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# APLIMAT - JOURNAL OF APPLIED MATHEMATICS

# VOLUME 5 (2012), NUMBER 1



## MATHEMATICS AND ART

# APLIMAT - JOURNAL OF APPLIED MATHEMATICS
# VOLUME 5 (2012), NUMBER 1

## NEW TRENDS IN MATHEMATICS EDUCATION

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# SCIENTIFIC ICONOGRAPHY BETWEEN MATHEMATICS AND ARTS IN THE AGE OF DIGITAL VISUALIZATION.

# FIRST OUTLINES.

**BRUNETTI Federico, Dott.Arch. PhD. (Italy),**

**Abstract.** Scientific research uses methodologies of representation - formerly acquired in the usual communication languages, but having to evolve and decipher the results of innovative experiments and opening new scenarios of investigation - and also configures innovative ways of figuration to obtain the most appropriate views and new kinds of images about the investigated phenomena. In a summary subdivision of skills, we could hypothetically say that science deals with discoveries and art deals with inventions. The "vision" in the investigation process anticipates and directs experimental research and may prelude to the logical sequence of a theory, which prefigures the creation of an experiment validating the theory itself.
The distinctions between art and science tend to get closer particularly in the digital era, being able to present more and more coming because of the similarities between the two routes, which remain, at least in methodological appearance, different and distant.

**Keywords and phrases.** Science, communication, design; arts, images, representation, vision; drawing, images, metaphors, iconology, theory.

*Mathematics Subject Classification:* Primary 00A66 Mathematics and visual arts, visualization;

*"Knowledge is an attitude, a passion. [...]*
*It is not true that the researcher follows the truth, it is the truth that follows the researcher."*
Robert Musil

*"If we try to keep the mirror itself, in the end we discover that the things on it;*
*if we want to grab things, do not we go back to find nothing but the mirror.*
*This is the more general history of knowledge."*
Friedrich Nietzsche

## 1. The scientific view and the images of research

Scientific research uses methodologies of representation - formerly acquired in the usual language of communication, but having to evolve and decipher the results of innovative experiments and opening new scenarios of investigation - and also configures innovative ways of figuration to obtain the most appropriate views of the investigated phenomena. The images generated by these views, both in the process of experimental testing or calibration, are configured to verify the effectiveness in representation of the acquired data, and become part of an contemporary imaginary of iconography, anticipating and foreshadowing, similarly to research in contemporary art, the visual codes future shared languages. In a summary subdivision of skills, we could hypothetically say that:

***science*** **deals with** ***discoveries***

i.e. to find laws and objects that are not known, which already exist in nature, or which has not yet been possible to redefine the respective taxonomies, and

***art*** **deals with** ***inventions***

i.e. to create artifacts rearranging already known elements, recombine the parts in order to propose a different relationship between the meanings and functions of parts of the work, who can innovate and change the state of pre-existing nature. Such a partition clearly skills leaves a wide margin of hybridization detectable in different fields of knowledge: one for all the technique (*tékne*) or any design process which could be placed as a cross-science between the disciplines of the "discovery" and those of the "invention", as oriented to know the natural qualities of the material components and recombine them to create artifacts aimed at re-creation of nature.
These two cognitive processes and can also be symbolized as:

**science** | **art**

**concept** | **perception**

↗ ↘ | ↗ ↘

**theory (science) design** | **image (art) critique**

↖ ↙ | ↖ ↙

**algorithm verification** | **algorithm processing**

**(scientific community)** | **(artistic community)**

**vision** (∗) | **vision** (∗)

**Science.** It could first be defined a cycle diagram within which takes place the course of scientific research: from the definition of a theory we can identify the possibility of the *concept* of a new experiment that will - immediately or in the course of progressive technological developments - verify these assumptions. The *design* and operational execution of this experiment will allow to test the theory through the development of these data in the light of *specific algorithms (and functions)* configured to recognize and verify objectives and data of the research; this can occur in the context of the *scientific community* appointed to validate any confirmations or refutations about the theory of departure.

**Art.** It could be similarly defined a cycle within which takes place the course of artistic exploration: each work is communicated through the proposition of an *image,* which is implicitly conformed to a interpretation theory of the reality experience; each new image alters the status of established languages, innovating the perception of the context in which the work is placed.

The role of artistic critique should be to understand the meaning of this innovation -originally implicit in the image- enabling the transformative algorithm (or function) and sharing innovative contents; this can occur in the context of the artistic community appointed to incorporate, share, confront or refute the image of this route where it has been originated.

In both cases the path of research departs and arrives from an experience of a new vision that evolves, by proposing a new *theory / image* until the completion of its explicit interpretative *paradigm (algorithm),* towards the scenery .of a new re-evolutionof discoveries or inventions.

The search for *beauty*, intuition underlying the nature of an *order* ("symmetry"), the investigation of a *logic* (mathematics) allows the human to intrude in the laws of nature allowing the researcher into the backdrop of his discoveries: these are some points of continuity that seem to be a common to these two areas.

*"There is something ineffable in the real, which, sometimes is described as something mysterious, and that inspires awe and reverence too, property which he alludes is undoubtedly its immediacy, spontaneity, the fact that occurs as the consequence of a perfect and complex of rational thought."(...) "On the other hand,mathematics, and especially geometry, are exactly those attributes of order, the elements of predictability, which actually seem to be missing. How do these alternatives can be assigned to units as contradictory to our experience?" (...) "In so far as the laws of mathematics refer to reality they are not satisfied, and insofar as they are certain, they do not refer to reality."*
Albert EINSTEIN, *Geometrie und Erfahrung* (1921).

## 2. Three different aspects of communication in scientific iconography: drawing, images and metaphors.[1]

Drawing and images occupy a relevant place among the languages and tools used to communicate science. This type of representation of scientific contents also has a fundamental role in developing the theoretical phase of the research which precedes the communication phase. Every discipline stems from a specific interest and establishes its ambit by naming, through precise definitions, the system of related objects. "Words" identify the "things" of the "world" that one decides to deal with, and the semantic system of representation becomes part of the procedures and of the descriptive, interpretive and pre-figurative methods of the specific discipline[2]. The words of science are often not literary expressions, but logical-mathematical paths expressed with alphanumeric formulas, taxonomies, theoretical hypotheses based on observation or requiring further observation to be validated[3]. The iconic component becomes particularly important in this type of path and, although it usually takes only the form of para-textual apparatus, when it is adequately

[1] **Federico Brunetti, *DISEGNO, IMMAGINI E METAFORE NELLA COMUNICAZIONE SCIENTIFICA;* in: ComunicareFisica 2007 Trieste, 1 – 6 ottobre 2007 http://comunicarefisica.ts.infn.it/presentazioni/brunetti.pdf**

[2] Michel Foucault, *Les Mots et les choses*, Paris 1966.

[3] Karl R. Popper, John C. Eccles, *The Self and its Brain*, Berlin 1977.

conceived, can be critically important and can even shape the thought that generated it. Among such iconic expressions three main elements can be identified: drawing, image and metaphor.
*Drawing.* Through the formalization process of drawing, events and concepts previously invisible become visible, in accordance with a compositional order that in itself stabilizes the analysis of the object of a research and makes it shareable. Like in geometry, the dot is taken as the essential element used to trace a sequence of events in describing physical phenomena; this minimal reference becomes the starting point for analyzing and investigating their dynamic development[4].
*Image.* The production of visual artefacts makes possible the interpretation of the collected data. The perception of such data is more focused and the information is integrated with contextual values and aesthetic meaning: thus the data can be more profoundly appropriate and also be shared[5]. During the heuristic phase of scientific (and artistic) research, the image is recognized as the mental configuration (even before becoming the visual configuration) of the synthetic comprehension of a problem and its data. Then the interpretation can be further analyzed logically and formally[6]. The representation of a real datum (or of a hypothetical datum) can be perceived easier through its image, within a system of shared codes of communication[7].
*Metaphor.* The classical origin of this rhetorical figure and its use, mainly in poetic expressions, has kept the scientific epistemology of the early modernity away from this concept, which has recently been reappraised in the analysis of phenomena that cannot be empirically perceived as in quantum physics[8]. Metaphor is a fundamental cognitive and communicative modality that uses a linguistic procedure in order to define, even if figuratively, the form or the concept of an object that is being studied, by introducing and referring to the figure of another object which is better known. This mode of communication, found both in text form and in iconography, uses a hypothetical model, not only to represent the formal structure of the phenomena, but also to enable the comprehension of the real datum according to specific explanatory hypotheses or at least according to certain elements that make such comprehension possible through shared experiences.
*Drawing, images and metaphors* can be so adopted as a first criterion to understand the multifaceted atlas of images used in scientific communication, and produced by researchers to understand each others in the complex world of "Big Science"[9].
Beyond the individual samples and in many strict research protocols, one can undoubtedly find several examples of crossbreeding and nuances among the simple categories that have been introduced as the initial iconological taxonomy of the universe of visual thought.

## 3. Images between science and arts

The contemporary scientific research inherits from its cultural history, and reasonably by fundamental human way of thinking, an attitude to collect synthetically form the results of its investigation into figures and images. Similarly, and more immediately, images are expressive and synthetic work realized by arts.

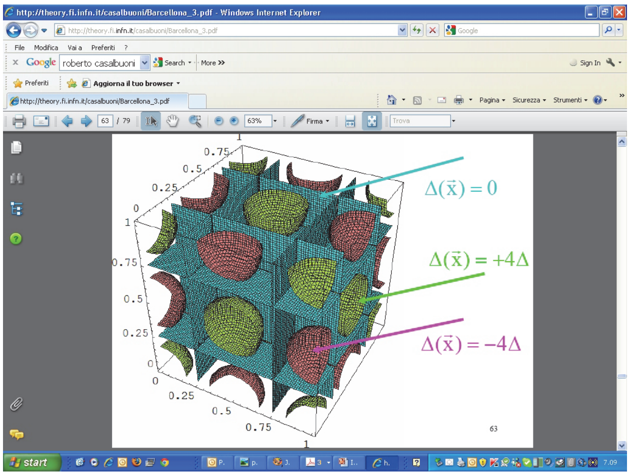
**Science ➔ *images* ➔ Arts**

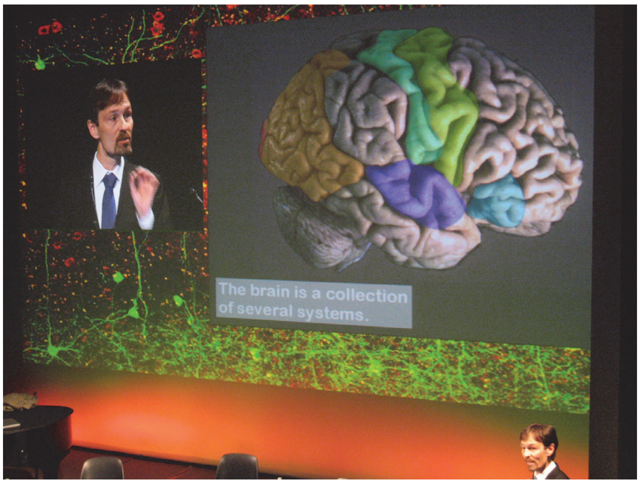
[4] Paul Klee, *Pädagogisches Skizzenbuch,* München 1925; Unendliche Naturgeschichte, Basel 1970.
[5] Jill Ackerman, *Origins, Imitation, and Conventions: Representation in the Visual Arts,* Cambridge, Mass. 2001
[6] Stephen M. Kosslyn, *Le immagini nella mente. Creare ed utilizzare immagini nel cervello,* Giunti, Firenze, 1999; S.T.Moulton, S. M. Kosslyn (2009), *Imagining predictions: mental imagery as mental emulation*, "Philosophical Transactions of the Royal Society" B, 364, 1273-1280. 6.
[7] Giovanni Anceschi, *L'oggetto della raffigurazione,* Milano 1992.
[8] *Elena Castellani,* Verità e scienze fisiche*, in Silvana Borutti Luca Fonnesu (a cura di), La verità. Scienza, filosofia, società,* Bologna 2005
[9] *Yurij Castelfranchi, Nico Pitrelli,* Come si comunica la scienza?*, Roma-Bari 2007*

It should be noted, however, that the point of view of scientific research, both in the past as in the present, around this approach has been, and remains, significantly critical. In fact logical deductive procedures and abstract and alphanumeric formulations oversee this area of knowledge, so as to make perfectly understandable the purely *alphanumeric results*, regardless of their *iconical visualization* or communication for researchers.

