beta \in \mathbb{C}$ , can be multiplied if and only if either  $\alpha = 0$  or  $\beta = 0$ . In this case  $R\mathcal{M} = \{0\}$ . The restriction to  $\mathcal{M} \times \mathcal{M}$  of any sesquilinear form  $\varphi$  on  $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$  such that  $\varphi(X, X) > 0$  is obviously representable ( $\pi_\varphi$  can be taken as the identity map of  $\mathcal{B}(\mathcal{H}_\varphi)$ ). The null subspace is clearly a precore but it is not a core.

Now we give, for the sake of completeness, a sketch of the GNS construction for a biweight. The corresponding statement was proven in [3, 4].

Let  $\varphi$  be a biweight on  $\mathfrak{A}$  with a core  $B(\varphi)$ . We put

$$\pi_\varphi^\circ(a)\lambda_\varphi(x) = \lambda_\varphi(ax), \quad a \in \mathfrak{A}, x \in B(\varphi). \quad (3.6)$$

Then it follows from (3.2) and Definition 3.1(v) that  $\pi_\varphi^\circ(a)$  is a well-defined linear operator of  $\lambda_\varphi(B(\varphi))$  into  $\mathcal{H}_\varphi$ . Furthermore, it follows from conditions (iv) and (v) of Definition 3.1 that  $\pi_\varphi^\circ$  is an  $*$ -representation of  $\mathfrak{A}$ . We denote by  $\pi_\varphi^B$  the closure of  $\pi_\varphi^\circ$ . Then the triple  $(\pi_\varphi^B, \lambda_\varphi, \mathcal{H}_\varphi)$  is called the *GNS construction* for the biweight  $\varphi$  on  $\mathfrak{A}$  with the core  $B(\varphi)$ . It is worth remarking that if  $B_1(\varphi)$ ,  $B_2(\varphi)$  are two different cores for the biweight  $\varphi$ , it might happen, of course, that  $\pi_\varphi^{B_1} = \pi_\varphi^{B_2}$ . However, the set of all cores that yield the same GNS representation for  $\varphi$  has a maximal element, namely:

$$B_L(\varphi) = \{x \in \mathcal{D}(\varphi) \cap R\mathfrak{A} : \lambda_\varphi(x) \in \mathcal{D}(\pi_\varphi^B), ax \in \mathcal{D}(\varphi), \lambda_\varphi(ax) = \pi_\varphi^B(a)\lambda_\varphi(x), \forall a \in \mathfrak{A}\}. \quad (3.7)$$

From the GNS construction outlined above, the following holds.

**PROPOSITION 3.6.** *Every biweight  $\varphi$  on  $\mathfrak{A}$  is representable.*

Let  $\mathfrak{A}_1, \mathfrak{A}_2$  be partial  $*$ -algebras,  $\varphi$  a representable form on  $\mathfrak{A}_2$  with domain  $\mathcal{D}(\varphi)$  and precore  $B(\varphi)$  and let  $\Phi$  be an  $*$ -homomorphism of  $\mathfrak{A}_1$  into  $\mathfrak{A}_2$ . We put

$$\mathcal{D}(\varphi_\Phi) = \{a \in \mathfrak{A}_1 : \Phi(a) \in \mathcal{D}(\varphi)\} \quad (3.8)$$

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and define

$$\varphi_\Phi(a, b) = \varphi(\Phi(a), \Phi(b)), \quad a, b \in \mathcal{D}(\varphi_\Phi). \quad (3.9)$$

Then  $\varphi_\Phi$  is a positive sesquilinear form on  $\mathcal{D}(\varphi_\Phi) \times \mathcal{D}(\varphi_\Phi)$ . The set

$$B(\varphi_\Phi) = \{x \in R\mathfrak{A}_1 : \Phi(x) \in B(\varphi)\} \quad (3.10)$$

is a precore for  $\varphi_\Phi$ . Indeed, the conditions of Definition 3.1 are satisfied. Condition (i), that is,  $B(\varphi_\Phi) \subset R\mathfrak{A}_1$ , is obvious. We prove condition (ii). Let  $a \in \mathfrak{A}_1$  and  $x \in B(\varphi_\Phi)$ . We have to prove that  $ax \in \mathcal{D}(\varphi_\Phi)$ , that is,  $\Phi(ax) \in \mathcal{D}(\varphi)$ . This results from  $\Phi(ax) = \Phi(a)\Phi(x)$  and  $\Phi(x) \in B(\varphi)$ .

As for conditions (iii) and (iv) in Definition 3.1, we have, for every  $a \in \mathfrak{A}$  and  $x, y \in B(\varphi_\Phi)$ ,

$$\begin{aligned} \varphi_\Phi(ax, y) &= \varphi(\Phi(ax), \Phi(y)) = \varphi(\Phi(a)\Phi(x), \Phi(y)) \\ &= \varphi(\Phi(x), \Phi(a)^* \Phi(y)) = \varphi(\Phi(x), \Phi(a^* y)) \\ &= \varphi_\Phi(x, a^* y), \end{aligned} \quad (3.11)$$

and for every  $a \in L(b)$ ,  $x, y \in B(\varphi_\Phi)$ ,

$$\begin{aligned} \varphi_\Phi(a^* x, by) &= \varphi(\Phi(a^* x), \Phi(by)) = \varphi(\Phi(a^*)\Phi(x), \Phi(b)\Phi(y)) \\ &= \varphi(\Phi(x), (\Phi(a)\Phi(b))\Phi(y)) \\ &= \varphi_\Phi(x, (ab)y), \end{aligned} \quad (3.12)$$

where we have used the fact that if  $ab$  is well defined, so is  $\Phi(a)\Phi(b)$ .

**PROPOSITION 3.7.** *Let  $\mathfrak{A}_1, \mathfrak{A}_2$  be partial  $*$ -algebras,  $\varphi$  a representable form on  $\mathfrak{A}_2$  with domain  $\mathcal{D}(\varphi)$  and precore  $B(\varphi)$  and let  $\Phi$  be a surjective  $*$ -homomorphism of  $\mathfrak{A}_1$  into  $\mathfrak{A}_2$ . Then  $\varphi_\Phi$  is representable.*

*Proof.* Let  $\lambda_{\varphi_\Phi}(\mathcal{D}(\varphi_\Phi))$  and  $\mathcal{H}_{\varphi_\Phi}$  be as before. Then we define a map  $\hat{\Phi} : \lambda_{\varphi_\Phi}(\mathcal{D}(\varphi_\Phi)) \rightarrow \lambda_\varphi(\mathcal{D}(\varphi))$  by

$$\hat{\Phi}(\lambda_{\varphi_\Phi}(x)) := \lambda_\varphi(\Phi(x)). \quad (3.13)$$

This map is well defined. Indeed, if  $\lambda_{\varphi_\Phi}(x) = 0$ , then  $\varphi_\Phi(x, x) = 0$ , which implies successively  $\varphi(\Phi(x), \Phi(x)) = 0$  and  $\lambda_\varphi(\Phi(x)) = 0$ . On the other hand,  $\hat{\Phi}$  is injective. Indeed, if  $\lambda_\varphi(\Phi(x)) = 0$ , then  $\varphi(\Phi(x), \Phi(x)) = 0$ , which implies  $\varphi_\Phi(x, x) = 0$  and  $\lambda_{\varphi_\Phi}(x) = 0$ . Moreover,

$$\|\hat{\Phi}(\lambda_{\varphi_\Phi}(x))\|^2 = \|\lambda_\varphi(\Phi(x))\|^2 = \varphi(\Phi(x), \Phi(x)) = \varphi_\Phi(x, x) = \|\lambda_{\varphi_\Phi}(x)\|^2. \quad (3.14)$$

Thus  $\hat{\Phi}$  is isometric. Since  $\Phi$  is surjective,  $\hat{\Phi}$  is also surjective and so it extends to a unitary operator, denoted by the same symbol, from  $\mathcal{H}_{\varphi_\Phi}$  onto  $\mathcal{H}_\varphi$ . Since  $\varphi$  is representable, there

exist a dense domain  $\mathcal{D}(\pi_\varphi) \subset \mathcal{H}_\varphi$  and an  $*$ -representation  $\pi_\varphi : \mathfrak{A}_2 \rightarrow \mathcal{L}^+(\mathcal{D}(\pi_\varphi), \mathcal{H}_\varphi)$  such that

$$\varphi(ax, by) = \langle \pi_\varphi(a)\lambda_\varphi(x) \mid \pi_\varphi(b)\lambda_\varphi(y) \rangle, \quad \forall a, b \in \mathfrak{A}_2, x, y \in B(\varphi). \quad (3.15)$$

Now, put  $\mathcal{D}_{\varphi_\Phi} := \hat{\Phi}^{-1}\mathcal{D}(\pi_\varphi)$ . Then  $\mathcal{D}_{\varphi_\Phi}$  is a dense domain in  $\mathcal{H}_{\varphi_\Phi}$  and

$$\hat{\Phi}\lambda_{\varphi_\Phi}(B(\varphi_\Phi)) = \lambda_\varphi(\Phi(B(\varphi_\Phi))) \subset \lambda_\varphi(B(\varphi)) \subset \mathcal{D}(\pi_\varphi). \quad (3.16)$$

Hence,  $\lambda_{\varphi_\Phi}(B(\varphi_\Phi)) \subset \mathcal{D}_{\varphi_\Phi}$ .

