The periods of the generalized Jacobian of a complex elliptic curve. *†‡

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Abstract

We show that the toroidal Lie group $\mathcal{G} = \mathbb{C}^2/\Lambda$, where Λ is the lattice generated by $(1,0)$, $(0,1)$ and $(\widehat{\tau}, \widetilde{\tau})$, $\widehat{\tau} \notin \mathbb{R}$, is isomorphic to the generalized Jacobian J_L of the complex elliptic curve $\mathcal{C}$ with modulus $\widehat{\tau}$, defined by any divisor class $L \equiv (M) + (N)$ of $\mathcal{C}$ fulfilling $M - N = [\wp(\widetilde{\tau}) : \wp'(\widetilde{\tau}) : 1] \in \mathcal{C}$. This follows from an apparently new relation between the Weierstrass sigma and elliptic functions.

1 Introduction

In order to determine the periods of the generalized Jacobian J_L of a complex elliptic curve $\mathcal{C}$ with modulus L, we consider complex functions of two variables with three $\mathbb{R}$-independent periods. These functions were studied for the first time by Cousin in 1910 [3] and are at the origin of the theory of generalized Jacobians. After having been explicitly mentioned by Severi in 1947 [10] in the case of the field $\mathbb{C}$ of complex numbers, generalized Jacobians were introduced by Rosenlicht in 1954 [8] in the general context of algebraic groups over an arbitrary field. The representation of the generalized Jacobian of a curve as an extension of a linear group by the ordinary Jacobian has been widely investigated in [9], and since then. More recently a growing interest on the computational aspects of generalized Jacobians over finite fields arose from [5].

When one considers an n-dimensional *toroidal group* $\mathcal{G} = \mathbb{C}^n/\Lambda$ (sometimes called quasi-torus) of real rank $r = n + 1$ (see, e.g., [1]), it is easy to see that the toroidal group is an extension of a $(n-1)$-dimensional maximal linear sub-torus by an elliptic curve $\mathcal{C}$ with period matrix $(1, \widehat{\tau})$, with $\widehat{\tau} \notin \mathbb{R}$, (cf. [4], Prop. 2.2.3, p. 26) and, because the functor $\mathrm{Ext}(\mathcal{C}, (\mathbb{C}^*)^{n-1})$ is additive, one can restrict to the case where $n = 2$ and $\mathcal{G}$ is defined by the lattice Λ generated by $(1,0)$, $(0,1)$ and $(\widehat{\tau}, \widetilde{\tau})$, for a given $\widetilde{\tau} \in \mathbb{C}$ such that the only pair $(a,b) \in \mathbb{Z}^2$ such that $a\widehat{\tau} + b\widetilde{\tau}$ is an integer is

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the pair $(0,0)$. As such, it is also an example of a complex commutative theta group (see [7] §23, cf. [2] §6.1). In this paper we consider the generalized Jacobian J_L corresponding to $\mathcal{G}$ and we bridge these two representations by finding of the modulus $L = (M) + (N)$ which gives in turn the period lattice Λ of $\mathcal{G}$ with a purely geometric description.

Roughly speaking, the generalized Jacobian J_L of a curve $\mathcal{C}$ relative to the module (effective divisor) $L = \sum n_P(P)$, $P \in \mathcal{C}$, is obtained when two divisors of $\mathcal{C}$ are identified if they differ for a principal divisor (g) of a function g such that $v_P(1-g) \geq n_P$ for any $P \in \mathcal{C}$, where v_P is the discrete valuation of the local ring $\mathcal{O}_P$ of the rational functions of $\mathcal{C}$ that are regular in P.