But in some cases, scientists openly declare themselves to have been able to full comprehension of certain problems, or the meaning of some results, using intuitive events very similar to the "vision" of a form, intended as a solution or synthetic perception (Gestalt Theorie) of a set of data.

This bipolarity is, and has been in the past, ground of deep epistemological diatribes[10] , and nowadays perhaps neurosciences investigations can only best describe and represent the competencies, and connected relationships, between different "knowledge regions" of the brain: logic-formal, language alphanumeric text, iconic and perceptual, theoretical and intuitive. But specifically some epistemological issues will be not likely to be resolved by observing the forms of thinking, but by developing new and more integrated forms of representation[11].



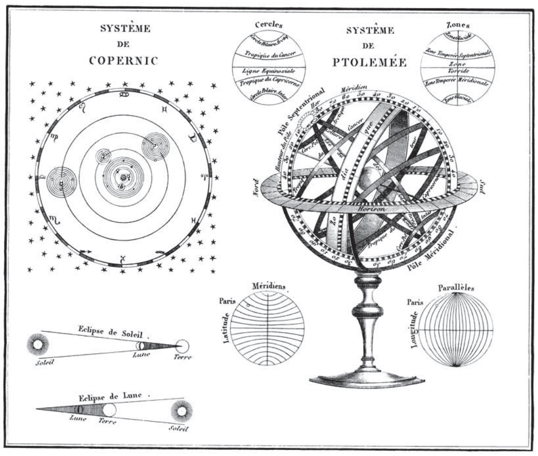




Left: Roberto Casalbuoni, *NGB and their parameters* - http://theory.fi.infn.it/casalbuoni/Barcellona_3.pdf
Right: *"Il colore del pensiero", Brainforum convention*, Milano 2011 http://www.brainforum.it/brainforum-2011.html

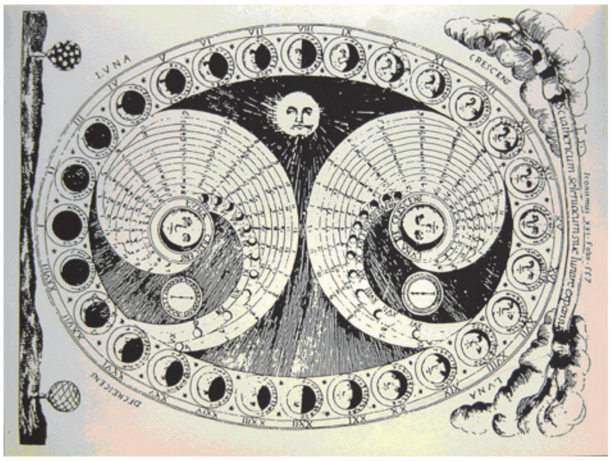
From this point of view the new information technologies are proposing integrated processes that transfigure each other computational data into qualitative images (color or multi-dimensional), allowing to explore elements of reality (microscopic or macroscopic, both in space and time) that otherwise could be not perceptively visible, amplifying detections of the signal through the powerful tools of measurement, and post-treatment of the obtained data.

The amplification of knowledge achieved through research tools was evident in the development of science, just think the results applied from optics by the telescope and then under the microscope, but actually the power of treatment possible through digital technology is opening up scenarios of representation and exploratory power otherwise unthinkable.

[10] Luca Zucchi, *Image, Language and Knowledge: Iconophobia and Iconophilia in the Epistemological Debate* in: Centre for Advanced Study (CAS) of Sofia, Bulgaria, 2/20/2006
[11] http://www.ilsussidiario.net/News/Scienze/2011/4/6/NEUROSCIENZE-Rizzolatti-non-e-normale-essere-narcisista-l-uomo-e-relazione/print/165307/ interview by Federico Brunetti, at Brainforum 2011, Milano.

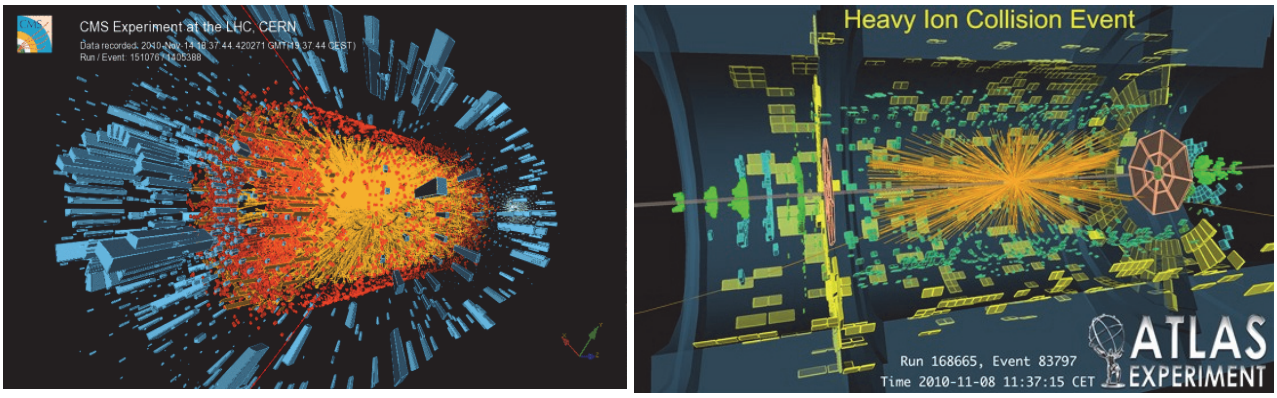



Systeme de Copernic – Systeme de Ptolomèe



Athanasius Kircher , *Ars magna lucis et umbrae* 1646

An interesting interweaving is in particular occurring about physics research, since some of the issues relating to micro-scale of particle physics or macro-scale cosmological studies are going to investigate, and possibly discover, a few subjects so far only theoretically assumed.
In fact the contemporary scientific Big Science research is verifying an unexpected and significant correlation of interests regarding to some fundamental questions concerning the antimatter, dark matter, through research in microscale[12] (particle collisions - the Higgs boson - the initial states of space-time - LHC @ CERN) and the evidence and hypothesis taken from the observation of cosmic phenomena from multispectral complex dynamics income of the relationship between matter and energy (i.e. “antimatter”, “blacks holes”) on a cosmic scale. These issues have become crucial term of comparison for understanding not only of matter or the universe, but the same effect as the previously established models of interpretation, for example in the Standard Model. In this sense, we are in an unexpected continuum of open questions and possible cross-solutions from the “Zepto-space”[13] to the new cosmology.




CMS and ATLAS experiment at LHC CERN, Geneva 2010. Particle collision images

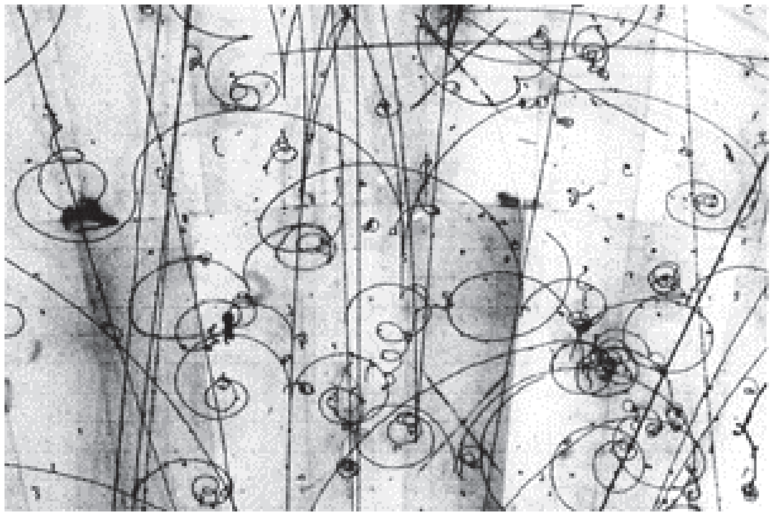
It is clear that these advances in experiments and research would not have been historically possible without an adequate technological advancement that would make acheivable the realization; it

[12] **Federico Brunetti,** *Gli anelli del sapere. INFN x LHC. Il contributo italiano alla più grande ricerca sulla fisica delle particelle al CERN di Ginevra.* **(Istituto Nazionale di Fisica Nucleare/ CERN / Dept. Indaco Politecnico di Milano / ed. Abitare Segesta RCS Mediagroup) Milano 2009; essays: Federico Brunetti:** ***Tra scienza ed architettura*** **pp. 10-13; La fisica come progetto*.*** ***Bellezza Rappresentazione, visione.*** **pp.18-37;** ***Disegno, metafore, immagini nella comunicazione scientifica.*** **pp. 38-43. www.theringsofknowledge.info**

[13] Gian Francesco Giudice, *A Zeptospace Odyssey: A Journey into the Physics of the LHC,* Oxford University Press, Oxford 2010

should be noted, however, as the theoretical foundation that led to a working hypothesis on which it operates now are based on previous experiments in pre-digital technology generations.


Max Bill, Sculpture and topological research



Images of track from bubble room (around 1960)

*"Mathematics is not only an essential tool of the primary thought, and therefore a need for knowledge of the appeals of the surrounding reality, but also, in its fundamental elements, the proportions of a science of behavior from object to object, from group to group , from movement to movement. And since this science has in itself these fundamental elements and relates them in a significant way, it is natural that such facts can be represented, transformed into images."*
Max Bill, *The Mathematical Way of Thinking in the Visual Art in our time,* 1949

*"... In conclusion, I believe that there is no physical or mathematical concept that is not open to the graphical representation of visual thinking and verbal are isomorphic each other."*
David Brisson, Rhode Island School of Design di Providence (USA),
artist, drawing teacher, interested in iperobjects, he organized in 1977 the first exhibition of hypergraphics (multi-dimensional graphics)

## 4. Scientific iconography in the age of digital visualization

It is quite easy to see how in the scenery of modernity, that we can backdate to the Renaissance[14] or to the more recent Industrial Revolution, researches conducted by science on the one hand, and the other in the experiments of the visual arts have anticipated, each on its behalf and sometimes from several decades, the figurations that later became conventional in ordinary language, generating

[14] Martin Kemp, *The science of art. Optical themes in western art from Brunelleschi to Seurat*, NewHaven and London 1990;
Martin Kemp, *Immagine e verità. Per una storia dei rapporti fra arte e scienza,* Milano 1999

images and forms of purposive way to their specific needs, both for discovery of models used to interpret the phenomena of nature that the invention of expressive languages[15]


A similar comparative study can be obtained between the images, generated for scientific purposes and the potential of developing a new visual culture, or to the ongoing trials in contemporary artistic research, unexpected reverberations could tell that, given the rapid technological evolution in language, we could even fail to capture in the present.

The opportunity demonstrated by our retrospective preliminary investigation leads us to this interest; obviously the amount of "works" both scientific and artistic fonts to be examined would be likely to exceed any discussion here, but we want to propose this theme and methodology, and the specific interest in these iconographic cases of study of research between mathematics and art[16].