Now, put  $\pi_{\varphi_\Phi}(a) = \hat{\Phi}^{-1}\pi_\varphi(\Phi(a))\hat{\Phi}$ . Then, we have, for every  $a, b \in \mathfrak{A}_1, x, y \in B(\varphi_\Phi)$ ,

$$\begin{aligned} \varphi_\Phi(ax, by) &= \varphi(\Phi(ax), \Phi(by)) \\ &= \langle \pi_\varphi(\Phi(a))\lambda_\varphi(\Phi(x)) \mid \pi_\varphi(\Phi(b))\lambda_\varphi(\Phi(y)) \rangle \\ &= \langle \pi_\varphi(\Phi(a))\hat{\Phi}(\lambda_{\varphi_\Phi}(x)) \mid \pi_\varphi(\Phi(b))\hat{\Phi}(\lambda_{\varphi_\Phi}(y)) \rangle \\ &= \langle \hat{\Phi}^{-1}\pi_\varphi(\Phi(a))\hat{\Phi}(\lambda_{\varphi_\Phi}(x)) \mid \hat{\Phi}^{-1}\pi_\varphi(\Phi(b))\hat{\Phi}(\lambda_{\varphi_\Phi}(y)) \rangle \\ &= \langle \pi_{\varphi_\Phi}(a)\lambda_{\varphi_\Phi}(x) \mid \pi_{\varphi_\Phi}(b)\lambda_{\varphi_\Phi}(y) \rangle. \end{aligned} \quad (3.17)$$

This proves the statement.  $\square$

Given a representable form  $\varphi$ , there is nothing in our previous discussion that prevents the possibility that the  $*$ -representation  $\pi_\varphi$  is trivial, since the precore  $B(\varphi)$  may reduce to  $\{0\}$ . Also, when dealing with an  $*$ -homomorphism  $\Phi$ , it may well happen that  $\pi_\varphi$  is nontrivial, while  $\pi_{\varphi_\Phi}$  is trivial. This unpleasant situation cannot occur when  $\varphi$  is a biweight, since in this case  $\lambda_\varphi(B(\varphi))$  is dense in  $\mathcal{H}_\varphi$ . If  $\Phi$  is an  $*$ -homomorphism and  $\varphi$  is a biweight on  $\mathfrak{A}_1$ , then  $\varphi_\Phi$  is certainly representable, but the corresponding representation may again be trivial. For these reasons, we look in the next subsection for conditions that guarantee that  $\varphi_\Phi$  is a biweight on  $\mathfrak{A}_1$ .

**3.2.  $*$ -Homomorphisms preserving biweights.** We begin with the simplest possible situation.

Let  $\mathfrak{A}$  be a partial  $*$ -algebra and  $\mathfrak{B}$  an  $*$ -subalgebra of  $\mathfrak{A}$ . This means that  $x, y \in \mathfrak{B}$  and  $(x, y) \in \Gamma$  imply  $xy \in \mathfrak{B}$ . The identity map  $i_\mathfrak{B} : \mathfrak{B} \rightarrow \mathfrak{A}$  is an  $*$ -homomorphism. If  $\varphi$  is a representable form on  $\mathfrak{A}$ , then we can construct, as before, the representable form  $\varphi_{i_\mathfrak{B}}$ , which we simply denote by  $\varphi_\mathfrak{B}$ . Clearly,  $\varphi_\mathfrak{B}$  can be viewed as the *restriction* of  $\varphi$  to  $\mathfrak{B}$ . Let  $\varphi$  be a biweight on  $\mathfrak{A}$  with domain  $\mathcal{D}(\varphi)$  and core  $B(\varphi)$ . In this case,

$$\mathcal{D}(\varphi_\mathfrak{B}) = \mathfrak{B} \cap \mathcal{D}(\varphi), \quad B(\varphi_\mathfrak{B}) = \mathfrak{B} \cap B(\varphi). \quad (3.18)$$

Then  $\varphi_\mathfrak{B}$  satisfies conditions (i), (ii), (iii), and (iv) of Definition 3.1. But, as the next example shows, (v) may fail and thus  $\varphi_\mathfrak{B}$  is not, in general, a biweight on  $\mathfrak{B}$ .

## 8 Biweights and $*$ -homomorphisms of partial $*$ -algebras

*Example 3.8.* We consider the partial  $*$ -algebra  $L^p[0, 1]$ , with  $2 \leq p \leq \infty$ , as discussed in [4]. Let us put

$$\begin{aligned}\mathcal{D}(\varphi) &= L^p[0, 1], \\ \varphi(x, y) &= \int_0^1 x(t)\overline{y}(t)dt, \quad x, y \in \mathcal{D}(\varphi).\end{aligned}\tag{3.19}$$

It is known [3] that  $\varphi$  is a biweight on  $L^p[0, 1]$  with largest core  $L^\infty[0, 1]$ . Let us consider the subspace  $V$  of  $L^p[0, 1]$  generated by a single real function  $f \in L^p[0, 1] \setminus L^r[0, 1]$ , for any  $r > p$ , that is  $V := \{\alpha f : \alpha \in \mathbb{C}\}$ .  $V$  can be viewed as a partial  $*$ -subalgebra of  $L^p[0, 1]$ , the partial product of  $\alpha f$  and  $\beta f$  being defined if and only if  $\alpha\beta = 0$ . We have

$$\mathcal{D}(\varphi_V) = V, \quad B(\varphi_V) = V \cap B(\varphi) = \{0\}.\tag{3.20}$$

Since  $L^\infty[0, 1]$  is the largest core of  $\varphi$ , it follows that  $B(\varphi_V)$  is the largest core of  $\varphi_V$ . Thus, if  $f \notin L^\infty[0, 1]$ ,  $\varphi_V$  cannot be a biweight.

Our first goal is to determine some conditions under which Definition 3.1(v) is satisfied. Since there is no need to suppose that  $\mathfrak{B}$  is a partial  $*$ -subalgebra of  $\mathfrak{A}$ , we will simply assume that  $\mathfrak{B}$  is a subspace of  $\mathfrak{A}$ . It is worth remarking that any  $*$ -invariant subspace  $\mathfrak{B}$  of  $\mathfrak{A}$  has always a natural structure of partial  $*$ -algebra: if  $x, y \in \mathfrak{B}$ , then  $(x, y) \in \Gamma_{\mathfrak{B}} \Leftrightarrow (x, y) \in \Gamma$  and  $xy \in \mathfrak{B}$ . Nevertheless,  $\mathfrak{B}$  is not necessarily a partial  $*$ -subalgebra of  $\mathfrak{A}$ .

Let  $\varphi$  be a biweight on  $\mathfrak{A}$  with domain  $\mathcal{D}(\varphi)$  and core  $B(\varphi)$  and  $\mathfrak{B}$  a subspace of  $\mathfrak{A}$ . We define  $\mathcal{D}(\varphi_{\mathfrak{B}})$  and  $B(\varphi_{\mathfrak{B}})$  as in (3.18). We also put

$$N_{\varphi_{\mathfrak{B}}} = \{x \in \mathcal{D}(\varphi_{\mathfrak{B}}) : \varphi_{\mathfrak{B}}(x, x) = 0\} = N_{\varphi} \cap \mathfrak{B}.\tag{3.21}$$

The set  $N_{\varphi_{\mathfrak{B}}}$  is a vector subspace of  $\mathcal{D}(\varphi_{\mathfrak{B}})$ , and  $\mathcal{D}(\varphi_{\mathfrak{B}})/N_{\varphi_{\mathfrak{B}}}$  is a pre-Hilbert space with respect to the inner product

$$\langle \lambda_{\varphi_{\mathfrak{B}}}(x) \mid \lambda_{\varphi_{\mathfrak{B}}}(y) \rangle = \varphi_{\mathfrak{B}}(x, y).\tag{3.22}$$

We denote  $(\mathcal{D}(\varphi_{\mathfrak{B}})/N_{\varphi_{\mathfrak{B}}}, \langle \cdot \mid \cdot \rangle)$  by  $\lambda_{\varphi_{\mathfrak{B}}}(\mathcal{D}(\varphi_{\mathfrak{B}}))$ .