Fixing a point $\Omega \in \mathcal{C}$, given a divisor D we denote simply by P the point of $\mathcal{C}$ and by $f \in \mathbb{C}(\mathcal{C})$ the function such that $D = (P) - (1-\deg(D))(\Omega) + (f)$. Thus, in the case where the module is $L = \sum_{i=0}^{n}(Q_i)$, two divisors D_1 and D_2 are equivalent when $D_1 = (P) - (1-\deg(D_1))(\Omega) + (f_1)$, $D_2 = (P) - (1-\deg(D_2))(\Omega) + (f_2)$ and $\frac{f_1(Q_i)}{f_1(Q_0)} = \frac{f_2(Q_i)}{f_2(Q_0)}$, for any $i = 1, \ldots, n$.

Clearly, the function L_{P_1,P_2} such that the sum of the two divisors $(P_1) - (\Omega)$ and $(P_2) - (\Omega)$ is equal to $(P_1 + P_2) - (\Omega) + (L_{P_1,P_2})$ is proportional to $\frac{\ell_{P_1,P_2}(X)}{\ell_{P_1+P_2,\Omega}(X)}$, where $\ell_{A,B}(X) = 0$ is the line through A and B, or the tangent in A to $\mathcal{C}$, if $A = B$. Thus,

$$D_1 + D_2 = (P_1 + P_2) - \big(1 - \deg(D_1) - \deg(D_2)\big)(\Omega) + (f_1 f_2 L_{P_1,P_2}). \tag{1}$$

For the ordinary Jacobian of $\mathcal{C}$ we do not keep track of the fact that $f_3 = f_1 f_2 L_{P_1,P_2}$. But now, for any $L = \sum_{i=0}^{n}(Q_i)$, the n–tuple $c_L(P_1,P_2) = \left(\frac{L_{P_1,P_2}(Q_1)}{L_{P_1,P_2}(Q_0)}, \ldots, \frac{L_{P_1,P_2}(Q_n)}{L_{P_1,P_2}(Q_0)}\right)$ keeps record of the cosets modulo the principal divisors (g) such that $v_{Q_i}(1-g) \geq 1$ for any $i = 0, \ldots, n$ and clearly from (1) it follows that c_L is a (non–regular) factor system. It must be noticed that the partial operation $(P_1, k_1) + (P_2, k_2) = \big(P_1 + P_2, k_1 \cdot k_2 \cdot c_L(P_1,P_2)\big)$ cannot be continuously extended to $X \times \mathbb{C}^*$ and gives therefore only a *birationally* isomorphic image Ξ_L of J_L (in the sense of [11]).

The non–regular factor system $c_L : \mathcal{C} \times \mathcal{C} \to \mathbb{C}^*$ corresponding to J_L induces a non–regular factor system $c_L P : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^*$, $(x_1, x_2) \mapsto c_L\big(P(x_1), P(x_2)\big)$, with $P(x) = [\wp(x) : \wp'(x) : 1] \in \mathcal{C}$. Since every commutative Lie group extension of $\mathbb{C}^*$ by $\mathbb{C}$ is splitting, we expect to unveil $c_L P$ as a co-boundary $\delta^1(g) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^*$, for some $g : \mathbb{C} \to \mathbb{C}^*$.

2 Triply periodic functions

About the geometric description we are looking for, it is worth noticing that, as we expect that $c_L P = \delta^1(g)$, it will not be possible to express $g(x)$ in function of $P(x) = [\wp(x) : \wp'(x) : 1] \in \mathcal{C}$, even if the factor system $c_L = \delta^1(g)$ in $\mathcal{C}$ is given in terms of the points $P(x)$ and $P(y)$. Nevertheless, g can be expressed as a function of the Weierstrass sigma function $\sigma(x)$ corresponding to $\wp(x)$, as we have:

Theorem. *With the above notations, let $M = P(v_M)$ and $N = P(v_N)$ be two different points of $\mathcal{C}$ such that $P(\widetilde{\tau}) = M - N$. The function $G(x,y) = \exp(2\pi i y) g(x)$, where*

$$g(x) = \exp\Big(2\eta_1(v_N - v_M)x\Big)\frac{\sigma(v_M)}{\sigma(v_N)}\,\frac{\sigma(x - v_N)}{\sigma(x - v_M)},$$

(with the constant η_1 defined, e.g., in [6], p. 150) is triply periodic with period lattice Λ.