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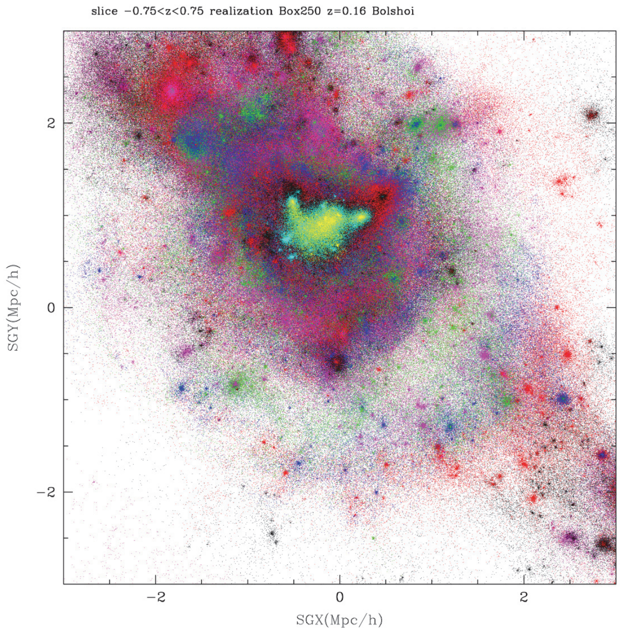
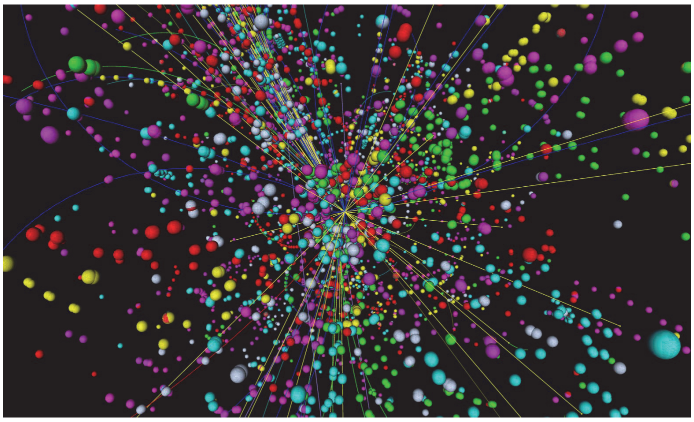
[15] Federico Brunetti, Facoltà del Design Politecnico di Milano, INDACO Department. *Excursus in Aveiro. Digressão em Aveiro. Arte-Design-DESEGNO-Técnica-Ciencia*. Workshop of Drawing and a theorical seminar about the relations between Drawing, Arts and Science. Invited by Associated Prof. Vasco Branco Co-ordinator of the Master Course in Design, Materiais e Gestão de Produto a cura di professor Graça Maria Alves dos Santos Magalhes 16 October 2006, University of Aveiro, Campus Universitário de Santiago 3810-193 Aveiro (Portugal) Department of Communication and Art

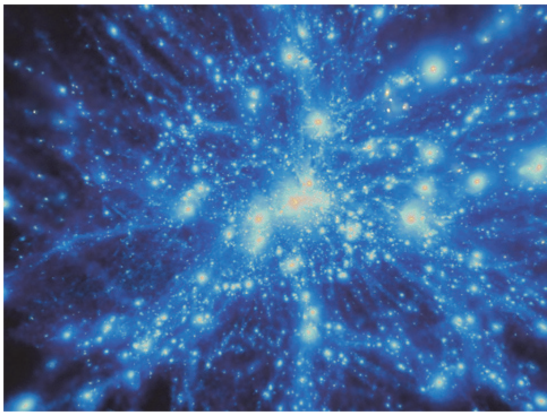
[16] Some work in progress of this interest will be examinated and displayed in: www.sciencecommunicationdesign.eu




Left: Distinctive Features in the Constellation Cygnus A view of Cygnus with the North America Nebula and the smaller Pelican Nebula to its right. The two nebulae are actually part of the same ionized cloud; the dark division between them is a foreground dust cloud. Astronomers have been using the Kepler Space Telescope to search Cygnus for signs of Earth-like planets. www.nytimes.com/interactive/2011/12/02/science/space/20111202-planetscapes.html?nl=todaysheadlines&emc=thab1#28

Right: Dark matter particles in a slice. Particles colored by their velocity, A slice of the whole 250Mpc box. Made by Stefan Gottlober (AIP) with IDL, http://hipacc.ucsc.edu/Bolshoi





Left: Simulation of a detection of Hidden Valley Z' decay into jets in the CMS experiment gen-2007-004_02
Right: Gas density distribution of the most massive galaxy cluster (cluster 001) in a high resolution resimulation, x-y-projection. (Kristin Riebe, PMviewer) http://hipacc.ucsc.edu/Bolshoi

“As T.S. Eliot wrote: *"The mind of the poet is in fact a repository that captures and stores endless feelings, phrases, remaining there until the particles, which can join together to create a new composite are present at any time."* This quote can be applied even to a scientist like me”
Benoit Mandelbrot, 2001

*"The proper and immediate object of science is the acquirement, or communication of truth: the proper and immediate object of poetry is the communication of immediate pleasure."*
Samuel Taylor Coleridge

### *5. " vision"* (∗)

The attention to scientific representations can also be motivated to capture or realize images of this kind are consequent to an earlier phase of "vision" (∗) that does not depend on either result from a process of visual optical perception or otherwise empirical exploration of the phenomenon, but rather come - or even precede - the ability to understand, rationalize and synthesize the multiplicity of factors and data interpretation because of an intuition as to reveal a "form". This ability to "vision" and preconditioning precedes the perception (sometimes at the limit of being able to affect their actual veracity [as in case history Schiapparelli and the *channel /canal* on Mars[17]]): it appears as a sort of clear demand that the objective fact is expected as a sort of response.

Scientific representations lead to the conceptual sharing, and therefore let "perceive" what the mind of the researcher is able to conceive as a vision of a phenomenon or a data system that has been observed experimentally, and that was able to understood as an morpho-logical structure. The "vision" of the investigation process anticipates and directs experimental research and may prelude to the logical sequence of a theory, outlining the elements based on reality, which prefigures the creation of an experiment validating the theory itself. The distinctions between art and science tend to get closer to being able to present more and more coming because of the similarities between the two routes, and visually assimilated by digital platforms processes, which remain at least in appearance methodologically different and distant. To confirm this evolution we can note some contemporary characteristics in actual scientific iconography:

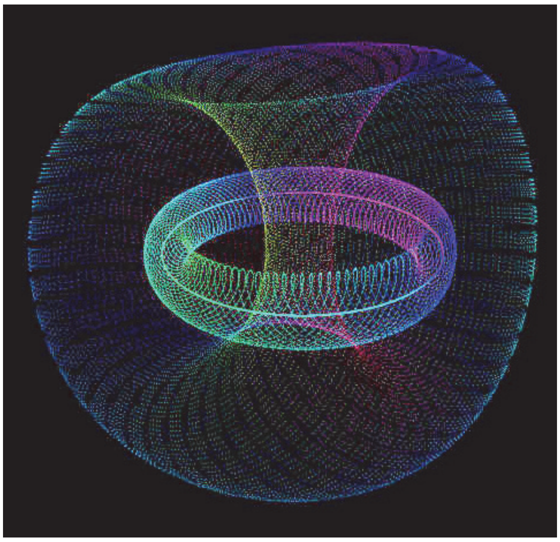
> The ***"algorithmic fusion",*** made possible by digital process, creates images and (virtual?) objects simulating the genesis of natural forms, reaching configurations likely to spontaneous and chaotic morphologies, previously non-deterministically representable, as elusive aspects considered marginal or parasites compared to a definition of art or science understood as "order".
In the light of higher mathematics, these topologies reveals re-generative aspects of nature (real or re-generated) -or neo-natural- and appears as "hyper-realistic artifacts" or "chimeras" open new conventions between the intelligence of the scientist and the re-creation of the artist .

> The ***"introspection of neurosciences".*** The recent possibility to view "motionless" activity of the brain - also made possible by digital processes and the CT (computed tomography) - explores the subtle distinction between the functional "organic" activity of the brain and the "intangible" of thought. This deeper knowledge of the human mind put forth the profound contiguity between what the sciences or humanities have so far described as a linguistic "disciplines" - with conceptual different scenarios, so far "relegated" into specific knowledge - tentatively divided and separated and, at most, sometimes metaphorically suggestive. The mapping of areas of thinking in the human brain - investigation clearly immense but not impossible, as similar genetic research have already shown- is leading to a "visibility"of what was previously presumed as "invisible" (moving beyond

[17] Canadelli, Elena. *"Some Curious Drawings". Mars through Giovanni Schiaparelli's Eyes: between Science and Fiction*. Nuncius, 2009 24(2), 439-464. http://hdl.handle.net/10281/9174 . Giovanni Virgilio Schiapparelli, *La vita sul pianeta Marte*, a cura di Pasquale Tucci, Agnese Mandrino, Antonella Testa, Milano 1998.

the knowledge/power of "science") and re-acquires the creative intuitivity in the field of scientific investigation, previously assumed and delegated to "art".

> ***Reconstruction forecast:*** Through mathematical algorithms and networks with computing power unimaginable until now (GRID etc.), the researchers have forecasting systems of remote events (i.e. remote configurations in the universe [Bolshoi] or simulations (and control) for the apparatus of collisions and decomposition of particles [LHC- CERN] that can both record, than predict, the possible and unknown configurations of theoretical scenarios, assuming their aggregation dynamics: analysing data, or previewing the eventual existence and actual morphology. In view of these forecasting models, the scientist can compare the possible actual data collected as part of confirmation, or deviation, from the assumption in reason of which an experiment was designed; the artist can deal with representations of objects/events space-time impracticable any tangible perception and experience, but no less representative of the concrete reality. The science of *Statistical probabilities* and the *arts of Fantastic* come together around imaginary objects and representations, about which we can not ask so much about “if", but of "when” and "where", we will have the confirmation of existence.

http://hipacc.ucsc.edu/Bolshoi/ [18] http://lhcathome.cern.ch/sixtrack/ - www.isgtw.org [19]

In these directions we enter in a process of understanding of the human and the world in which I think we will always be more likely to meet (in-)voluntary dynamics of creativity in science and the (pro-)vocations of imaginative art, acquiring new technologies and images of science as iconographic messages in the scenario of the perceptible world.

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**18** Visualization of the dark matter in 1/1000 of the gigantic Bolshoi cosmological simulation, zooming in on a region centered on the dark matter halo of a very large cluster of galaxies. Visualized by Chris Henze, NASA Ames Research Center.

19 Particle motion in accelerator is at least four-dimensional, the coordinates being x (horizontal displacement), y (vertical displacement) and the two conjugate momenta. The Figure shows an example of motion in the vicinity of a stable fix-line (thin line), stable motion around the fix-line (small torus) and motion just outside the resonance structure (large torus). The color (rainbow scale) indicates the fourth coordinate allowing for a 3D visualization of particle motion in the four dimensional phase space.
www.isgtw.org/feature/accelerate-physics-your-own-computer

Science and Arts seem to be like two incurable Narcissus who will see one another through the reflection, unstable into the mirror-images of the apparently stagnant water,: thinking of looking only at themselves, but prodigiously remaining two. Nature, calm or stormy, it remains to be observed.

*"The artist in fact is often the inventor or discoverer of science: all these three actors looking for new relationships between man and his world. The relationships discovered by the artist are emotional or cognitive rather than practical. The creative artist does not want, on the one hand, to copy everything that surrounds him, nor on the other hand, shows us through his eyes. He is a specialist who allows us to see in his work, as in a mirror, on our behalf that we were not able to grasp the condition of our soul. He is the outward symbols for feelings that really dominate us, but in reality remain for us only chaotic stimuli, and thus disturbing and obsessive. This is why artists are still needed, despite the difficulties that threaten their place in the modern world (p.422)*

Siegfried Giedion*, Space, time, and architecture: the growth of a new tradition,* 1941
Cambridge, Harvard University Press, 1941 , 1 vol. (XVI-601 p.)