Let us consider the following map:

$$I_{\mathfrak{B}} : \lambda_{\varphi_{\mathfrak{B}}}(x) \in \lambda_{\varphi_{\mathfrak{B}}}(\mathcal{D}(\varphi_{\mathfrak{B}})) \longmapsto \lambda_{\varphi}(x) \in \lambda_{\varphi}(\mathcal{D}(\varphi)).\tag{3.23}$$

This map is well defined and injective. Indeed,

$$\lambda_{\varphi_{\mathfrak{B}}}(x) = 0 \iff \varphi_{\mathfrak{B}}(x, x) = 0 \iff \varphi(x, x) = 0 \iff \lambda_{\varphi}(x) = 0.\tag{3.24}$$

Clearly,  $I_{\mathfrak{B}}$  is linear and, moreover, is isometric too. Indeed,

$$\|\lambda_{\varphi_{\mathfrak{B}}}(x)\|^2 = \varphi_{\mathfrak{B}}(x, x) = \varphi(x, x) = \|\lambda_{\varphi}(x)\|^2.\tag{3.25}$$

Thus it extends to an isometric operator  $\hat{I}_{\mathfrak{B}}$  from  $\mathcal{H}_{\varphi_{\mathfrak{B}}}$  into  $\mathcal{H}_{\varphi}$ , where  $\mathcal{H}_{\varphi_{\mathfrak{B}}}$  and  $\mathcal{H}_{\varphi}$  denote the Hilbert space completions of  $\lambda_{\varphi_{\mathfrak{B}}}(\mathcal{D}(\varphi_{\mathfrak{B}}))$  and  $\lambda_{\varphi}(\mathcal{D}(\varphi))$ , respectively. It follows



that  $\hat{I}_{\mathfrak{B}}\mathcal{H}_{\varphi_{\mathfrak{B}}}$  is a closed subspace of  $\mathcal{H}_{\varphi}$ . Let  $P$  be the projection from  $\mathcal{H}_{\varphi}$  onto  $\hat{I}_{\mathfrak{B}}\mathcal{H}_{\varphi_{\mathfrak{B}}}$ . Since  $\varphi$  is a biweight, it follows that  $P\lambda_{\varphi}(B(\varphi))$  is dense in  $P\mathcal{H}_{\varphi} = \hat{I}_{\mathfrak{B}}\mathcal{H}_{\varphi_{\mathfrak{B}}}$ .

We notice that we have, in general,

$$I_{\mathfrak{B}}\lambda_{\varphi_{\mathfrak{B}}}(B(\varphi_{\mathfrak{B}})) \subseteq P\lambda_{\varphi}(B(\varphi)). \quad (3.26)$$

Indeed, if  $x \in B(\varphi_{\mathfrak{B}})$ , then, by definition,  $I_{\mathfrak{B}}\lambda_{\varphi_{\mathfrak{B}}}(x) = \lambda_{\varphi}(x) \in \lambda_{\varphi}(B(\varphi))$ .

**PROPOSITION 3.9.** *The following statements are equivalent:*

- (i)  $\lambda_{\varphi_{\mathfrak{B}}}(B(\varphi_{\mathfrak{B}}))$  is dense in  $\mathcal{H}_{\varphi_{\mathfrak{B}}}$ ;
- (ii)  $\overline{I_{\mathfrak{B}}\lambda_{\varphi_{\mathfrak{B}}}(B(\varphi_{\mathfrak{B}}))} \supseteq P\lambda_{\varphi}(B(\varphi))$ .

*Proof.* (i)  $\Rightarrow$  (ii). If  $\lambda_{\varphi_{\mathfrak{B}}}(B(\varphi_{\mathfrak{B}}))$  is dense in  $\mathcal{H}_{\varphi_{\mathfrak{B}}}$ , it follows that  $I_{\mathfrak{B}}\lambda_{\varphi_{\mathfrak{B}}}(B(\varphi_{\mathfrak{B}}))$  is dense in  $\hat{I}_{\mathfrak{B}}\mathcal{H}_{\varphi} = P\mathcal{H}_{\varphi}$ . Hence  $\overline{I_{\mathfrak{B}}\lambda_{\varphi_{\mathfrak{B}}}(B(\varphi_{\mathfrak{B}}))} = \hat{I}_{\mathfrak{B}}\mathcal{H}_{\varphi_{\mathfrak{B}}} \supseteq P\lambda_{\varphi}(B(\varphi))$ .

(ii)  $\Rightarrow$  (i). If  $P\lambda_{\varphi}(B(\varphi)) \subseteq \overline{I_{\mathfrak{B}}\lambda_{\varphi_{\mathfrak{B}}}(B(\varphi_{\mathfrak{B}}))}$ , since  $P\lambda_{\varphi}(B(\varphi))$  is dense in  $\hat{I}_{\mathfrak{B}}\mathcal{H}_{\varphi}$ , it follows that  $\overline{I_{\mathfrak{B}}\lambda_{\varphi_{\mathfrak{B}}}(B(\varphi_{\mathfrak{B}}))} = \hat{I}_{\mathfrak{B}}\mathcal{H}_{\varphi}$ , that is,  $\overline{\lambda_{\varphi_{\mathfrak{B}}}(B(\varphi_{\mathfrak{B}}))} = \mathcal{H}_{\varphi_{\mathfrak{B}}}$ .  $\square$

In the special case where  $B(\varphi) \subseteq \mathfrak{B}$ , it follows that  $B(\varphi) = B(\varphi_{\mathfrak{B}})$ . In this case the restriction of  $I_{\mathfrak{B}}$  to  $\lambda_{\varphi_{\mathfrak{B}}}(B(\varphi_{\mathfrak{B}}))$  is one-to-one; therefore, the density of  $\lambda_{\varphi}(B(\varphi))$  in  $\mathcal{H}_{\varphi}$  implies the density of  $\lambda_{\varphi_{\mathfrak{B}}}(B(\varphi_{\mathfrak{B}}))$  in  $\mathcal{H}_{\varphi_{\mathfrak{B}}}$ .

**Example 3.10.** Consider the space  $L^p(I)$   $p \geq 2$ , where  $I$  is a finite interval. Then  $\varphi(x, y) = \int_I x(t)y(t)dt$  is a biweight on  $L^p(0, 1)$  with  $\mathcal{D}(\varphi) = L^p(I)$  and  $B(\varphi) = L^\infty(I)$ . If  $q > p$ , then  $L^q(I) \subset L^p(I)$  and  $L^\infty(I) \subset L^q(I)$ . The application of the previous result gives the known result that  $\varphi$  restricted to  $L^q(I)$  is again a biweight.

Example 3.8 shows that, in general, if  $\mathfrak{A}_1, \mathfrak{A}_2$  are partial  $*$ -algebras,  $\varphi$  is a biweight on  $\mathfrak{A}_2$  with domain  $\mathcal{D}(\varphi)$  and core  $B(\varphi)$ , and  $\Phi$  is an  $*$ -homomorphism of  $\mathfrak{A}_1$  into  $\mathfrak{A}_2$ , then  $\varphi_\Phi$  is not necessarily a biweight on  $\mathfrak{A}_1$ . We give some additional examples in this direction. They all exhibit a common feature: the corresponding precore  $B(\varphi_\Phi)$  is too small to allow a significant representation.

**Example 3.11.** Let us define the space  $L_I^2(0, 1)$  as the set of all functions of  $L^2(0, 1)$  with support in a fixed proper subinterval  $I \subsetneq (0, 1)$ , for example,  $I = (0, 1/2)$ . If  $f \in L^2(0, 1) \setminus L_I^2(0, 1)$ , we define the following vector space:

$$\mathfrak{A}_f := \{g \in L^2(0, 1) : g := \alpha f + \beta f\psi + \phi, \alpha, \beta \in \mathbb{C}, \psi \in C_I(0, 1)\}, \quad (3.27)$$

where  $C_I(0, 1)$  is the space of continuous functions on  $(0, 1)$  with support in  $I$ . The partial multiplication in  $\mathfrak{A}_f$  is defined for pairs  $(g_1, g_2) \in \mathfrak{A}_f \times \mathfrak{A}_f$  such that either  $\alpha_1 = \beta_1 = 0$  or  $\alpha_2 = \beta_2 = 0$ . Then  $R\mathfrak{A}_f = C_I(0, 1)$ . Let us consider the positive sesquilinear form

$$\varphi(g_1, g_2) = \int_0^1 g_1(x)\overline{g_2(x)}dx \quad \text{on } D(\varphi) = \mathfrak{A}_f. \quad (3.28)$$

We have

$$N_\varphi = \left\{g \in \mathfrak{A}_f : \int_0^1 |g(t)|^2 dt = 0\right\} = \{\bar{0}\}. \quad (3.29)$$

The completion of  $D(\varphi) = \mathfrak{A}_f$  with respect to the inner product  $\varphi(g_1, g_2)$  is  $\mathcal{H}_\varphi = \{g \in L^2(0, 1) : g = \alpha f + h, h \in L^2_I(0, 1)\}$ , that is, the space  $\{\alpha f\} + L^2_I(0, 1)$  with  $f \in L^2(0, 1) \setminus L^2_I(0, 1)$ . Since  $R\mathfrak{A}_f$  is not dense in  $\mathcal{H}_\varphi$ , then there cannot exist a core  $B(\varphi)$  dense in  $\mathcal{H}_\varphi$ , that is,  $\varphi$  is not a biweight on  $\mathfrak{A}_f$ . Hence, the identity map  $i : \mathfrak{A}_f \rightarrow L^2(0, 1)$  is an \*-homomorphism which does not preserve biweights.