The generalized Jacobian J_L of $\mathcal{C}$ defined by the modulus $L = (M) + (N)$ is isomorphic to $\mathbb{C}^2/\Lambda$, and we find that $\delta^1(g) = c_L P$ is the rational non-regular factor system of the extension *corresponding to* L *and the map* $(x,y)+\Lambda \mapsto \big(P(x), G(x,y)\big)$ *is an isomorphism from* $\mathbb{C}^2/\Lambda$ *onto a birationally isomorphic image of* J_L*.*

Proof. The first part of the claim could be verified *a posteriori*, but we want to give a constructive proof, which better explains the link between the quotient Lie group $\mathbb{C}^2/\Lambda$ and the generalized Jacobian J_L.

Let the 2–dimensional toroidal group $\mathcal{G}$ be defined by the lattice Λ generated by $(1,0)$, $(0,1)$ and $(\widehat{\tau}, \widetilde{\tau})$, with $\widehat{\tau} \notin \mathbb{R}$. Let $\sigma(x)$, $\zeta(x) = \frac{\sigma'(x)}{\sigma(x)}$ and $\wp(x) = -\zeta'(x)$ be the Weierstrass sigma function, zeta function and elliptic function, respectively, corresponding to the lattice in $\mathbb{C}$ generated by 1 and $\widehat{\tau}$. Fix the epimorphism

$$z \mapsto P(z) := [z^3\wp(z) : z^3\wp'(z) : z^3]$$

from $\mathbb{C}$ onto the corresponding elliptic curve $\mathcal{C} = \mathcal{C}(\widehat{\tau})$ of the projective plane of coordinates $[X : Y : T]$, having $\mathcal{O} = [0 : 1 : 0]$ as the zero element.

Let $M = [X_M : Y_M : T_M]$ and $N = [X_N : Y_N : T_N]$ be two distinct non-zero points of $\mathcal{C}$, let $L = (M) + (N)$ and let $c_L(P_1, P_2) := \frac{\ell_{P_1,P_2}(M)}{\ell_{P_1+P_2,\mathcal{O}}(M)} \frac{\ell_{P_1+P_2,\mathcal{O}}(N)}{\ell_{P_1,P_2}(N)}$ be the non–regular factor system which defines on $\mathcal{C} \times \mathbb{C}^*$ a birational group Ξ_L, birationally isomorphic to J_L, where $\ell_{A,B}(X) = 0$ is the line through A and B, or the tangent in A to $\mathcal{C}$, if $A = B$ (cf. [9], §7 n. 16).

We look for a homomorphism $\mathbb{C}^2 \longrightarrow \Xi_L$ with Λ as the kernel. We can assume $(0,y) \mapsto (\mathcal{O}, \mathrm{e}^{2\pi i y})$. Let ϵ be a complex multiplication such that $(x,0) \mapsto (P(\epsilon x), g(x))$, for a suitable $g : \mathbb{C} \longrightarrow \mathbb{C}^*$ such that $\delta^1(g) = c_L P\epsilon$, that is, $g(x_1 + x_2) = g(x_1)g(x_2)c_L(P(\epsilon x_1), P(\epsilon x_2))$.

As $P(0) = \mathcal{O}$, the derivative at $x_2 = 0$ of the last equation must be performed in the chart $Y \neq 0$ containing $\mathcal{O}$ and, deriving with respect to x_2 in the chart $\left[\frac{\wp(z)}{\wp'(z)} : 1 : \frac{1}{\wp'(z)}\right]$ we find

$$g'(x_1 + x_2) = g(x_1)(g'(x_2)c_L(P(\epsilon x_1), P(\epsilon x_2)))+$$

$$g(x_1)g(x_2)\Big(\epsilon\big(1 - \frac{\wp(\epsilon x_2)\wp''(\epsilon x_2)}{\wp'(\epsilon x_2)^2}\big)\frac{\partial}{\partial X_{P(\epsilon x_2)}} c_L(P(\epsilon x_1), P(\epsilon x_2)) - \epsilon\frac{\wp''(\epsilon x_2)}{\wp'(\epsilon x_2)^2}\frac{\partial}{\partial T_{P(\epsilon x_2)}} c_L(P(\epsilon x_1), P(\epsilon x_2))\Big).$$