Tr.it *: Sigfried Giedion, Spazio, Tempo ed Architettura. Lo sviluppo di una nuova tradizione, Milano 1984.*

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**References**

***1. The scientific view and the images of research***

[1] Jill Ackerman, *Origins, Imitation, and Conventions: Representation in the Visual Arts*. Cambridge, MA; MIT Press, 2001 Michel Foucault, *Le parole e le cose. Un'archeologia delle scienze umane*, Milano 1988.
[2] GOMBRICH, Ernst Hans, J. Hochberg, M. Black, *Arte percezione e realtà, come pensiamo le immagini,* Torino 1992 Gombrich E.H., *The Sense of Order*, Phaidon Press, Oxford-1979
[3] Gregory, Richard LANGTON, *Occhio e cervello: la psicologia del vedere* - Nuova ed.. - Milano 1991
[4] KANIZSA, Gaetano *Grammatica del vedere: saggi su percezione e gestalt*/ Bologna : Il mulino, 1995.
[5] Paul KLEE *Il pensiero figurativo* 1956; "*Teoria della forma e della figurazione"* 1970

[6] KOSSLYN, Stephen Michael, *Le immagini nella mente. Creare e utilizzare immagini nel cervello*, Firenze 1989.
[7] Erwin PANOFSKY *Meaning in the Visual Arts,* Papers in and on Art History, Garden City, New York 1955 e 1957
[8] M. RIGHETTI, A. STRUMIA, *L'arte del pensare.* Appunti di logica, ESD, Bologna 1998;
[9] SALAM, *L'unificazione delle forze fondamentali. Lo sviluppo e gli obiettivi della fisica moderna,* Rizzoli, Milano 1990;
[10] V.F. WEISSKOPF, *Conoscenza e meraviglia. La descrizione umana del mondo della natura*, Zanichelli, Bologna 1966;


***2. Tree different aspect of communication in scientific iconography. Drawing, images and metaphors***


[1] Federico BRUNETTI, DISEGNO, IMMAGINI E METAFORE NELLA COMUNICAZIONE SCIENTIFICA; in: ComunicareFisica 2007 Trieste, 1 – 6 ottobre 2007 http://comunicarefisica.ts.infn.it/presentazioni/brunetti.pdf
[2] M. DAUMAS (a cura di), *Storia della Scienza,* Laterza, Bari 1969;
[3] T.S. KUHN, *La struttura delle rivoluzioni scientifiche,* Einaudi, · Torino 1969;
[4] M. JAMMER, *Storia del concetto di massa,* Feltrinelli, Milano.1982;
[5] J. MARITAIN, *La filosofia della natura*, Morcelliana, Brescia 1974;
[6] MINKOWSKI, H. *The principle of relativity,* Metheun, 1923. London
[7] M. JAMMER, *Storia del concetto di spazio*, Feltrinelli, Milano 1966;
[8] G. NICOLIS, I. PRIGOGINE, *La complessità. Esplorazioni nei nuovi campi della scienza*, Einaudi, Torino 1991;
[9] STRUMIA*, Il problema della creazione e le cosmologie scientifiche,* "Divus Thomas" (1992), pp. 82-94;
[10] R. THOM, *Stabilità strutturale e morfogenesi. Saggio di una teoria generale dei modelli*, Einaudi, Torino 1980;
[11] C. VON WEIZSÄCKER, *Die Einheit der Natur*, Hanser, München 1987$^3$;
[12] H. WEYL, *La simmetria*, Feltrinelli, Milano 1975; E. SOBER, *Simplicity,* Clarendon Press, Oxford 1975;


***3. Images between science and arts***


[1] Cesare BAIRATI. *La simmetria dinamica: scienza ed arte nell'architettura classica*, Tamburini, Milano, 1952.
[2] F. BERTOLA ET AL. (a cura di), *La bellezza nell'universo*, Il Poligrafo, Padova 1996;
[3] J.D. BARROW, *The Artful Universe*, Oxford Univ. Press, Oxford 1995;
[4] Marco Beretta, *Imaging a Career in Science The Iconography of Antoine Laurent Lavoisier Uppsala Studies in History of Science,* Volume 29, Bologna Studies in Scientific Heritage, Volume *1* 2001, Watson Publishing International LLC,
[5] *"La divina proporzione",* A. C. Cimoli, F. Irace (a cura di), atti del congresso Triennale 1951: *Riflessioni su saggi inediti del congresso del 1951*, Milano 2007;
[6] G. COHEN-TANNOUDJI, M. SPIRO, *La materia-spazio-tempo. La logica delle particelle elementari,* Jaca Book, Milano 1988;
[7] S. CHANDRASEKHAR, *The Beauty and the Quest for Beauty in Science*, "Physics today", July 1979, pp. 25-30; tr.it.: CHANDRASEKHAR, *Verità e bellezza. Le ragioni dell'estetica nella scienza*, Garzanti, Milano 1990;
[8] M. CINI, *Un paradiso perduto. Dall'universo delle leggi naturali al mondo dei processi evolutivi,* Feltrinelli, Milano 1994;

[9] D'Arcy THOMSON, *On Growth and Form*, Cambridge 1917 (ed. ital. *Crescita e forma*, Torino 1992).

[10] EINSTEIN, L. INFELD, *L'evoluzione della fisica,* Einaudi, Torino 1953 (riedizione1999);

[11] L. DE BROGLIE, E. SCHRODINGER, W. HEISENBERG, *Onde e particelle in armonia. Alle sorgenti della meccanica quantistica,* a cura di Sigfrido Boffi, Jaca Book, . Milano 1991;

[12] P.W. BRIDGMAN, *La logica della fisica moderna,* Boringhieri, Torino, 1965;

[13] DIRAC P.A:M, *Principles of quantum physics,* Clarendon Press, Oxford 1958;

[14] DIRAC P.A.M., *The Evolution of Physicist's Picture of Nature*, "Scientific American" 208 (1963), n. 5, p. 45-63;

[15] H. FIRETZSCH, *Quark. I mattoni del mondo,* Boringhieri, Torino 1983;
A. EINSTEIN, *Relatività: esposizione divulgativa*, Boringhieri, Torino 1960;

[16] R.P. FEYNMAN, QED. *La strana teoria della luce e della materia,* Adelphi, Milano1989;

[17] D.R. HOFSTADTER, *GodeI, Escher Bach: un'eterna ghirlanda brillante,* Adelphi, Milano 1984;

[18] H.E. HUNTLEY, *The Divine Proportion. A Study in Mathematical Beauty*, Dover, New York 1970;

[19] S. HILDEBRANDT, A. TROMBA, *The Parsimonious Universe. Shape and Form in Natural World*, Copernicus, New York 1997;

[20] JACQUARD, *L'equazione della Ninfea*, Piemme, Casale Monferrato 1999.

[21] N.I. LOBACEVSKIJ, *Nuovi principi della geometria* (1835-1838), Boringhieri, Torino·1974;

[22] L. MAIANI (a cura di), *Campi, forze e particelle,* Le Scienze Editore, Milano 1991;

[23] B.B. MANDELBROT, *La geometria della natura. Sulla teoria dei frattali,* Theoria, Roma-Napoli 1989;

[24] R. MASI, *Struttura della materia. Essenza metafisica e·costituzione fisica,* Morcelliana, Brescia 1957;

[25] E. NAGEL, J.R. NEWMAN, *La prova di Godel,* Boringhieri, Torino 1982;

[26] W. PAULI, *Collected Scientific Papers*, Interscience, New York 1964;

[27] H.O. PEITGEN, P.H. RICHTER, *La bellezza dei frattali,* Bollati Boringhieri, Torino 1987;

[28] H. POINCARÉ, *Geometria e caso. Scritti di matematica e fisica,* a cura di C. Bartocci, Boringhieri, Torino 1995;

[29] P.A. SCHILPP (a cura di), *Albert Einstein scienziato e filosofo;* Boringhieri Torino 1958;

[30] KOYRÉ, *Introduzione a Platone,* Vallecchi, Firenze 1973;

[31] PAIS, *"Sottile è il Signore ... "* La vita della scienza di Albert Einsten, Boringhieri, Torino1986;

[32] J.J. SANGUINETI, *La filosofia del cosmo in Tommaso d'Aquino,* Ares, Milano 1986;

[33] M. ARTIGAS, J.J. SANGUINETI, *Filosofia della natura,* Le Monnier, Firenze 1989;

[34] T, e I, ARECCHI, *I simboli e la realtà,* Jaca Book, Milano 1990;

[35] G. BASTI, *Il rapporto mente-corpo nella filosofia e nella scienza*, ESD, Bologna 1991;

[36] STEWART, M. GOLUBITSKY, *Terribili simmetrie. Dio è un geometra?*, Bollati Boringhieri, Torino 1995;

[37] STRUMIA, *Introduzione alle filosofia delle scienze,* ESD, Bologna . 1992;

[38] R. RUSSELL, N. MURPHY, A. PEACOCKE (a cura di), *Caos and Complexity. Scientific Perspectives on Divine Action, LEV* and Center for Theology and The Natural Sciences, Città del Vaticano - Berkeley 1995;

[39] LAMOUCHE, *La Théorie Harmonique. Le principe de simplicité dans les mathématiques et dans les sciences physiques*, Gauthier-Villars, Paris 1955;

[40] H. PEITGEN, P. RICHTER, *La bellezza dei frattali*, Bollati Boringhieri, Torino 1987;

[41] J.W. MCALLISTER, *Truth and Beauty in Scientific Reason*, "Sinthese" 78 (1989), pp. 25-51;

[42] J.W. MCALLISTER, *Dirac and Aesthetical Evaluation of Theories*, "Methodology and Science" 23 (1990), pp. 87-100;
[43] SHANKER S.G., *Il teorema di GodeI. Una messa a fuoco,* Muzzio, Padova 1988;
[44] H.U. VON BALTHASAR, *Gloria. Un'estetica teologica*, vol. I: *"La percezione della forma"* (1961), Jaca Book, Milano 1985;
[45] J. VON NEUMANN, *I fondamenti matematici della meccanica quantistica,* II Poligrafo, · Padova 1999. .
[46] Luca ZUCCHI, *Image, Language and Knowledge: Iconophobia and Iconophilia in the Epistemological Debate* in: Centre for Advanced Study (CAS) of Sofia, Bulgaria, 2/20/2006

***4. Iconografia/iconologia***

[1] J.D. BARROW, *Il mondo dentro il mondo*, Adelphi, Milano 1992;
[2] E. CANTORE, *L'uomo scientifico. Il significato umanistico della scienza*, EDB, Bologna 1987;
[3] Martin KEMP, *The science of art. Optical themes in western art from Brunelleschi to Seurat*, NewHaven and London 1990; tr. It: Martin Kemp, *Immagine e verità. Per una storia dei rapporti fra arte e scienza*, Milano 1999.
[4] Richard MASON, with Tony Morphet, and Sandra Prosalendis, *Reading Scientific Images, the iconography of evolution,* HRSC Human science research council 2006

***5. " vision"***

[1] F. BERTELÈ, A. OLMI, A. SALUCCI e A. STRUMIA, *Scienza, analogia, astrazione. Tommaso d'Aquino e le scienze della complessità,* Il Poligrafo, Padova 1999.
[2] Wilhelm F. BOYENS, Ulrike Mertens, *Visionaries and Achitects,* Egon Zendler International, 2001Dusseldorf.
[3] Mari-Haude CARES Nicole Marchand-ZANARTU, *Image de pansée* , Reunion de Musées Nationaux, Paris 2011
[4] Albert EINSTEIN, *Autobiografia scientifica,* con interventi di Wolfgang Pauli, Max Born, Walter Heitler, Niels
[5] BOHR, Henry MARGENAU, Hans REICHENBACH, Kurt GODEL, Titolo originale dell'edizione integrale: *Albert Einstein: Philosopher-Scientist,* a cura di Paul A. Schilpp, The Library of Living Philosophers, Evanston (Ili.) I949; Edizione integrale, *Albert Einstein scienziato e filosofo,* nella Biblioteca di cultura scientifica 1958 Ristampa marzo 2007
[6] *Arte+Scienza, nuove tecnologie dell'immagine;* a cura del Prof. Tommaso Trini. 1996; Milano, Accademia di Brera: Seminario nazionale di studio Partecipazione al convegno e coordinamento della sezione: *"Genesi e procedure di archiviazione dell'immagine digitale",* di A. De Andreis;
[7] GIBSON, James Jerome*, The ecological approach to visual perception* - London 1986. -
[8] SIEGFRIED Giedion, *Space, time, and architecture: the growth of a new tradition,* 1941; Cambridge, Harvard University Press, 1941 , 1 vol. (XVI-601 p.) Tr.it : Sigfried Giedion, *Spazio, Tempo ed Architettura. Lo sviluppo di una nuova tradizione,* Milano 1984.
[9] MINKOWSKI, H. *The principle of relativity,* Metheun, 1923. London.
[10] WILLIAM J. MITCHELL, *The Reconfigured Eye. Visual Truth in the Post-Photographic Era* 1997 MIT Press Aldo Montù, *Fi = 1.61803398875... Appunti e annotazioni su Natura e geometria: progressione aurea e forme pentagonali*, Melocchi, Milano 1970.