*Example 3.12.* Let us consider a selfadjoint operator  $X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$  such that  $X(\mathcal{D}) \not\subseteq \mathcal{D}$ , for example,  $X = P = i(d/dx)$  and  $\mathcal{D} = \{AC[0, 1] : g(0) = g(1)\}$ ; then  $X^\dagger = X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ . We can define the subspace

$$\mathcal{M}_X := \{Y \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) : Y = \alpha X + c1, \alpha, c \in \mathbb{C}\}. \quad (3.30)$$

It follows that  $\mathcal{M}_X$  is a partial  $O^*$ -algebra, with partial multiplication

$$(Y_1, Y_2) \in \Gamma \iff \alpha_1 \alpha_2 = 0, \quad (3.31)$$

where  $Y_i = \alpha_i + c_i 1$ ,  $(i = 1, 2)$ . Clearly,  $R\mathcal{M}_X = \{c1 : c \in \mathbb{C}\}$  is not dense in  $\mathcal{M}_X$ . If  $\xi \in \mathcal{D}$ , let us consider the positive sesquilinear form

$$\begin{aligned} \omega_\xi(Y_1, Y_2) &= (Y_1^\dagger \xi \mid Y_2^\dagger \xi); \\ \mathcal{D}(\omega_\xi) &= \{Y \in \mathcal{M}_X : \xi \in \mathcal{D}(Y^\dagger)\} = \mathcal{M}_X. \end{aligned} \quad (3.32)$$

The null space of  $\omega_\xi$  is given by

$$N_{\omega_\xi} := \{Y \in \mathcal{D}(\omega_\xi) : \omega_\xi(Y, Y) = 0\} = \{Y \in \mathcal{D}(\omega_\xi) : (\alpha X + c1)\xi = 0\}. \quad (3.33)$$

If  $\xi$  is not an eigenvector of  $X$ , it follows that  $N_{\omega_\xi} = \{0\}$ . Since the completion of  $\mathcal{M}_X$  with respect to the inner product  $\omega_\xi(Y_1, Y_2)$  is  $\mathcal{M}_X$  too, and  $B(\omega_\xi) \subset R\mathcal{M}_X$ , it follows that  $\omega_\xi$  is not a biweight. Now we consider the identity map

$$i : \mathcal{M}_X \longrightarrow \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}). \quad (3.34)$$

It is known that  $\omega_\xi$  is a biweight in  $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$  for all  $\xi \in \mathcal{H}$  (see also Example 5.3). Furthermore, the identity map is a faithful \*-homomorphism of the partial  $O^*$ -algebra  $\mathcal{M}_X$  into  $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$  (see [4, Lemma 2.5.17]).

The following proposition shows that, under certain conditions, biweights can be pulled back from one partial \*-algebra to another one by an \*-homomorphism.

**PROPOSITION 3.13.** *Let  $\mathfrak{A}_1, \mathfrak{A}_2$  be partial \*-algebras,  $\varphi$  a positive sesquilinear form on  $\mathfrak{A}_2$  defined on  $\mathcal{D}(\varphi) \times \mathcal{D}(\varphi)$  and  $B(\varphi)$  a precore for  $\varphi$ . Let  $\Phi$  be an \*-homomorphism of  $\mathfrak{A}_1$  into  $\mathfrak{A}_2$ . Define*

$$\begin{aligned} \mathcal{D}(\varphi_\Phi) &= \{a \in \mathfrak{A}_1 : \Phi(a) \in \mathcal{D}(\varphi)\}, \\ B(\varphi_\Phi) &= \{x \in R\mathfrak{A}_1 : \Phi(x) \in B(\varphi)\}, \\ \varphi_\Phi(a, b) &= \varphi(\Phi(a), \Phi(b)). \end{aligned} \quad (3.35)$$

If  $\Phi(B(\varphi_\Phi))$  is a core for  $\varphi$ , then  $B(\varphi_\Phi)$  is a core for  $\varphi_\Phi$  and, therefore,  $\varphi_\Phi$  is a biweight on  $\mathfrak{A}_1$  with domain  $\mathcal{D}(\varphi_\Phi)$  and core  $B(\varphi_\Phi)$ . In particular, if  $\Phi$  is surjective, then  $\Phi(B(\varphi_\Phi))$  is a core for  $\varphi$  if and only if  $B(\varphi_\Phi)$  is a core for  $\varphi_\Phi$ .

*Proof.* We already know from Proposition 3.7 that  $B(\varphi_\Phi)$  is a precore. We consider once more the isometric map  $\hat{\Phi}$  introduced in the proof of that proposition. Since, by definition,  $\hat{\Phi}(\lambda_{\varphi_\Phi}(B(\varphi_\Phi))) = \lambda_\varphi(\Phi(B(\varphi_\Phi)))$ , the density of  $\lambda_\varphi(\Phi(B(\varphi_\Phi)))$  in  $\mathcal{H}_\varphi$  implies the density of  $\lambda_{\varphi_\Phi}(B(\varphi_\Phi))$  in  $\mathcal{H}_{\varphi_\Phi}$ .

If  $\Phi$  is surjective, then  $\hat{\Phi}$  is unitary and the converse implication also holds.  $\square$

*Remark 3.14.* If  $\Phi : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$  is an  $*$ -homomorphism and  $\varphi$  is a biweight on  $\mathfrak{A}_2$ , then  $\Phi(\mathfrak{A}_1)$  is a vector subspace (but not necessarily a partial  $*$ -subalgebra of  $\mathfrak{A}_2$ ). We may consider, as before, the restriction of  $\varphi$  to  $\Phi(\mathfrak{A}_1)$ , denoted as  $\varphi_{\Phi(\mathfrak{A}_1)}$ . Then,

$$\begin{aligned} \mathcal{D}(\varphi_{\Phi(\mathfrak{A}_1)}) &= \Phi(\mathfrak{A}_1) \cap \mathcal{D}(\varphi) = \Phi(\mathcal{D}(\varphi_\Phi)), \\ B(\varphi_{\Phi(\mathfrak{A}_1)}) &= \Phi(\mathfrak{A}_1) \cap B(\varphi) = \Phi(B(\varphi_\Phi)), \\ N_{\varphi_{\Phi(\mathfrak{A}_1)}} &= \{\Phi(x), x \in \mathfrak{A}_1 : \varphi(\Phi(x), \Phi(x)) = 0\} = \Phi(N_{\varphi_\Phi}). \end{aligned} \quad (3.36)$$

We define the map

$$\tilde{\Phi} : \lambda_{\varphi_\Phi}(\mathcal{D}(\varphi_\Phi)) \longrightarrow \lambda_{\varphi_{\Phi(\mathfrak{A}_1)}}(\mathcal{D}(\varphi_{\Phi(\mathfrak{A}_1)})) \quad (3.37)$$

by

$$\tilde{\Phi}(\lambda_{\varphi_\Phi}(x)) := \lambda_{\varphi_{\Phi(\mathfrak{A}_1)}}(\Phi(x)). \quad (3.38)$$

It is easily seen that  $\tilde{\Phi} = \hat{I}_{\Phi(\mathfrak{A}_1)}^{-1} \hat{\Phi}$ . Then, as expected, if  $\lambda_{\varphi_{\Phi(\mathfrak{A}_1)}}(B(\varphi_{\Phi(\mathfrak{A}_1)}))$  is dense in  $\lambda_{\varphi_{\Phi(\mathfrak{A}_1)}}(\mathcal{D}(\varphi_{\Phi(\mathfrak{A}_1)}))$ , then  $\varphi_\Phi$  is a biweight on  $\mathfrak{A}_1$ , with domain  $\mathcal{D}(\varphi_{\Phi(\mathfrak{A}_1)})$  and core  $B(\varphi_{\Phi(\mathfrak{A}_1)})$ .

Proposition 3.13 suggests the following.

*Definition 3.15.* Let  $\mathfrak{A}_1, \mathfrak{A}_2$  be partial  $*$ -algebras,  $\Phi : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$  an  $*$ -homomorphism, and  $\varphi$  a biweight on  $\mathfrak{A}_2$  with domain  $\mathcal{D}(\varphi)$  and core  $B(\varphi)$ . Define  $\mathcal{D}(\varphi_\Phi) = \{a \in \mathfrak{A}_1 : \Phi(a) \in \mathcal{D}(\varphi)\}$  and  $B(\varphi_\Phi) = \{x \in R\mathfrak{A}_1 : \Phi(x) \in B(\varphi)\}$ .

The  $*$ -homomorphism  $\Phi$  is said to *preserve biweights* if, for any biweight  $\varphi$  on  $\mathfrak{A}_2$  and for any core  $B(\varphi)$  for  $\varphi$ ,  $\Phi(B(\varphi_\Phi))$  is a core for  $\varphi$ .

Moreover,  $\Phi$  is said to *strictly preserve biweights* if, for any biweight  $\varphi$  and for any core  $B(\varphi)$ , the equality  $\Phi(B(\varphi_\Phi)) = B(\varphi)$  holds.

Assume now that  $\Phi$  is an  $*$ -isomorphism of  $\mathfrak{A}_1 \rightarrow \mathfrak{A}_2$  and  $\varphi$  a biweight on  $\mathfrak{A}_2$  with domain  $\mathcal{D}(\varphi)$  and core  $B(\varphi)$ .