Putting $x_2 = 0$, since $\big(1 - \frac{\wp(0)\wp''(0)}{\wp'(0)^2}\big) = -\frac{1}{2}$, $\frac{\wp''(0)}{\wp'(0)^2} = 0$, and $c_L(P(\epsilon x_1), \mathcal{O}) = 1$, we find

$$\frac{g'(x_1)}{g(x_1)} = g'(0) - \frac{1}{2}\epsilon\Big(\frac{\partial}{\partial X_{P(\epsilon x_2)}} c_L(P(\epsilon x_1), \mathcal{O})\Big).$$

Since $c_L(P_1, P_2) = \frac{\ell_{P_1,P_2}(M)}{\ell_{P_1+P_2,\mathcal{O}}(M)} \frac{\ell_{P_1+P_2,\mathcal{O}}(N)}{\ell_{P_1,P_2}(N)}$, the computation of $\frac{\partial}{\partial X_{P(\epsilon x_2)}} c_L(P(\epsilon x_1), P(\epsilon x_2))$ is inconvenient, whereas a direct computation shows that

$$\lim_{\delta \to 0} \frac{1}{\delta}\Big(c_L\big([X : 1 : T], [\delta : 1 : 0]\big) - 1\Big) =$$

$$\frac{-Y_M T(-T_N X + X_N T) - T_M(X_N T + Y_N X T) + X_M(T_N T + Y_N T^2)}{(-T_M X + X_M T)(-T_N X + X_N T)}.$$

Hence we have

$$\frac{g'(x_1)}{g(x_1)} = g'(0) - \frac{X_M Y_N - Y_M X_N + (Y_M T_N - T_M Y_N)\wp(\epsilon x_1) + (X_M T_N - T_M X_N)\wp'(\epsilon x_1)}{2(X_M - T_M\wp(\epsilon x_1))(X_N - T_N\wp(\epsilon x_1))}.$$

Thus we have to solve the separable differential equation

$$\frac{g'(x)}{g(x)} = g'(0) + \frac{Y_M}{2(X_M - T_M\wp(\epsilon x))} - \frac{Y_N}{2(X_N - T_N\wp(\epsilon x))} - \frac{(X_M T_N - T_M X_N)\wp'(\epsilon x)}{2(X_M - T_M\wp(\epsilon x))(X_N - T_N\wp(\epsilon x))}.$$

Let $\log z$ be the principal value of the logarithm. Integrating by substitution, the last summand gives, up to an additive constant, the primitive integral

$$\int \frac{(X_M T_N - T_M X_N)\wp'(\epsilon x)}{(X_M - T_M\wp(\epsilon x))(X_N - T_N\wp(\epsilon x))}\,\mathrm{d}x = \frac{1}{\epsilon}\log\frac{T_M\wp(\epsilon x) - X_M}{T_N\wp(\epsilon x) - X_N}.$$

For the second and the third primitive integral, let $v_M, v_N \in \mathbb{C}$ be such that $M = [\wp(v_M) : \wp'(v_M) : 1]$ and $N = [\wp(v_N) : \wp'(v_N) : 1]$, that is $X_M = T_M\wp(v_M), Y_M = T_M\wp'(v_M)$ and $X_N = T_N\wp(v_N), Y_N = T_N\wp'(v_N)$.