[11] R. PENROSENROSE, *Ombre della mente. Alla ricerca della coscienza,* Rizzoli, Milano 1996;
[12] William R. SHEA, *Science and the Visual Image in the Enlightenment* Watson Publishing International LLC 2000,
[13] Roelf van STRATEN, *Introduzione all'iconologia,* ed. It a cura di Roberto Cassanelli, Milano 2002
[14] E. TIEZZI, *La bellezza e la scienza. Il valore dell'estetica nella conoscenza scientifica*, R. Cortina, Milano 1998;
[15] Van FRAASSEN, B. C., *An introduction to the philosophy of time and space*, Random House, New York 1970.
[16] Van FRAASSEN, B. C., *The Scientific Image*, Oxford University Press, Oxford 1980.
[17] Van FRAASSEN, B. C., *Laws and Symmetry*, Oxford University Press, Oxford 1989.
[18] Van FRAASSEN, B. C., *Quantum Mechanics: An Empiricist View*, Oxford University Press, Oxford 1991.
[19] Van FRAASSEN, B. C., *The Empirical Stance* , Yale University Press, New Haven and London 2002.
[20] Van FRAASSEN, B. C., *Scientific Representation: Paradoxes of Perspective*, Oxford
[21] S. ZECCHI, E. FRANZINI, *Storia dell'estetica*, 2 vl., Il Mulino, Bologna 1995;
[22] Bruno ZEVI, *Saper vedere l'architettura. Saggi sull'interpretazione spaziale dell'architettura*, Torino 1989.

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# SERGIO MUSMECI'S " FORMS WITH NO NAME" AND "ANTI-POLYHEDRONS"

**CAPANNA Alessandra, (I)**

**Abstract.** SERGIO MUSMECI (1926-1981) is one of the most daring and unconventional engineers born in the twentieth century; he was master equally of music, astronomy, aeronautics, mathematics, and philosophy, all of which informed his structures, whose shape was determined by the spatial distribution of static actions. Musmeci thought that he could reach the expression of "modernity" through the Science.
His activity is the answer to those who believe that the studies concerning the relationships between architecture and mathematics are much too theoretical.
His works are substantially of two types.
The first one is referred to a "geometry of the continuous", as in the constructive technique of the lightweight stressed skin structures and improved in what he called "forms with no name" elaborated for the bridge on the Basento river. The forms he conceived can be truly defined as three-dimensional because they are endowed with a different kind of curvature and orientation in space, in every point. As a consequence, the construction of spatial forms can be directly connected with the concept of structural minimum.
The other subject of Musmeci's researches concerns those aggregated structures that are the expression of a geometry of discontinuity, represented by the crystallographic conformation of trussed structures. In 1979 Musmeci exhibited full-size models representing various aggregated structures illustrating those "enigmatic and sharp space frames", geometrically constructed through the same formative process of regular and irregular polyhedrons and their reciprocal transformation from one into the others. His studies on polyhedrons culminated with the definition of the anti-polyhedron, a potentially unlimited and undetermined geometrical figure, even though it was generated from regular figures.

**Key words and phrases.** Architecture Mathematics Polyhedrons Structural minimum

*Mathematics Subject Classification:* Primary 00A67 52B99; Secondary 97M80 01A70

## 1. Genesis. Looking for the shape, this is the question.

*Awareness means to unveil the (correct) shape.*

Musmeci, explaining his aesthetic and philosophical theories, illustrated the scientific and mathematical basis of his researches bringing to light the historical origins and the processes for obtaining ever greater knowledge from which his architectures come. His point of view was that science is a tool for ideas, thought and invention, therefore for design, and it is not only a means to verify structural strength. Science must lead us to discover the best geometry for that particular static (or dynamic) condition, without using surplus of materials analyzing stress limit states of the structure. As a matter of fact, he began this research with the solution of a scientific problem in which he was engaged since he was a student: the determination of the arch-limit shape. *"Its equation is y = log cos X (a part from some multiplicative constants depending on the resistance of building materials)"* he stated, and it looks like a very extended parabola. This curve has some very interesting properties, particularly that the angle between the median axis of the parabola and the horizontal line is proportional to its abscissa, that is, the distance from the vertical axis. The limit span of this arch is that distance corresponding to a 90° angle.

## 2 Strains are not unknown. Structural minimum and its geometry.

Musmeci thought that the building science, so careful and accurate in checking structural safety, was not enough to find the correct figurative solution; he believed that «strains are not unknown (quantities)», the real unknown factor is the shape of a structure, so, strains must be calculated from the specification of their shape on.
On the contrary, he accounted as belonging to the past Nineteenth century period even those beautiful shapes born by Pier Luigi Nervi's and Riccardo Morandi's genius. During that period the construction of ribbed structures conforming domes and vaults was obtained through automorphic transpositions of symmetries through the rotation or the translation of plane figures, so that one can consider those volumes almost two-dimensional because it is their generative matrix. He had the positive ambition to create shapes which have not yet named and to conceive structures in which each portion had its specific shape. Forms that can be truly defined as three-dimensional because they are endowed with a different kind of curvature and orientation in space, in every point, in view of the fact that in space the intensity and the direction of strength at each point will be different. Different forms for different situations, that is complex forms of modernity.

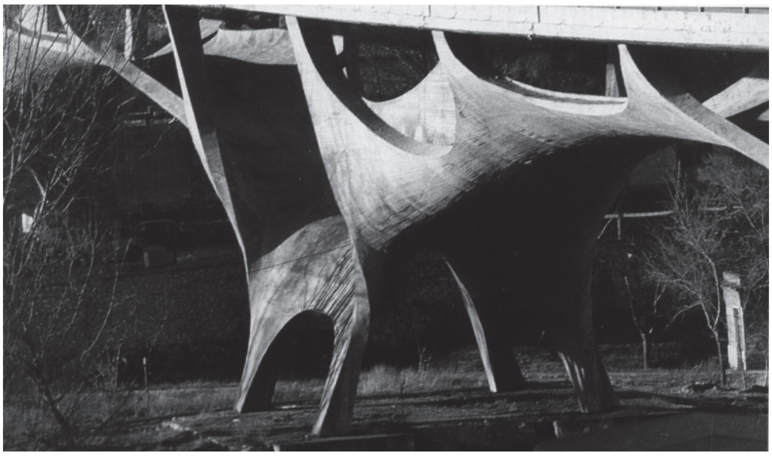


Fig.1 – Bridge on the Basento river – historical photo

So is the bridge over the Basento river, the main entrance to the city of Potenza. It looks like a thin line slightly inclined to the city, supported by a complex three-dimensional shape, and while holding the belt road, rotates and moulds itself, and finally settles to the ground almost disappearing in the vegetation. The concrete structure is modeled after an alternation of concave and convex to

form four arches, alternately slightly touching the ground or the floor under the driveway, as leaning on the tips of the fingers of one hand.
Musmeci stated that his bridge on the Basento river was not settled like an arch (in terms of strucural template), but like a vault, or better saying an equally compressed reinforced concrete membrane, 30 cm thick (excluding ribs along the edges) that was intended to be used as a walkway for pedestrians, meeting requirements about minimal structure, a mathematical and technical problem he examined at the end of the book *La statica e le strutture*, he edited in 1971.

## Minimal surfaces. Structural minimum and its geometry.

In mathematics we know that minimal surfaces are surfaces with a mean curvature of zero, including, but not limited to, surfaces of minimum area subject to various constraints that is minimizing surface area.
The project for this particular bridge is a development of previous designs: for "Tor di Quinto" viaduct in Rome, on the Tevere river, and for a viaduct in Calabria.
For the Basento bridge the minimal surface is unique and develops for a length of 280 mt.
"*The shape is determined by outlines designed in such a way as to exclude any bending stress. The calculation results have been checked with models consisting of soap blow bubbles and highly stretched rubber and finally verified with static tests over a metacrylate model and over a micro-concrete model constructed in Bergamo by ISMES (Experimental Institute for Models and Structures)* (in *Sergio Musmeci*, Pordenone 1979 p. 22).

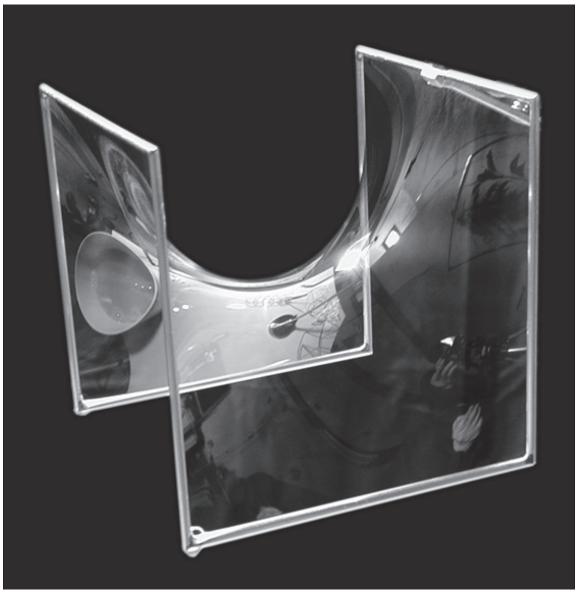


Fig. 2 – Basento river soap blow bubble check it is the minimal surface, set from this peculiar edge frame.

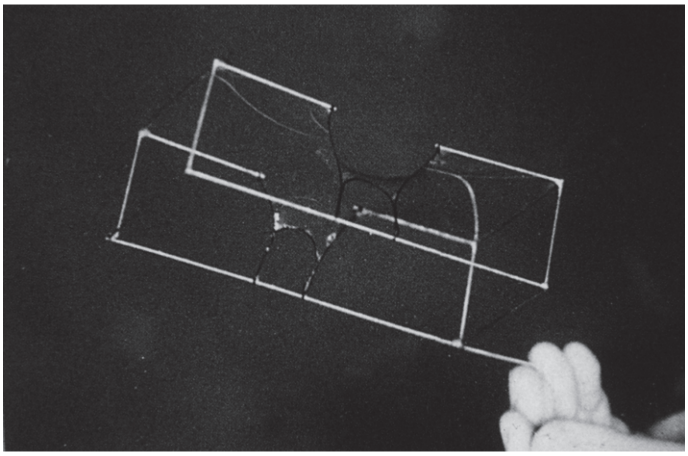
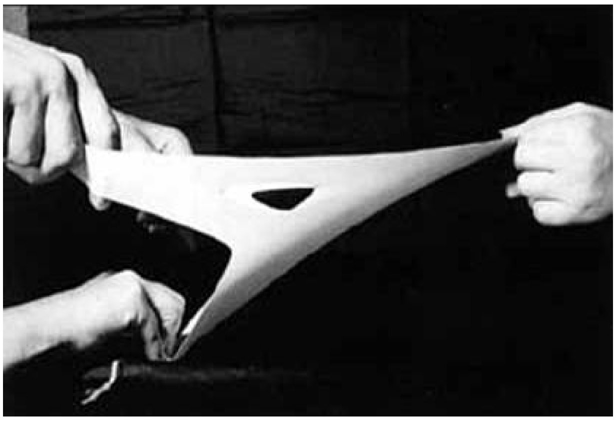


Fig. 3 – Scherk surface shaped with a soap film:

The whole structure, with its 4 spans of 70 mt. each, crosses the tracks of Potenza railway station, the river and two roads inside the industrial area near the city and resisted Irpinia and Basilicata huge earthquake in 1980 November 23 (6,89 $M_w$ - *Richter magnitude scale*)