As before, we put  $\mathcal{D}(\varphi_\Phi) = \{a \in \mathfrak{A}_1 : \Phi(a) \in \mathcal{D}(\varphi)\}$ . We consider the following set:

$$B'(\varphi_\Phi) := \{x \in \mathfrak{A}_1 : \Phi(x) \in B(\varphi)\}. \quad (3.39)$$

One has  $B'(\varphi_\Phi) = B(\varphi_\Phi) = \{x \in R\mathfrak{A}_1 : \Phi(x) \in B(\varphi)\}$ . Indeed, if  $x \in B'(\varphi_\Phi)$ , then  $\Phi(x) \in B(\varphi) \subset R\mathfrak{A}_2$ . It follows that the product  $\Phi(a)\Phi(x)$  is well defined for all  $a \in \mathfrak{A}_1$ . Since  $\Phi^{-1}$

is an  $*$ -homomorphism,  $\Phi^{-1}(\Phi(a)) \cdot \Phi^{-1}(\Phi(x))$  is well defined, so  $ax$  is well defined for all  $a \in \mathfrak{A}_1$ , and  $x \in R\mathfrak{A}_1$ . Hence,  $x \in B(\varphi_\Phi)$ . The equality just proven is equivalent to  $\Phi(B(\varphi_\Phi)) = B(\varphi)$ . Therefore, we have proved the following.

PROPOSITION 3.16. *Every  $*$ -isomorphism strictly preserves biweights.*

A slight modification of the previous argument shows the following.

PROPOSITION 3.17. *Every  $*$ -homomorphism  $\Phi : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$  such that  $\Phi(R\mathfrak{A}_1) = R\mathfrak{A}_2$  strictly preserves biweights.*

#### 4. Topological considerations

As shown in [10], biweights having a common core may be used to define certain unbounded  $C^*$ -seminorms on partial  $*$ -algebras, in the sense of [4]. Unbounded  $C^*$ -seminorms play a relevant role in the theory, since they often provide precious information on the structure of the partial  $*$ -algebra and on its  $*$ -representations. This is true *a fortiori* when the partial  $*$ -algebra is endowed with some locally convex topology. The interplay between unbounded  $C^*$ -seminorms and the original topology can be useful when looking for continuity properties of  $*$ -homomorphisms or  $*$ -representations. This is the subject of this section. For simplicity, we limit ourselves to the case of norm topologies.

Let  $\mathfrak{A}$  be a partial  $*$ -algebra. We denote by  $BW(\mathfrak{A})$  the set of all biweights of  $\mathfrak{A}$ . Moreover, if  $\mathfrak{B}$  is a subspace of  $R\mathfrak{A}$ , following [10], we call  $BW(\mathfrak{A}; \mathfrak{B})$  the subset of all  $\varphi \in BW(\mathfrak{A})$  having  $\mathfrak{B}$  as core.

If  $\mathfrak{B}$  is a subspace of  $R\mathfrak{A}$ , we put

$$\begin{aligned} p_{\mathfrak{B}}(a)^2 &= \sup \{ \varphi(ax, ax) : \varphi \in BW(\mathfrak{A}; \mathfrak{B}), x \in \mathfrak{B}, \varphi(x, x) = 1 \}, \\ \mathcal{D}(p_{\mathfrak{B}}) &= \{ a \in \mathfrak{A} : p_{\mathfrak{B}}(a) < +\infty \}. \end{aligned} \quad (4.1)$$

If  $BW(\mathfrak{A}; \mathfrak{B}) = \{0\}$ , we put  $p_{\mathfrak{B}}(a) = 0$  for every  $a \in \mathfrak{A}$  and  $\mathcal{D}(p_{\mathfrak{B}}) = \mathfrak{A}$ .

If  $\varphi \in BW(\mathfrak{A}; \mathfrak{B})$ , we denote by  $\pi_{\varphi}^{\mathfrak{B}}$  the GNS representation for  $\varphi$  with core  $\mathfrak{B}$ . It is easily seen that  $\mathcal{D}(p_{\mathfrak{B}})$  consists of all  $a \in \mathfrak{A}$  for which  $\pi_{\varphi}^{\mathfrak{B}}(a)$  is bounded for every  $\varphi \in BW(\mathfrak{A}; \mathfrak{B})$  and

$$\sup \{ \|\overline{\pi_{\varphi}^{\mathfrak{B}}(a)}\| : \varphi \in BW(\mathfrak{A}; \mathfrak{B}) \} < +\infty. \quad (4.2)$$

Moreover,

$$p_{\mathfrak{B}}(a) = \sup \{ \|\overline{\pi_{\varphi}^{\mathfrak{B}}(a)}\| : \varphi \in BW(\mathfrak{A}; \mathfrak{B}) \}, \quad a \in \mathcal{D}(p_{\mathfrak{B}}). \quad (4.3)$$

Then (see also [10])

- (1)  $\mathcal{D}(p_{\mathfrak{B}})$  is a partial  $*$ -subalgebra of  $\mathfrak{A}$ ;
- (2)  $p_{\mathfrak{B}}$  is an unbounded  $C^*$ -seminorm on  $\mathcal{D}(p_{\mathfrak{B}})$ , that is,  $p_{\mathfrak{B}}(a^*a) = p_{\mathfrak{B}}(a)^2$ , for every  $a \in \mathcal{D}(p_{\mathfrak{B}})$  and  $a^* \in L(a)$ ;
- (3) the set  $N(p_{\mathfrak{B}}) = \{ a \in \mathfrak{A} : p_{\mathfrak{B}}(a) = 0 \}$  is a *partial  $*$ -ideal* in the sense that if  $b \in \mathfrak{A}$ ,  $a \in N(p_{\mathfrak{B}})$ , and  $ba$  is well defined, then  $ba \in N(p_{\mathfrak{B}})$ .

We prove the following.

LEMMA 4.1. *Let  $\mathfrak{B}$  be a subspace of  $R\mathfrak{A}$  and  $\mathfrak{B}_0$  a subspace of  $\mathfrak{B}$  such that  $\lambda_\varphi(\mathfrak{B}_0)$  is dense in  $\mathcal{H}_\varphi$ , for every  $\varphi \in BW(\mathfrak{A}, \mathfrak{B})$ . Then*

- (i)  $\mathcal{D}(p_{\mathfrak{B}_0}) \subseteq \mathcal{D}(p_{\mathfrak{B}})$  and  $p_{\mathfrak{B}}(a) \leq p_{\mathfrak{B}_0}(a)$ , for all  $a \in \mathcal{D}(p_{\mathfrak{B}_0})$ ;
- (ii) if  $a \in \mathcal{D}(p_{\mathfrak{B}})$ , then

$$p_{\mathfrak{B}}(a)^2 = \sup \{ \varphi(ax, ax) : \varphi \in BW(\mathfrak{A}; \mathfrak{B}), x \in \mathfrak{B}_0, \varphi(x, x) = 1 \}. \quad (4.4)$$

*Proof.* Since  $\mathfrak{B}_0$  is a subspace of  $\mathfrak{B}$  with  $\lambda_\varphi(\mathfrak{B}_0)$  dense in  $\mathcal{H}_\varphi$ , for every  $\varphi \in BW(\mathfrak{A}, \mathfrak{B})$ , then  $BW(\mathfrak{A}, \mathfrak{B}) \subseteq BW(\mathfrak{A}, \mathfrak{B}_0)$ . If  $\pi_\varphi^{\mathfrak{B}}$  and  $\pi_\varphi^{\mathfrak{B}_0}$  denote, respectively, the GNS representations for  $\varphi$  with cores  $\mathfrak{B}$  and  $\mathfrak{B}_0$ , then we have  $\pi_\varphi^{\mathfrak{B}_0} \subseteq \pi_\varphi^{\mathfrak{B}}$ . For each  $a \in \mathfrak{A}$  such that  $\pi_\varphi^{\mathfrak{B}}(a)$  is bounded, we have, clearly,

$$\| \pi_\varphi^{\mathfrak{B}}(a) \| = \| \pi_\varphi^{\mathfrak{B}_0}(a) \|. \quad (4.5)$$

This, in turn, implies that  $\mathcal{D}(p_{\mathfrak{B}_0}) \subseteq \mathcal{D}(p_{\mathfrak{B}})$  and

$$p_{\mathfrak{B}}(a) \leq p_{\mathfrak{B}_0}(a), \quad \forall a \in \mathcal{D}(p_{\mathfrak{B}_0}). \quad (4.6)$$

This proves (i).

As for (ii), if  $a \in \mathcal{D}(p_{\mathfrak{B}})$ , we have, taking into account (4.5),

$$\begin{aligned} p_{\mathfrak{B}}(a) &= \sup \{ \| \pi_\varphi^{\mathfrak{B}}(a) \| : \varphi \in BW(\mathfrak{A}; \mathfrak{B}) \} \\ &= \sup \{ \| \pi_\varphi^{\mathfrak{B}_0}(a) \| : \varphi \in BW(\mathfrak{A}; \mathfrak{B}) \} \\ &= \sup \{ \varphi(ax, ax)^{1/2} : \varphi \in BW(\mathfrak{A}; \mathfrak{B}), x \in \mathfrak{B}_0, \varphi(x, x) = 1 \}. \end{aligned} \quad (4.7)$$

□

Let now  $\mathfrak{A}_1, \mathfrak{A}_2$  be partial  $*$ -algebras and  $\Phi : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$  an  $*$ -homomorphism preserving biweights. Let  $\mathfrak{B}_2$  be a subspace of  $R\mathfrak{A}_2$  and

$$\mathfrak{B}_1 = \{ x \in R\mathfrak{A}_1; \Phi(x) \in \mathfrak{B}_2 \}. \quad (4.8)$$

Then, by Proposition 3.13,  $\mathfrak{B}_1$  is a core for all biweights  $\varphi_\Phi$ , with  $\varphi \in BW(\mathfrak{A}_2; \mathfrak{B}_2)$ . Moreover,  $\Phi(\mathfrak{B}_1)$  is a core for  $\varphi_2$ .