Since $\zeta(x) = \frac{\sigma'(x)}{\sigma(x)}$ and $\wp(x) = -\zeta'(x)$, applying the addition formula for ζ (see, e. g., [6], (6.8.4), p. 161), we find, up to an additive constant,

$$\int \frac{Y_M}{X_M - T_M\wp(\epsilon x)}\,\mathrm{d}x = \frac{1}{\epsilon}\log\frac{\sigma(\epsilon x + v_M)}{\sigma(\epsilon x - v_M)} - 2x\zeta(v_M),$$

and the same for the second summand. Thus, up to a constant α of integration, we have

$$\log g(x) = \alpha + Hx + \frac{1}{2\epsilon}\log\left(\frac{T_N\wp(\epsilon x) - X_N}{T_M\wp(\epsilon x) - X_M}\,\frac{\sigma(\epsilon x + v_M)}{\sigma(\epsilon x - v_M)}\,\frac{\sigma(\epsilon x - v_N)}{\sigma(\epsilon x + v_N)}\right), \tag{2}$$

where we have put $g'(0) - \zeta(v_M) + \zeta(v_N) = H$.

In order to simplify $\log g(x)$, we write

$$\frac{\sigma(\epsilon x + v_M)}{\sigma(\epsilon x - v_M)}\,\frac{\sigma(\epsilon x - v_N)}{\sigma(\epsilon x + v_N)} = \frac{\sigma(\epsilon x + v_M)\sigma(\epsilon x - v_M)}{\sigma(\epsilon x - v_M)^2}\,\frac{\sigma(\epsilon x - v_N)^2}{\sigma(\epsilon x + v_N)\sigma(\epsilon x - v_N)},$$

and we apply the addition formula for σ (see, e. g., [6], (6.7.5), p. 158), obtaining that the last is equal to

$$\frac{\big(\wp(v_M) - \wp(\epsilon x)\big)\sigma(\epsilon x)^2\sigma(v_M)^2}{\sigma(\epsilon x - v_M)^2}\,\frac{\sigma(\epsilon x - v_N)^2}{\big(\wp(v_N) - \wp(\epsilon x)\big)\sigma(\epsilon x)^2\sigma(v_N)^2}.$$

Now equation (2) becomes

$$\log g(x) = \alpha + Hx + \frac{1}{2\epsilon}\log\left(\frac{T_N\wp(\epsilon x) - X_N}{T_M\wp(\epsilon x) - X_M}\,\frac{\wp(v_M) - \wp(\epsilon x)}{\wp(v_N) - \wp(\epsilon x)}\,\frac{\sigma(\epsilon x - v_N)^2}{\sigma(\epsilon x - v_M)^2}\,\frac{\sigma(v_M)^2}{\sigma(v_N)^2}\right) \tag{3}$$

and, substituing $X_M = T_M\wp(v_M)$ and $X_N = T_N\wp(v_N)$, we get

$$\log g(x) = \alpha + Hx + \frac{1}{2\epsilon}\log\left(\frac{T_N}{T_M}\,\frac{\sigma(\epsilon x - v_N)^2}{\sigma(\epsilon x - v_M)^2}\,\frac{\sigma(v_M)^2}{\sigma(v_N)^2}\right). \tag{4}$$

From the membership of $(0, 1)$ in the lattice, it follows that $g(0) = 1$, that is,

$$\log g(x) = Hx + \frac{1}{2\epsilon}\log\left(\frac{\sigma(\epsilon x - v_N)^2}{\sigma(\epsilon x - v_M)^2}\,\frac{\sigma(v_M)^2}{\sigma(v_N)^2}\right). \tag{5}$$

Now, since

$$\log\Big(c_L(P(\epsilon x_1),P(\epsilon x_2))\Big)=\log\Big(\delta^1(g)\,(x_1,x_2)\Big)=\frac{1}{\epsilon}\log\delta^1\left(\frac{\sigma(\epsilon x-v_N)}{\sigma(\epsilon x-v_M)}\frac{\sigma(v_M)}{\sigma(v_N)}\right)$$

and since the first terms of the MacLaurin series for $c_L\big(P(\epsilon x_1),P(\epsilon x_2)\big)$ are the same of the ones for $\delta^1\left(\frac{\sigma(\epsilon x-v_N)}{\sigma(\epsilon x-v_M)}\frac{\sigma(v_M)}{\sigma(v_N)}\right)$, that is,

$$1+\epsilon^2(\wp(v_M)-\wp(v_N))x_1x_2+\frac{\epsilon^3}{2}(\wp'(v_N)-\wp'(v_M))(x_1^2x_2+x_1x_2^2),$$

it follows that $\epsilon=1$. From (5) we get now $g(x)=\exp(Hx)\frac{\sigma(v_M)}{\sigma(v_N)}\,\frac{\sigma(x-v_N)}{\sigma(x-v_M)}$.