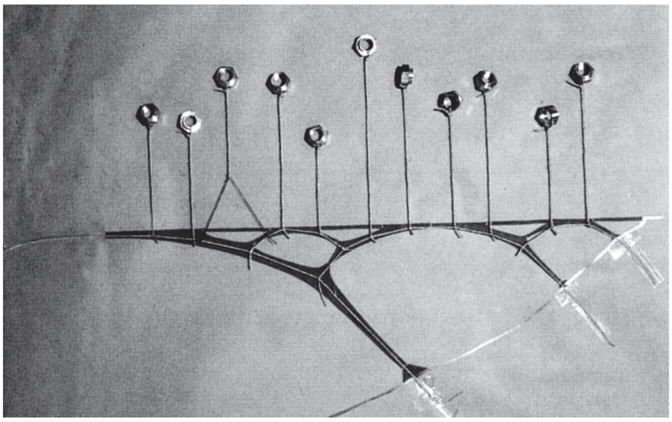
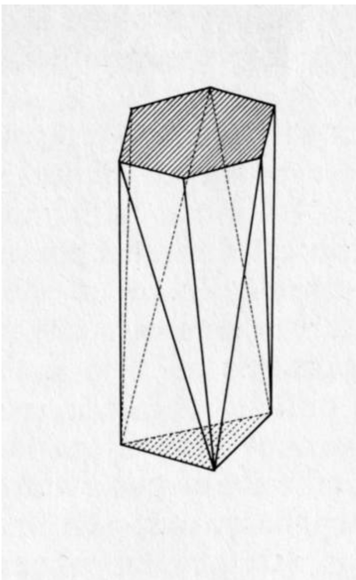


Fig. 4 e 5 – Study-models for Tor di Quinto viaduct (1959) and for the bridge over the Astico river(1956)

## 3 Intriguing polyhedrons. Origins of spaces frames.

The other subject of Musmeci's researches concerns those aggregated structures that are the expression of a geometry of discontinuity, represented by the crystallographic conformation of trussed structures, consisting of a set of framework that follows, in strict accuracy, a structural space arrangement according to which usually identical elements are assembled. Trussed structures are the best known among discrete systems in architecture; they consists in a set of bars whose modulus is a prism. The most widely used, because its *non-deformability,* is the one presenting triangular faces. These space frames are light and flexible and generally work better than other structural systems. Musmeci began to study this system with the aim to test an innovative material, provided with high resistance to the strains. A particular type of concrete produced with the addition of synthetic compound polymer. In addition his intuition to try out some new configuration of the most important and complex parts of the trussed structure being the nodes, produced the remarkable result. Musmeci decided to eliminate the nodes replaced by empty polyhedrons upon whose immaterial faces bars final sections converge. Fixed joints soldered together with concrete.
The most important characteristic is that the shape of any single bar arises from the condition that each node does not involve coupling of elements not related with the whole geometrical shape.



These kind of structures can be arranged in various way depending on the polyhedron node (in Musmeci’s lexicon: “poliedro di nodo”) from which they are originated.
Regular polyhedrons in 3d space are only five (tetrahedron, cube, octahedron, dodecahedron and icosahedrons): as Lewis Carrol said, it is a small number so provocative. Could it have to do with the fact that perfection is so rare? Musmeci discovered that there are only three regular polyhedrons (tetrahedron, cube, octahedron) and two semi-regular (triangular bipyramid and rhombic dodecahedron), among the 13 known in 3d space, fitting the role of the node. Another intriguing small number indeed. He also noted that polyhedron node can join both prismatic and anti-prismatic bars
If in geometry, a n-sided antiprism is a polyhedron composed of two parallel copies of some particular n-sided polygon, connected by an alternating band of triangles, in a framework it is composed of prismatic or anti-prismatic identical bars (Fig 6)

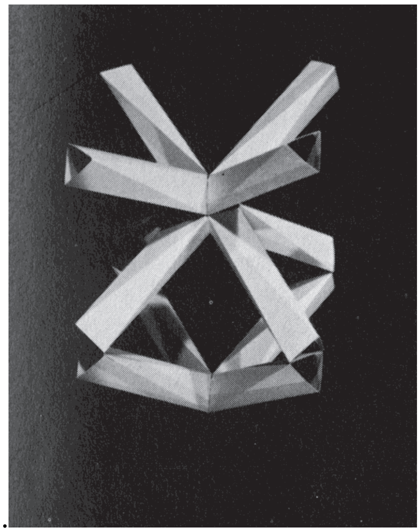


Fig.7 – Detail of octahedral framework system with anti-prismatic bars.

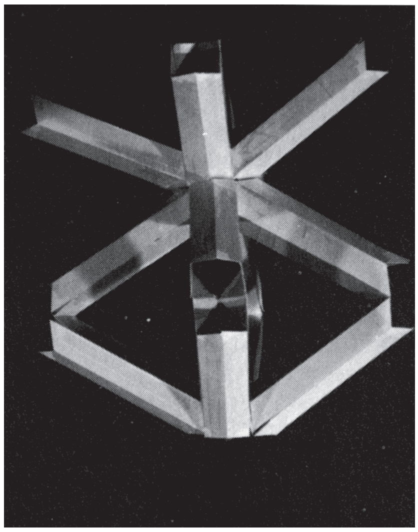


Fig. 8 – Detail of octahedral framework system with prismatic bars.

Antiprisms are similar to prisms but the bases are twisted relative to each other, and the side faces are triangles, rather than quadrilaterals.

**Anti-polyhedrons. Musmeci's definition.**

Musmeci called anti-polyhedron all those structures made with vertex, edges and faces just like regular polyhedrons, but supplied with an excess of polygons around each vertex. *As a matter of fact frameworks made with prismatic or even anti-prismatic bars are anti-polyhedron (...) In Bukminster-Fuller dome, for instance, each node has 6 bars except for few that has 5 (those icosahedrons originating the volume of the dome. (...)In the same way, if we insert 7 bars in anti-polyhedrical nodes ... we can obtain the generative points of the inversion of curvature. And if anti-polyhedrical nodes have increasing reciprocal distances starting from the origin of that specific point, we can obtain a framework shaping a saddle surface like a Hyperbolic paraboloid* ("Parametro" n. 80, 1979, p. 30)

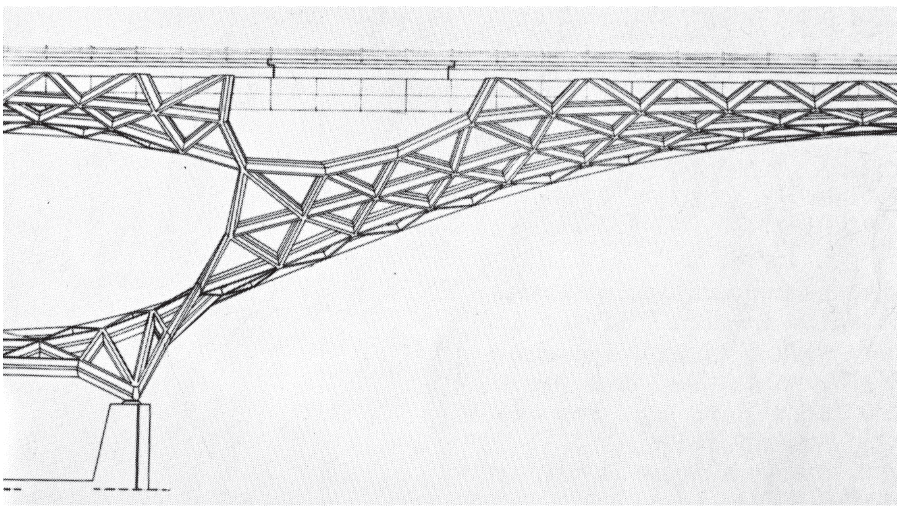


Fig. 9 bridge over the Niger river

**Roma 1979. Piazza S. Salvatore in Lauro. Exhibiting space frames.**

In 1979 Musmeci exhibited full-size models representing various aggregated structures illustrating those huge, enigmatic and sharp space frames, that are systems geometrically constructed through the same formative process of regular and irregular polyhedrons and their reciprocal transformation from one into the others. His studies on polyhedrons culminated with the definition of the anti-polyhedron, a potentially unlimited and undetermined geometrical figure, even though it was generated from regular figures.
At S. Salvatore in Lauro square, in the very middle of the city of Rome, Musmeci exhibited the only five possible spatial systems with the joint shaped voids - just described - and bars all equivalent. According to Musmeci, the significance of the system was in determining the so called "coordination number" of each of the five systems, that is the maximum number of bars meeting in a node (or joint). The easier is the cubic system, whose "coordination number" is 6 and one of the more complex is the rhombic dodecahedic with "coordination number" = 12; the octahedric system ("coordination number" = 8) the nodes are situated on the vertex of an ideal cube whose diagonals are antiprismatic bars. The same characteristic has the tetrahedric system whose nodes have the same spatial disposition of carbon atoms in a diamond crystal. So it is the shape for the project of Oikos pavilion in Bologna (work completed by his wife Zenaide Zanini after Musmeci's death).

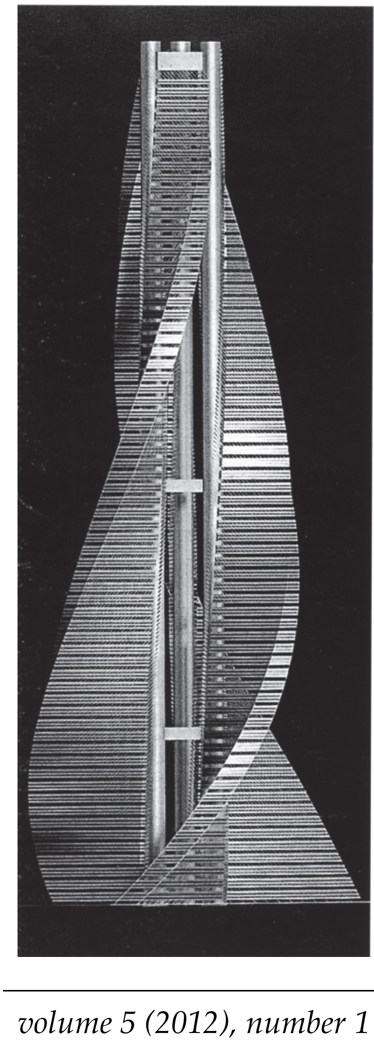
Fig. 10 – Tetrahedral framework system with anti-prismatic bars.

## 4 Consequences, (partial) conclusions and open questions.

To sum up, any structure can belong to one of those two main systems: the continuous or the discontinuous one, but also to a combination between them The former are those structures made with concrete or else with plastic materials, the others are trussed structures and space frames.
Musmeci applied the minimal structure concept to both, as we can realize in the bridge over the Basento and in the space frames.
The fact that in Musmeci's researches the role of geometry is no more exploited as an instrument for controlling architectural form, but for its liberation, is clearly highlighted in the projects presented, but also - and even much more - in the project for an helicoidal skyscraper which consisted of a succession of warped wings developed on the layout of the logarithmic spiral. It is a structure which considers the use of high-limit steel cables placed in accordance with helicoidal spirals in the space. It has an aerodynamic form and structure inspired by high technology, but it also assumes a strong organic metaphor referring itself to the extremely effective endurance due to the relationship between bone and muscle: a compressed central core and external tensioned fascias. Three warped wing anchored to three 565mt-high mast elements define a form minimizing the wind load.