PROPOSITION 4.2. *Given  $a_1 \in \mathcal{D}(p_{\mathfrak{B}_1})$ , then  $\Phi(a_1) \in \mathcal{D}(p_{\mathfrak{B}_2})$  and*

$$p_{\mathfrak{B}_2}(\Phi(a_1)) \leq p_{\mathfrak{B}_1}(a_1). \quad (4.9)$$

*Proof.* Making use of Lemma 4.1(ii) and of the fact that  $\Phi(\mathfrak{B}_1)$  is a core for  $\varphi_2$ , for every  $\varphi_2 \in BW(\mathfrak{A}_2; \mathfrak{B}_2)$ , we have

$$\begin{aligned}
& p_{\mathfrak{B}_2}(\Phi(a_1))^2 \\
&= \sup \{ \varphi_2(\Phi(a_1)z, \Phi(a_1)z) : \varphi_2 \in BW(\mathfrak{A}_2; \mathfrak{B}_2), z \in \mathfrak{B}_2, \varphi_2(z, z) = 1 \} \\
&= \sup \{ \varphi_2(\Phi(a_1)\Phi(x_1), \Phi(a_1)\Phi(x_1)) : \varphi_2 \in BW(\mathfrak{A}_2; \mathfrak{B}_2), \\
&\quad x_1 \in \mathfrak{B}_1, (\varphi_2 \circ \Phi)(x_1, x_1) = 1 \} \\
&= \sup \{ \varphi_2(\Phi(a_1x_1), \Phi(a_1x_1)) : \varphi_2 \in BW(\mathfrak{A}_2; \mathfrak{B}_2), x_1 \in \mathfrak{B}_1, (\varphi_2 \circ \Phi)(x_1, x_1) = 1 \} \\
&= \sup \{ (\varphi_2 \circ \Phi)(a_1x_1, a_1x_1) : \varphi_2 \in BW(\mathfrak{A}_2; \mathfrak{B}_2), x_1 \in \mathfrak{B}_1, (\varphi_2 \circ \Phi)(x_1, x_1) = 1 \} \\
&\leq \sup \{ \varphi(a_1x_1, a_1x_1) : \varphi \in BW(\mathfrak{A}_1; \mathfrak{B}_1), x_1 \in \mathfrak{B}_1, \varphi(x_1, x_1) = 1 \} \\
&= p_{\mathfrak{B}_1}(a_1)^2.
\end{aligned} \tag{4.10}$$

□

Proposition 4.2 has an obvious counterpart for  $*$ -algebras. In that case, the seminorms to consider are those constructed by Yood [11] using families of positive linear functionals and it holds, of course, for arbitrary  $*$ -homomorphisms  $\Phi$ . It is worth noticing that, in the case of  $C^*$ -algebras, the analogue of (4.9) implies the continuity of  $\Phi$ . Here, however, we need additional assumptions in order to obtain similar results.

Let now  $\mathfrak{A}$  be a normed partial  $*$ -algebra with norm  $\|\cdot\|$  and  $\mathfrak{B}$  a subspace of  $R\mathfrak{A}$  (for shortness, we write, from now on,  $\mathfrak{B} \leq R\mathfrak{A}$ ).

Let  $\varphi \in BW(\mathfrak{A}, \mathfrak{B})$ , then, following [10], we say that  $\varphi$  has a  $\|\cdot\|$ -continuous  $\mathfrak{B}$ -orbit if

$$\forall x \in \mathfrak{B}, \quad \exists \gamma_x > 0 : |\varphi(ax, bx)| \leq \gamma_x \|a\| \cdot \|b\|, \quad \forall a, b \in \mathfrak{A}. \tag{4.11}$$

The set of all these biweights is denoted by  $CO(\mathfrak{B})$ . We define

$$CO^e(\mathfrak{B}) = \{ \varphi \in CO(\mathfrak{B}) : |\varphi(ax, bx)| \leq \varphi(x, x) \|a\| \cdot \|b\|, \quad \forall a, b \in \mathfrak{A}, x \in \mathfrak{B} \}. \tag{4.12}$$

We denote by  $q_{\mathfrak{B}}$  the  $C^*$ -seminorm

$$q_{\mathfrak{B}}(a) = \sup \{ \varphi(ax, ax)^{1/2} : \varphi \in CO^e(\mathfrak{B}), \varphi(x, x) = 1 \}. \tag{4.13}$$

**PROPOSITION 4.3.** *The following statements hold:*

- (i)  $q_{\mathfrak{B}}(a) \leq \|a\|$ , for all  $a \in \mathfrak{A}$  (therefore  $\mathcal{D}(q_{\mathfrak{B}}(a)) = \mathfrak{A}$ );
- (ii) for every  $\varphi \in CO^e(\mathfrak{B})$ , one has

$$|\varphi(ax, bx)| \leq \varphi(x, x) q_{\mathfrak{B}}(a) q_{\mathfrak{B}}(b). \tag{4.14}$$

*Proof.* Both statements come directly from the definitions. □

Now assume that the family of seminorms  $\{q_{\mathfrak{B}}; \mathfrak{B} \preceq R\mathfrak{A}\}$  satisfies the condition

$$a \in \mathfrak{A}, \quad q_{\mathfrak{B}}(a) = 0, \quad \forall \mathfrak{B} \preceq R\mathfrak{A} \implies a = 0. \quad (4.15)$$

Then  $\{q_{\mathfrak{B}}; \mathfrak{B} \preceq R\mathfrak{A}\}$  defines a locally convex Hausdorff topology  $\tau_{C^*}$  on  $\mathfrak{A}$  (which we will call the  $C^*$ -like topology). The topology  $\tau_{C^*}$  is coarser than the norm topology because of Proposition 4.3(i).

*Example 4.4.* What we have done so far applies, of course, to the particular case where  $\mathfrak{A}$  is a Banach  $*$ -algebra with unit. Let  $\varphi$  be a biweight on the Banach  $*$ -algebra  $\mathfrak{A}$  with domain  $\mathcal{D}(\varphi)$  and core  $\mathfrak{B}$ . If  $x \in \mathfrak{B}$ , then  $\varphi_x(a, b) := \varphi(ax, bx)$  is a sesquilinear form on  $\mathfrak{A} \times \mathfrak{A}$ . Therefore, we can define  $\omega_x(a) = \varphi(ax, x)$ ,  $a \in \mathfrak{A}$ . Then  $\omega_x$  is a positive linear functional on  $\mathfrak{A}$  and hence one has

$$|\omega_x(a)| \leq \omega_x(e)\|a\|, \quad \forall a \in \mathfrak{A}. \quad (4.16)$$

Therefore,

$$|\varphi_x(a, b)| = |\omega_x(b^*a)| \leq \varphi(x, x)\|a\|\|b\|. \quad (4.17)$$

Thus, for any core  $\mathfrak{B}$ , one has  $\varphi \in CO^e(\mathfrak{B})$  and  $p_{\mathfrak{B}}(a) = q_{\mathfrak{B}}(a)$  for all  $a \in \mathfrak{A}$ . If  $\mathfrak{A}$  is an  $*$ -semisimple Banach  $*$ -algebra, the topology  $\tau_{C^*}$  reduces to that defined by the unique norm

$$p(a) = \sup_{\omega \in S(\mathfrak{A})} \omega(a^*a)^{1/2}, \quad (4.18)$$

where  $S(\mathfrak{A})$  denotes the set of states of  $\mathfrak{A}$ . This topology is coarser than the norm topology, in general. When  $p \sim \|\cdot\|$ , then  $\mathfrak{A}$  is an  $A^*$ -algebra, whereas  $\mathfrak{A}$  is a  $C^*$ -algebra if  $p(a) = \|a\|$  for all  $a \in \mathfrak{A}$ .

*Remark 4.5.* Let  $\mathfrak{A}_1, \mathfrak{A}_2$  be Banach partial  $*$ -algebras and let  $\Phi : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$  be a continuous  $*$ -homomorphism. Then, even in this case, we cannot be sure that for a given biweight  $\varphi$  on  $\mathfrak{A}_2$ , with domain  $\mathcal{D}(\varphi)$  and core  $B(\varphi)$ , the corresponding form  $\varphi_{\Phi}$  is a biweight on  $\mathfrak{A}_1$ . Indeed, let  $f \in L^2(0, 1)$ , and consider the following space:

$$\mathfrak{A}_f := \{g \in L^2(0, 1) : g = \alpha f + c, \alpha, c \in \mathbb{C}\}. \quad (4.19)$$

In this space, a partial multiplication is defined:  $(g_1, g_2) \in \Gamma$  if and only if  $\alpha_1 \alpha_2 = 0$ . Furthermore,  $\mathfrak{A}_f$  and  $L^2(0, 1)$  are Banach partial  $*$ -algebras in obvious way.