From the quasi-periodicity of σ:

$$\sigma(z+n\widehat{\tau}+m)=(-1)^{(n+m)}\exp\Big(2\eta_3(n^2\widehat{\tau}/2+nz)+2\eta_1(m^2/2+m(z+n\widehat{\tau}))\Big)\sigma(z),$$

(where η_1 and $\eta_3=\eta_1\widehat{\tau}-i\pi$ are constants, for details see [6] §6.2, p. 150) and from the membership of $(1,0)$ in the lattice, it follows $g(1)=1$, hence we have:

$$1=\exp(H)\frac{\sigma(v_M)}{\sigma(v_N)}\,\frac{\exp\Big(2\eta_1(1/2-v_N)\Big)\sigma(-v_N)}{\exp\Big(2\eta_1(1/2-v_M)\Big)\sigma(-v_M)}=\exp\Big(H+2\eta_1(v_M-v_N)\Big),$$

that is, for some $h_1\in\mathbb{Z}$ we obtain

$$2h_1\pi i=H+2\eta_1(v_M-v_N). \tag{6}$$

Finally, from the membership of $(\widehat{\tau},\widetilde{\tau})$ in the lattice, it follows $g(\widehat{\tau})\,\exp(2\pi i\widetilde{\tau})=1$, hence we have:

$$1=\exp(H\widehat{\tau})\frac{\sigma(v_M)}{\sigma(v_N)}\exp(2\pi i\widetilde{\tau})\,\frac{\sigma(\widehat{\tau}-v_N)}{\sigma(\widehat{\tau}-v_M)}=\exp(H\widehat{\tau})\frac{\sigma(v_M)}{\sigma(v_N)}\exp(2\pi i\widetilde{\tau}+2\eta_3(v_M-v_N))\,\frac{\sigma(v_N)}{\sigma(v_M)},$$

that is,

$$\exp(H\widehat{\tau}+2\pi i\widetilde{\tau}+2\eta_3(v_M-v_N))=1.$$

Thus, for some $h_2\in\mathbb{Z}$, we get

$$H\widehat{\tau}+2\pi i\widetilde{\tau}+2\eta_3(v_M-v_N)=2h_2\pi i. \tag{7}$$

Recalling that $\eta_1\widehat{\tau}=\eta_3+i\pi$, from (7) and (6) it follows that

$$2h_1\pi i\widehat{\tau}=2h_2\pi i-2\pi i\widetilde{\tau}+2\pi i(v_M-v_N),$$

hence the point $P(\widetilde{\tau})=[\wp(\widetilde{\tau}):\wp'(\widetilde{\tau}):1]$ is equal to $M-N$.

Remark: The relationship $\delta^1(g)=c_LP$ gives now

$$\frac{\sigma(v_N)}{\sigma(v_M)}\,\frac{\sigma(x_1+x_2-v_N)\sigma(x_1-v_M)\sigma(x_2-v_M)}{\sigma(x_1+x_2-v_M)\sigma(x_1-v_N)\sigma(x_2-v_N)}=\frac{\ell_{P(x_1),P(x_2)}(M)}{\ell_{P(x_1+x_2),\mathcal{O}}(M)}\,\frac{\ell_{P(x_1+x_2),\mathcal{O}}(N)}{\ell_{P(x_1),P(x_2)}(N)},$$

an apparently new relationship between the Weierstrass sigma and elliptic functions.

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