**A rose is a rose is a rose. Back to Aristotle's Law of Identity?**

Fig. 11 – Helicoidal skyscraper



Everything that exists has a specific nature. Each entity exists as something in particular and it has characteristics that are a part of what it is. "This paper is about Sergio Musmeci's structures and is 8 pages long."
To have an identity means to have a single identity; an object cannot have two identities. Or not?
In the particular Musmeci's figurative researches generated from one subject matter: minimal surface, one can say that a singular condition can generate one and only one entity/identity, such as for the skyscraper, but also that different conditions (continuity of concrete structures and discontinuity of trussed frameworks) can reach the same conclusion, each with its identity. Are they or not very close one another, if we imagine the framework of the bridge over the Niger river, covered with the foreseen precast shell, to the Basento bridge?
And if in the Aristotelian logic, the law of identity is the first of the so-called three classic laws of thought that states: if an object is the same as itself: A $\rightarrow$ A (if you have A, then you have A), then Gertrude Stein's *a rose is a rose is a rose* meaning "things are what they are" is a popular version of the statement of the law of identity.
Deleuze's main philosophical project in the works he wrote prior to his collaborations with Guattari can be briefly summarized as a systematic inversion of the traditional metaphysical relationship between identity and difference. Traditionally, difference is seen as derivative from identity: e.g., to say that "X is different from Y"

assumes some X and Y with at least relatively stable identities. To the contrary, Deleuze claims that all singular identity is effect of difference. Does it means that

*When all is said and done, a thing is what it is*

**Suggestions required**

**Selected bibliography**

There are very few publications about Musmeci, so Musmeci's research is often hardly known. With the exception of a very rare book edited by Sergio Musmeci himself in 1979 and of a monographic number of the italian review "Parametro"

[1] *Sergio Musmeci*, Pordenone 1979

[2] *Sergio Musmeci* o *delle tensioni cognite*, in "Parametro" n. 80, 1979 (monographic number)

and some interesting presentation of single projects on magazines,
three more hints:

[3] M. Nicoletti, *Sergio Musmeci.. Organicità di forme e forze nello spazio*, Torino 1999 (brief monography)

[4] M. Guccione (a cura), *Il ponte e la città. Sergio Musmeci a Potenza*, Roma 2004

[5] R Capomolla, *Le "forme organiche strutturali". Materia e spazio nelle opere di Sergio Musmeci*, in "Rassegna di architettura e urbanistica", n.121/122, gennaio-agosto 2007, pp.135-148 e 171-172

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# EVOLUTION OF THE GEOMETRY THROUGH THE ARTS

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**Abstract. -** We deal with "*geometrical transformations*" which appear in the development of architecture. We propose a teaching approach which is subsidiary to the treatment of "*Euclidean geometry*" in the High-School. We move from *prospective*, as a tool to represent in the plane objects from the 3-dimensional space, and consider homological transformations as mathematical characterization of Representation.

**Key words.** architecture, art, projective geometry, perspective, history

*Study Science first, and then follow the practice born from science itself: those who fall in love with practice without the diligence are like sailors on ships that enter the sea without rudder or compass, that never know where they are going.*
*Always, the practice must be built on a good theory, which the PROSPECT is guide and door: without it, nothing you do well, so in painting as in every other profession.*
*Leonardo da Vinci*

## 1. Iroduction

***Perspective*** is widely considered the theoretical foundation of pictorial art, from the Latin word ***perspective*** (optics). The ***Perspective*** method, in geometry, is among those used to represent figures of the space above a plane.

The representation of the technical details of the figure occurs using the ***Descriptive geometry***; it's up to the ***Projective geometry***, instead, to represent the transformations the figure is subjected to, with the operations of ***projection*** and ***section***.

The first ones who tried to solve some representation problems were probably Greek artists and it's not to be discarded that in the Classical Age it would have been existed a representation system not so different from what was developed during the Renaissance. It seems excluded, instead, any study for spatial definition that is mathematical and uniform, in prehistoric figures.

It's in the Greek age that was born a study group, whose results are "***the Optics of the ancients***", from the desire to study the light phenomena, for the purpose to discern "*appearance*" from "*reality*".

Starting from the postulate "*light travels in a straight line*", many theorems are set (most of them by Euclid) that are still considered among the foundations of the mathematical study of light;

other propositions are judged unacceptable because they are consequent upon the principle – claimed by Plato but rejected by contemporary physics – that vision happens because of rays traveling not from the object observed but from the eye of the observer.

Already in a XII century work, translated from arabic to latin by ***Gherardo da Cremona*** (1114, 1187), but published by ***Pietro Rama*** (1515, 1572) in 1572, we meet for the first time the idea (later on admitted by all) that "***from each point of an illuminated body, rays start traveling in all directions, so that the eye of an observer (whose pupil becomes center of a perspective cone) is center of a star of rays directed towards all the points of the objected observed***".

Supposing, then, that between the eye and the object is interposed a transparent surface (*picture*) where we can identify the intersecting points with each ray of light, we will see on the picture a set of points that will have on the observer the same impression of the object itself: this is the **perspective** that is, then, "*the art of represent objects upon a picture so to hold its external appearance*" and it is the theoretical foundation of pictorial art.

## 2. The first rules of Descriptive Geometry

Evidence of the ancient Greeks interest for the representation as a basis for pictorial arts are some writings of ***Vitruvio*** (Marco Vitruvio Pollone, lived probably in the I century b.C.), who can be considered the most meaningful essay author in the latin world.

About Vitruvio we know little, it is even questioned about the authenticity of his work "***De Architectura***" (27 b.C.) where he describes the Basilica of Fano, of which he could have been the builder (I chap. V book).

During the Renaissance, the *De Architectura* of Vitruvio was considered a model by all the essay authors in the Architecture field, who obtained knowledge and often followed drafts and standards from that.

In the I book of the work, it's outlined the shape of the architect and the limitations of Architecture as a science; in the II book is developed a brief history of Architecture and in the following books (10 in all) the treatise becomes more and more analytical and technical.

It's in the XII century with the ***Architectura Gothica***, however, that we begin to glimpse a principle of representation more strictly rational; *the main problem set by gothic Architecture was to achieve the highest brightness possible and the widest roominess with the smallest bulk of structures and wall masses.*

Already in the XI century the makers of the Ile de France thought to increasingly remove from buildings and churches every inert mass, that date back to Romanesque art, in order to embrace the pointed arch, which allows to soften the lateral loads. With the gothic architecture it was accentuated the sense of verticality and the space was configured in an undefined shape, so to offer, through the complexity of plans with more aisles and radical chapels, ***ever changing perspectives*** under the different actions of light, often passing through stained glass windows.

Therefore the gothic cathedral, around which flourished generations of builders and decorators, became an extremely logic building, into which are embossed all those parts actually having static function, almost a beam of forces without inert mass. Gothic character was given to the Fossanova and Casamari abbeys, to the church of Servants in Bologna and to the San Francesco d'Assisi.

Then, we discover the first step to leave behind the medieval scheme in the works of ***Giotto*** (Colle di Vespignano, 1226 – Firenze, 1337), ***Duccio di Buoninsegna*** (whose birth year is still unknown but from his works it's deduced the time he worked, in Siena, between 1278 and around 1318) and, some decade later, in the works of ***Ambrogio Lorenzetti*** (birth date unknown but it's believed he died in Firenze or Siena during the plague of 1348), where it becomes clear *the study to define the space holding the several elements of the representation.*

But the adoption of a precise method of ***LINEAR GEOMETRIC PERSPECTIVE*** dates back to 1400, with the Florentine ***Filippo Brunelleschi*** (Firenze, 1377-1446), who first looked at the rules of ***Perspective*** and opened a new era that symbolized the beginning of the Renaissance.
The first works of Brunelleschi, around 1400 or so, didn't let foresee the revolution that would be later put in place, because these works were still much in the gothic tradition, as evidenced by the busts and silver statues for the altar of S. Jacopo, in the Cathedral of Pistoia, realized around 1400.
Is with the Crucifix of Santa Maria Novella (1409, or perhaps later) that we start to see a new work for the perfect proportions and for the symmetric distribution of parts, because during that period Brunelleschi was studying the means to make clearly represented (and thus rationally knowable) the bodies of space: these means he found them in the ***Perspective*** (the evidence is in the ***perspective views*** of S. Giovanni and of Palazzo Vecchio) of which he set the rules.
Along the same lines expressed ***Leon Battista Alberti*** (Genova around 1404 - 1472) who also dedicated to Brunelleschi his treatise "***De Pictura***" in which he wrote that *in the florentine art of those years we already saw the passing of the works of antiquity*. His thinking is based on the concept of Perspective by Brunelleschi, which was posed again more clearly and precisely, exploiting also the knowledge of mathematics and philosophy, as evidenced by the great architectural work "***The Temple of Malatesta of Rimini***".
Unlike Brunelleschi that followed step by step the works he had planned, Alberti, convinced that the architect should not have a crafter task, entrusted to others the realization of his drawings and plans.
Leon Battista Alberti also spread among the painters of his time, the procedure (perhaps already known to the ancient Egyptians), which consists of using a square grid to reproduce a given design in a different scale; it is a procedure based on the concept of similitude and in which we can find a first approach to analytic geometry and projective geometry that will be developed two centuries later.
The genius and culture of Alberti also manifested in the literary and pedagogical works; he can be defined *the scholar of the XV century's art*, almost in opposition to another great artist of the same period, ***Piero della Francesca*** (Arezzo, around 1415 - 1492), that represent *the mathematic of the XV century's art*, so that Giorgio Vasari (1511–1574) in the "*Lives of most excellent sculptors, painters and architects*" wrote that "*Piero never draw back from mathematics in which he had been held a rare master, so that the books deservedly have acquired him the name of the best building surveyor that should be in the dais of him*".
In the decade 1470-1480, Piero della Francesca wrote a complete treatise on Perspective, "*De perspectiva pingendi*" where, by applying the concept of Alberti's "perspective of a body", he used some processes that only in today geometry find their complete descriptive development and he was the first to use, for drawing the prospect of a solid bounded by a curved surface, the corresponding sections of a convenient set of plane sections; therefore, before it was established and perceived the concept of *envelope of a family of lines in the plane*, he saw that all the curves born this way were tangent to a given line, that is the projection of the outline of the considered surface.
In this group of eminent italian painters-surveyors emerges, then, Leonardo da Vinci (Firenze, 1452-1519) himself also convinced of the idea that painting should have a scientific foundation, as shown in Chapter VII of his work *Treatise on Painting*, which reads as follows: *those who fall in love with practice without the diligence (the science) are like the mariners on ships that enter the sea without rudder or compass, which would never be sure where they are heading*.
According to Vasari, Piero della Francesca and Leonardo da Vinci were two antithetical personalities: on one side Leonardo as "*wonderful and heavenly*" as "*varied and unstable*" passing from the frescoes to the mirrors, from the anatomy to the machines, on the other hand Piero,

rational, systematic, which *builds theorem after theorem, problem after problem, his many books* written in a style very close to that used by Euclid.
And it's in the wake of Piero della Francesca that in the XVI century the perspective is passed from the hands of the artists to those of scientists, mainly thanks to the work of an eminent commentator, ***Federico Commandino*** (Urbino, 1509-1575). In a treatise on *Linear Perspective* written with the intent to establish the principles upon which it's based the projection method due to the greek astronomer Claudio Tolomeo (II century AD.) and that we now call "*stereographic projection*", Commandino supposed to refer all the figures seen to two planes orthogonal each other, the horizontal and the vertical (with the language of architects the *plan* and the *elevation*) and to assume as a framework a plane perpendicular to both and as a point of view a point on the vertical plane; he then imagine that the framework is reversed on the vertical plane by rotating around its intersection with the plane itself, so as to fix also the joint elements: *this is a first idea of the composition of two perspective operations that, with the development of projective geometry, will lead to the concept of homology*.

## 3. Contribution to Perspective by foreign authors

Meanwhile, the teaching of the great Italian artists, from Brunelleschi to Alberti, from Piero della Francesca to Leonardo da Vinci, was assimilated by other non-italian painters, among whose a place of particular importance in the history of mathematics is surely due to the german artist *Dürer* (1471-1528) who, in his country, has taught many things related to Perspective, probably learned from Piero della Francesca, during a long stay he made in Italy.
The contribution of the Belgian and Dutch to the Perspective is due to some mathematicians, the most significant of which are *Simon Stevin* (Bruges Hague 1548-1620) and *Francesco D'Aguillon* (Bruxelles 1566-Anversa 1617), even if, already at the beginning of the XV century, the Dutch painter *Giovanni Van Dick* (Maastricht 1385-Bruges 1440) had shown to know and to use the *vanishing point* of a system of straight lines parallel to each other, while the other Dutch painters learned of its existence only two centuries later by Italian writers of Perspective.
The belgian Francesco D'Aguillon was inspired in his work to a student of Commandino, *Guido Ubaldo dei Marquis del Monte* (Pesaro 1545-1607) author of the treatise "*Perspective*", written in the purest Euclidean style and therefore rightly *considered one of the classics of mathematical literature.*
In France, we must remember the painter and sculptor *Giovanni Cousin* (1500-1590) with his treatise "*The Livre de Pourtraiture*", but new horizons for the geometry are outlined some decades later with *Girard Desargues* (1591-1661), considered the father of the general method for *descriptive geometry* and the precursor of *projective geometry*. Using the *coordinates method* in the time in which *René Descartes* (1596-1650) and *Pierre de Fermat* (1601-1675) were developing *analytic geometry*, he suggested a new analytic method of construction that is also at base of the concept of axonometric.
The Desargues completed his treatise with a work in 1639 in which inaugurated the *method of central projections*, which introduced for the first time the concept of *point at infinity*, laying the foundations for the development of *projective geometry*, ie, *the discipline that studies the properties of figures that aren't altered by projection and section*.
Projective geometry was developed, then, in the next century, first with *Gaspard Monge* (1746-1818) and then with his pupil, *Jean-Victor Poncelet* (1788-1867) to find his rigour in 1872 with "*The Erlangen program*" of Felix Klein (1849-1925) who, appointed professor at the University of Erlangen, published his "*Program*" where he looked upon the *geometrical properties of the figures*

*with respect to transformation groups*. This consists of applying the "*Group Theory*" (in the arrangement given later by Jordan), to geometric theories.
And it is in *projective geometry* that it is highlighted the importance of the *points at infinity* and the analogy between *points* and *lines* expressed by the *principle of duality*.