In this case,  $R\mathfrak{A}_f$  consists only of constant functions. Hence,  $R\mathfrak{A}_f$  is not dense in  $\mathfrak{A}_f$ . The everywhere-defined positive sesquilinear form on  $\mathfrak{A}_f$

$$\varphi_1(g_1, g_2) = \int_0^1 g_1(x) \overline{g_2(x)} dx, \quad (4.20)$$

is not a biweight on  $\mathfrak{A}_f$ . Indeed,  $N_{\varphi_1} = \{0\}$ . Then  $\mathcal{H}_{\varphi_1} = \mathfrak{A}_f$ , and  $\lambda(R\mathfrak{A}_f) = R\mathfrak{A}_f$ , which is not dense in  $\mathcal{H}_{\varphi_1}$ . It is clear that  $\varphi_1 = \varphi_{\Phi}$ , where  $\varphi$  is the following positive sesquilinear

form on  $L^2(0, 1)$ :

$$\int_0^1 f(x) \overline{h(x)} dx, \quad f, h \in L^2(0, 1), \quad (4.21)$$

which is a biweight with largest core given by  $L^\infty(0, 1)$  and  $\Phi$  is the identity map of  $\mathfrak{A}_f$  into  $L^2(0, 1)$ , which is trivially a continuous \*-homomorphism of  $\mathfrak{A}_f$  into  $L^2(0, 1)$ .

Let  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$  be Banach partial \*-algebras, and  $\Phi : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$  a continuous \*-homomorphism, hence  $\|\Phi(a)\|_{\mathfrak{A}_2} \leq \|\Phi\| \|a\|_{\mathfrak{A}_1}$  for all  $a \in \mathfrak{A}_1$ . Assume that  $\Phi$  preserves biweights.

We have the following.

**PROPOSITION 4.6.** *Let  $\mathfrak{B}$  be a subspace of  $R\mathfrak{A}_2$  and put*

$$\mathfrak{B}_\Phi := \{a \in R\mathfrak{A}_1 : \Phi(a) \in \mathfrak{B}\}. \quad (4.22)$$

*If  $\varphi \in CO(\mathfrak{B})$ , then  $\varphi_\Phi \in CO(\mathfrak{B}_\Phi)$ . Moreover, if  $\varphi \in CO^e(\mathfrak{B})$ , then there exists  $\gamma > 0$  such that  $\gamma\varphi_\Phi \in CO^e(\mathfrak{B}_\Phi)$ .*

*Proof.* Let  $\varphi \in CO(\mathfrak{B})$ . Then, we have

$$\begin{aligned} |\varphi(\Phi(a)\Phi(x), \Phi(b)\Phi(x))| &\leq \gamma_{\Phi(x)} \|\Phi(a)\|_{\mathfrak{A}_2} \|\Phi(b)\|_{\mathfrak{A}_2} \\ &\leq C^2 \gamma_{\Phi(x)} \|a\|_{\mathfrak{A}_1} \|b\|_{\mathfrak{A}_1}, \quad \forall a, b \in \mathfrak{A}_1, x \in \mathfrak{B}_\Phi. \end{aligned} \quad (4.23)$$

Let now  $\varphi \in CO^e(\mathfrak{B})$ . Then

$$\begin{aligned} |\varphi(\Phi(a)\Phi(x), \Phi(a)\Phi(x))| &\leq \varphi(\Phi(x), \Phi(x)) \cdot \|\Phi(a)\|_{\mathfrak{A}_2} \|\Phi(b)\|_{\mathfrak{A}_2} \\ &\leq \varphi_\Phi(x, x) \|\Phi\|^2 \|a\|_{\mathfrak{A}_1} \|b\|_{\mathfrak{A}_1}, \quad \forall a, b \in \mathfrak{A}_1, x \in \mathfrak{B}_\Phi. \end{aligned} \quad (4.24)$$

Thus  $(1/\|\Phi\|^2)\varphi_\Phi \in CO^e(\mathfrak{B}_\Phi)$ . □

The next question we want to consider is the following: under which conditions is an \*-homomorphism continuous from  $\mathfrak{A}_1[\|\cdot\|_{\mathfrak{A}_1}]$  to  $\mathfrak{A}_2[\|\cdot\|_{\mathfrak{A}_2}]$ ?

We begin with the case where  $\Phi$  is an \*-isomorphism. Then, as already proven, for every core  $B(\varphi)$  for  $\varphi$ , one has  $\Phi(B(\varphi_\Phi)) = B(\varphi)$ .

Now let  $\mathfrak{B} \leq R\mathfrak{A}_2$  and consider the family of biweights  $CO^e(\mathfrak{B})$ . Then, for every  $a, b \in \mathfrak{A}_1$  and for every  $x \in \mathfrak{B}_1 := \{x \in R\mathfrak{A}_1 : \Phi(x) \in \mathfrak{B}\}$ , we have

$$\begin{aligned} |\varphi_\Phi(ax, bx)| &= |\varphi(\Phi(ax), \Phi(bx))| = |\varphi(\Phi(a)\Phi(x), \Phi(b)\Phi(x))| \\ &\leq \varphi_\Phi(x, x) \|\Phi(a)\|_{\mathfrak{A}_2} \|\Phi(b)\|_{\mathfrak{A}_2}. \end{aligned} \quad (4.25)$$

Thus, in general,  $\varphi_\Phi$  need not be an element of  $CO^e(\mathfrak{B}_1)$ . But, according to Proposition 4.6, this is a necessary condition for continuity. The same discussion as above applies to the case where  $\Phi$  is an \*-homomorphism strictly preserving biweights.

As for the continuity of an \*-homomorphism, we can now state the following preliminary result.



PROPOSITION 4.7. Let  $\mathfrak{A}_1, \mathfrak{A}_2$  be normed partial  $*$ -algebras and  $\Phi$  an  $*$ -homomorphism of  $\mathfrak{A}_1$  into  $\mathfrak{A}_2$  strictly preserving biweights and with the property that, for every  $\mathfrak{B} \leq R\mathfrak{A}_2$ , the following implication holds:

$$\varphi \in CO^e(\mathfrak{B}) \implies \gamma\varphi_\Phi \in CO^e(\mathfrak{B}_\Phi) \quad \text{for some } \gamma > 0, \quad (4.26)$$

where  $\mathfrak{B}_\Phi = \{x \in R\mathfrak{A}_1 : \Phi(x) \in \mathfrak{B}\}$ . Then  $\Phi$  is continuous for the  $\tau_{C^*}$  topologies of  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$ , respectively.

*Proof.* Let  $q_{\mathfrak{B}}$  be any one of the seminorms defining the topology  $\tau_{C^*}$  of  $\mathfrak{A}_1$ . Then, we have

$$\begin{aligned} q_{\mathfrak{B}}(\Phi(a)) &= \sup \{ \varphi(\Phi(a)x_2, \Phi(a)x_2)^{1/2} : \varphi \in CO^e(\mathfrak{B}), x_2 \in \mathfrak{B}, \varphi(x_2, x_2) = 1 \} \\ &= \sup \{ \varphi(\Phi(ax_1), \Phi(ax_1))^{1/2} : \varphi \in CO^e(\mathfrak{B}), x_1 \in \mathfrak{B}_\Phi, \varphi_\Phi(x_1, x_1) = 1 \} \\ &\leq \frac{1}{\gamma} q_{\mathfrak{B}_1}(a). \end{aligned} \quad (4.27)$$

□

## 5. The $*$ -radical

The seminorms  $\{p_{\mathfrak{B}} : \mathfrak{B} \leq R\mathfrak{A}\}$ , defined as in the previous section by families of biweights on  $\mathfrak{A}$  having  $\mathfrak{B}$  as core, can be used to introduce a notion of  $*$ -radical for a partial  $*$ -algebra. We will not carry out here a detailed analysis of this notion, but we will only show that the proposed definition leads to what is expected in some special situations.

If  $p$  is any seminorm on  $\mathfrak{A}$ , we put  $N(p) = \{a \in \mathfrak{A} : p(a) = 0\}$ .

*Definition 5.1.* Given a partial  $*$ -algebra  $\mathfrak{A}$ , the *algebraic  $*$ -radical* of  $\mathfrak{A}$  is defined as follows:

$$\text{Rad}^*(\mathfrak{A}) = \begin{cases} \mathfrak{A} & \text{if } BW(\mathfrak{A}) = \{0\}; \\ \bigcap_{\mathfrak{B} \subset R\mathfrak{A}} N(p_{\mathfrak{B}}) & \text{if } BW(\mathfrak{A}) \neq \{0\}. \end{cases} \quad (5.1)$$

A partial  $*$ -algebra  $\mathfrak{A}$  for which  $\text{Rad}^*(\mathfrak{A}) = \{0\}$  is called *algebraically  $*$ -semisimple*.

PROPOSITION 5.2. The algebraic  $*$ -radical  $\text{Rad}^*(\mathfrak{A})$  coincides with the intersection of the kernels of all GNS representations  $\pi_\varphi$  constructed from biweights  $\varphi \in BW(\mathfrak{A})$ .

*Proof.* Let  $\varphi$  be a biweight with domain  $\mathcal{D}(\varphi)$  and core  $B(\varphi)$ . Then, we have for the GNS representation  $\pi_\varphi^B$ ,

$$\|\pi_\varphi^B(a)\lambda_\varphi(x)\|^2 = \varphi(ax, ax). \quad (5.2)$$

Thus, if  $a \in \text{Rad}^*(\mathfrak{A})$ , then  $\pi_\varphi^B(a)\lambda_\varphi(x) = 0$ , for all  $x \in B(\varphi)$ , and so  $\pi_\varphi^B(a) = 0$ .

Conversely, if  $\pi_\varphi^B(a) = 0$  for every biweight  $\varphi$  with core  $B(\varphi)$ , then  $\varphi(ax, ax) = 0$ , for all  $x \in B(\varphi)$ , and therefore  $a \in \text{Rad}^*(\mathfrak{A})$ . □

*Example 5.3.* Let  $\mathfrak{M}$  be a partial  $O^*$ -algebra on  $\mathcal{D}$  in  $\mathcal{H}$  and  $\xi \in \mathcal{H}$ . As in [4, Example 9.1.12], we construct a biweight  $\varphi_\xi$  on  $\mathfrak{M}$  in the following way. We put

$$\begin{aligned} D(\varphi_\xi) &= \{X \in \mathfrak{M}; \xi \in \mathcal{D}(X^{+*})\}, \\ \varphi_\xi(X, Y) &= (X^{+*}\xi \mid Y^{+*}\xi), \quad X, Y \in \mathcal{D}(\varphi_\xi). \end{aligned} \quad (5.3)$$

Then  $\varphi_\xi$  is a positive sesquilinear form on  $\mathcal{D}(\varphi_\xi) \times \mathcal{D}(\varphi_\xi)$ .

(1) Suppose that  $\xi \in \mathcal{D}$  and put

$$B(\varphi_\xi) = \{X \in R^w(\mathfrak{M}); X\xi \in \mathcal{D}^{**}(\mathfrak{M})\}. \quad (5.4)$$

Assume that  $B(\varphi_\xi)\xi$  is dense in  $\overline{D(\varphi_\xi)\xi}$ . Then  $\varphi_\xi$  is a biweight on  $\mathfrak{M}$  with core  $B(\varphi_\xi)$ .

(2) On the other hand, suppose that  $\xi \in \mathcal{H} \setminus \mathcal{D}$  and put

$$\begin{aligned} B_o(\varphi_\xi) &= \{X \in R^w(\mathfrak{M}); \xi \in \mathcal{D}(\overline{X}), \overline{X}\xi \in \mathcal{D}\}, \\ B(\varphi_\xi) &= \text{linear span of } B_o(\varphi_\xi). \end{aligned} \quad (5.5)$$

Then,  $B(\varphi_\xi)$  is a subspace of  $\mathcal{D}(\varphi_\xi)$  satisfying the conditions (i), (ii), (iv), and (v) of Definition 3.1.

Whenever  $B(\varphi_\xi)\xi$  is dense in  $\overline{D(\varphi_\xi)\xi}$ , then  $\varphi_\xi$  is a biweight on  $\mathfrak{M}$  with a core  $B(\varphi_\xi)$ .

For every vector  $\xi \in \mathcal{H}$ ,  $\varphi_\xi$  is a biweight on the maximal partial  $O^*$ -algebra  $\mathcal{L}^+(\mathcal{D}, \mathcal{H})$ . This holds true because  $\{\eta \otimes \xi; \eta \in \mathcal{D}\} \subset B(\varphi_\xi)$  and  $\{(\eta \otimes \xi)\xi; \eta \in \mathcal{D}\} = \mathcal{D}$ .

From the previous example, the following holds.

**PROPOSITION 5.4.**  $\mathcal{L}^+(\mathcal{D}, \mathcal{H})$  is algebraically  $*$ -semisimple.

*Proof.* Let  $A \in \mathcal{L}^+(\mathcal{D}, \mathcal{H})$  and assume that for every biweight  $\varphi$  on  $\mathcal{L}^+(\mathcal{D}, \mathcal{H})$  and every  $X$  in a core  $B(\varphi)$ ,  $\varphi(AX, AX) = 0$ . Then this is true, in particular, for the biweights of the type  $\varphi_\xi$ ,  $\xi \in \mathcal{D}$ , defined in the previous example. Since  $I \in B(\varphi_\xi)$ , we get  $\|A\xi\|^2 = \varphi_\xi(A, A) = 0$ , for every  $\xi \in \mathcal{D}$ . Thus  $A = 0$ .  $\square$

Consider now the special case where  $\mathfrak{A}$  is a Banach  $*$ -algebra. Then there is a standard notion of  $*$ -radical (see, e.g., [8, Chapter 4]), denoted here by  $R^*(\mathfrak{A})$ , that can be expressed as

$$R^*(\mathfrak{A}) = \left\{ a \in \mathfrak{A} : \sup_{\omega \in S(\mathfrak{A})} \omega(a^*a) = 0 \right\}, \quad (5.6)$$

where  $S(\mathfrak{A})$  is the set of all states on  $\mathfrak{A}$ . The notion of algebraic  $*$ -radical introduced in Definition 5.1 is a genuine generalization of this standard one, as it should. We have indeed the following.

**PROPOSITION 5.5.** Let  $\mathfrak{A}$  be a Banach  $*$ -algebra. Then,

$$R^*(\mathfrak{A}) = \text{Rad}^*(\mathfrak{A}). \quad (5.7)$$

*Proof.* Let  $\varphi$  be a biweight with domain  $\mathcal{D}(\varphi)$  and core  $B(\varphi)$ . If  $x \in B(\varphi)$ , then  $\varphi_x(a, b) = \varphi(ax, bx)$  is a sesquilinear form on  $\mathfrak{A} \times \mathfrak{A}$ . Define  $\omega_x(a) := \varphi(ax, x)$ . This is a positive

linear functional on  $\mathfrak{A}$ . Indeed,

$$\omega_x(a^*a) = \varphi(a^*ax, ax) = \varphi(ax, ax) \geq 0, \quad (5.8)$$

by the condition (iv) of Definition 3.1. Moreover,  $\omega_x$  is a state if and only if,  $\varphi(x, x) = 1$ . So if  $a \in \text{Rad}^*(\mathfrak{A})$ , we have  $\omega_x(a^*a) = 0$ , then  $\varphi(ax, ax) = 0$ , which implies  $a \in \text{Rad}^*(\mathfrak{A})$ .

Conversely, assume that  $a \in \text{Rad}^*(\mathfrak{A})$  and let  $\omega \in S(\mathfrak{A})$ . Put  $\varphi_\omega(a, b) = \omega(b^*a)$ . Then  $\varphi_\omega$  is clearly a biweight with domain  $\mathcal{D}(\varphi_\omega) = B(\varphi_\omega) = \mathfrak{A}$ . Thus, for every  $x \in \mathfrak{A}$ ,  $\varphi_\omega(ax, ax) = 0$ . In particular, for  $x = e$ , we get  $\varphi_\omega(a, a) = 0$ , which implies  $\omega(a^*a) = 0$ .  $\square$

In the previous example, all sesquilinear forms  $\varphi_x$ ,  $x \in \mathfrak{B}$ , are, in fact, continuous and the corresponding GNS representations are bounded. This fact suggests considering for normed partial  $*$ -algebras a stronger notion of radical, using this time the seminorms  $\{\varphi_{\mathfrak{B}} : \mathfrak{B} \leq \mathfrak{R}\mathfrak{A}\}$ , which are defined by biweights  $\varphi$  for which  $\varphi_x, x \in \mathfrak{B}$ , are bounded.

**Definition 5.6.** Given a normed partial  $*$ -algebra  $\mathfrak{A}$ , the *topological  $*$ -radical* of  $\mathfrak{A}$  is defined as follows:

$$\text{Rad}_{\text{top}}^*(\mathfrak{A}) = \begin{cases} \mathfrak{A} & \text{if } CO^e(\mathfrak{B}) = 0, \forall \mathfrak{B} \in \mathfrak{R}\mathfrak{A}; \\ \bigcap_{\mathfrak{B} \in \mathfrak{R}\mathfrak{A}} N(\varphi_{\mathfrak{B}}). \end{cases} \quad (5.9)$$

If  $\text{Rad}_{\text{top}}^*(\mathfrak{A}) = \{0\}$ , then  $\mathfrak{A}$  is said to be  *$*$ -semisimple*.

Of course, if  $\mathfrak{A}$  is  $*$ -semisimple, it follows that  $\mathfrak{A}$  is algebraically  $*$ -semisimple. The converse is not true in general.

An interesting question, which we leave open, is the following: for which normed partial  $*$ -algebras do the algebraic  $*$ -radical and the topological  $*$ -radical coincide?

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