## 4. The "projective geometry" in the High School

### *4.1 - Projection and section*

We define projection of a point P on a line r, from a projection center S, the point P' ≡ SP ∩ *r*. (fig. 4.1).
If the segment SP is perpendicular to r, the projection is called orthogonal. The point P' is also known as a section of the SP with the straight line r.

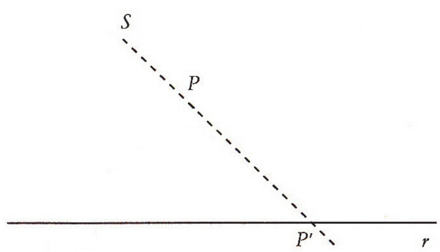


fig. 4.1

### *4.2. Perspectivity between straight lines: limit point, point at infinity*

Given two straight lines *r* and *s* of the plane and a point S outside both *r* and *s*, we define the *perspectivity center S between the straight lines r and s* as the correspondence that matches each point P ∈ *s* to the point P' ≡ SP ∩ *r* (ie the projection of the point P of r from the center of projection) (fig. 4.2).
If *s* is parallel to *r*, the correspondence is complete and bi-univocal since the half-line that connects the center of projection S with any point of *s* intersects certainly the line *r*, and vice versa.
If *s* impacts on *r* at a point U, it can be observed that:

1) to the point U corresponds itself: U is called fixed point.
2) there is a point I∈*s* such that the line IS is parallel to r, so the point I doesn't correspond to any point of r.

It's possible to complete the bi-univocal correspondence between *s* and *r*, associating an element $J_\propto$ to the point I of r, which is called *improper point* or *point at infinity* and is identified by the direction of the straight line r.
In fact, if we *consider* the perspectivities centered in S between the line *s* and any parallel to *r*, we can see that to the point I∈*s* still corresponds $J_\propto$ (fig. 4.3).

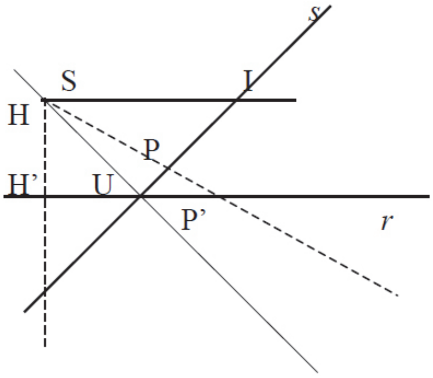


fig. 4.2

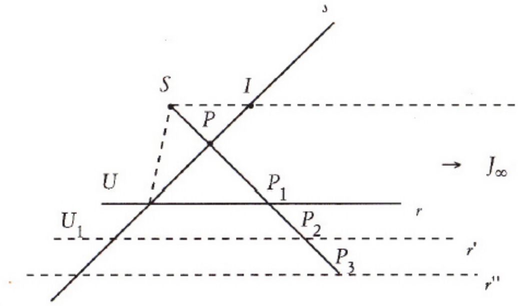


fig. 4.3

The point I∈*s* is called *limit point* in the perspectivity of center in S, between the straight lines *s* and *r*.

### *4.3. Perspectivity between planes: limit line, improper line*

Let α and β be two planes of Euclidean space and let S be a point external to α and β.
We define *perspectivity centered in S between the planes β and* α, the correspondence that associates each point P∈β to point P'∈α such that P'≡ α ∩ SP (i.e. the projection of point P of *β* on plane α from the center of projection S).

- If β is parallel to α, the correspondence is complete and bi-univocal because the half-line that connects the center of projection S with any point of β certainly intersects the plane α, and vice versa
- If β impacts α along a line *u*, it can be observed that:

1) to the points of *u* correspond the points themselves: *u* is called *line of fixed points* in the perspectivity of center S between planes α and β;

2) it exists a line *t* ∈β whose points are placed in the plane passing for S and parallel to α, so that projections from S on α are improper points.

It is completed the correspondence between β and α by associating to the points of the line t the improper points, which are, then, a straight line to infinity, called *improper line*. (fig. 4.4).
The improper line is identified by the angular position of plane α (or by the beam of planes parallels to α).
In fact, if we consider the perspectivity of center S, between the plane β and any plane parallel to α, we can observe that the points of the line *t* of β still correspond to the improper points of $t_\infty$. The line *t* is called *limit line* in the perspectivity of center S, between the planes α and β.

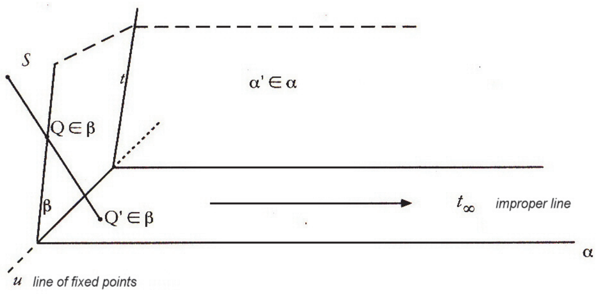


fig. 4.4

Euclidean space, with the addition of the improper elements, is called *projective space*; the geometry that studies its properties is the *projective geometry*; the group of transformations is the *group of homographies*.

In projective geometry are still applied postulates of elementary geometry relating to the conditions of belonging between points, lines and planes; they even take on a broader meaning through the *principle of duality*, which characterizes the geometric entities using two propositions obtained from each other by exchanging the words *point* and *plane*.

*A perspectivity ω between planes always subordinates a perspectivity between lines.*
In fact, if α and β are corresponding to each other in the perspectivity ω of center S, a plane passing by S, not parallel to α neither to β, intersects α and β along two lines *t1* and *t2* that are also corresponding each other in ω. (fig. 4.5)

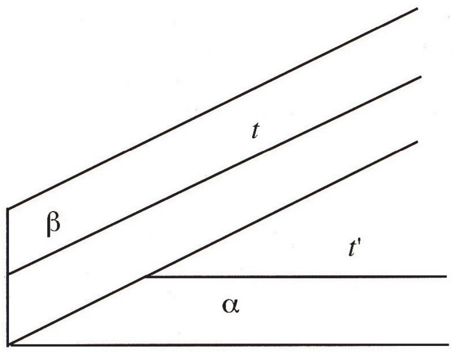


Fig. 4.5

Generally speaking, a biunivocal correspondence ω between two planes α and β, such that if a point P ∈ α describes a straight line on α, also its corresponding P' = ω(P) also describes a line on β, it's called a ***homography*** [5].
The set of homographies forms a group, according to Klein's conception:
*the group of transformations in which are not altered the properties of figures subjected to operations of projection and section.*

If α coincides with β, we say that *ω is an homography of the plane itself.*

Se α coincide con β, diciamo che *ω è una omografia del piano in sé.*

An homography ω of the plane α onto itself, that exhibits a fixed point U and a line *u* of fixed points is called ***homology*** [5] [6].

## 5. The homology

The importance of studying the ***homological transformations*** [6] in modern education is due to the use of them to teach the Design for Representation in Architecture.
In fact, the homology is a projective transformation of the plane itself determined by the product of two perspectivities in the space, generated by different centers that can be either proper or improper.
Thus, *although the process of building the homology is spatial the process takes place in the plane* and, therefore, its study is carried out independently of the spatial configuration that generated it.
To understand the spatial configuration that allows build a homology in the plane, let's consider the planes α and β that are corresponding in the perspectivities (fig. 5.1):

- $\omega_1$: $\alpha \rightarrow \beta$, of center S', that associates the ABC triangle of $\alpha$ to the A'B'C' triangle of $\beta$.
- $\omega_2$: $\beta \rightarrow \alpha$, of center S'', that associates the A'B'C' triangle of $\beta$ to the A''B''C'' triangle of $\alpha$.

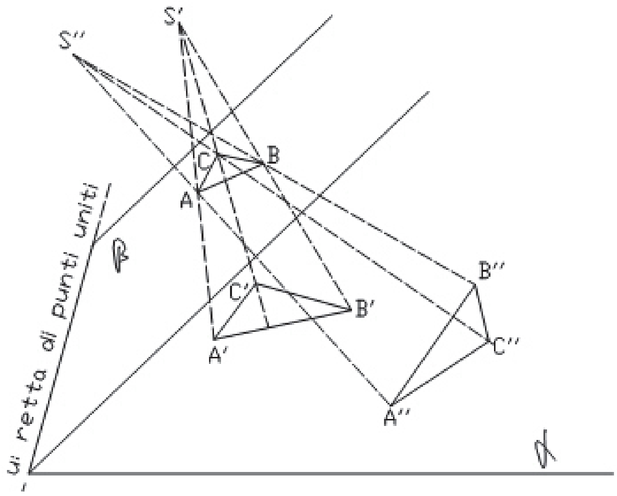


Fig. 5.1

The transformation: $\omega = \omega_1 \circ \omega_2$, composition of the two perspectivities centered respectively in S' and S'', is an *homology of the plane* $\alpha$ *in itself*, because the points of the line $u \equiv \alpha \cap \beta$ ($u$ can be proper or improper) and the point (proper o improper) U $\equiv \alpha \cap$ SS' are fixed in $\omega$.
If the axis is improper, the homology is a ***translation*** when the center is improper too, instead is an ***homothety*** if the center is proper. In both cases, the plane of the figure is parallel to the projection plane; in the case of the translation the line joining the centers of projection S 'and S' is parallel to the projection plane. In the case of homothety, the center is proper because The line joining the centers of projection impacts the projection plane, while the axis is improper because the plane of the figure is parallel to the projection plane. The centers of projection may be both proper or one proper and the other improper.
The homothety and the translation are also affine transformations because to pairs of parallel lines correspond parallel lines.
Besides the case of translation and homothety, other homological transformations that are also affine, are when the axis is proper and the center it is improper. In this case, the homology is called "*homological affinity*" or "*affine homology*".
If the direction of the improper center is perpendicular to the plane of projection, the transformation is called "*orthogonal homological affinity*".
A special case of homological affinity is the axial symmetry, where the reference plane is perpendicular to the plane of the figure, which in turn is perpendicular to the line joining the centers of projection *S′* and *S″*, and equidistant from them.
Through the use of computer, we can obtain representations of a figure or an object, from different points of view, if are known the equations of the homology leading one figure into the other.
It is required for this operation, the knowledge of the rudiments of analytic geometry [8] and basic concepts of algebra of matrices [7], in order to identify the transformation matrix of the particular homology, whose spatial genesis gives rise to perspectivities from different centers of projection.

## 6. The representation in the projective plane

For the representation of a point and a line in the projective plane, we should start from the representation of a point P($\bar{x}, \bar{y}$) in the Cartesian plane then, by expanding the projective plane, find the improper point with the addition of a third coordinate.
With reference to the figure, let's consider a direction δ of the plane (improper point $P_\infty$), be $r$ the straight line through the origin that identifies $P_\infty$ and be $P_1$ ($\bar{x}, \bar{y}$) a point of $r$.

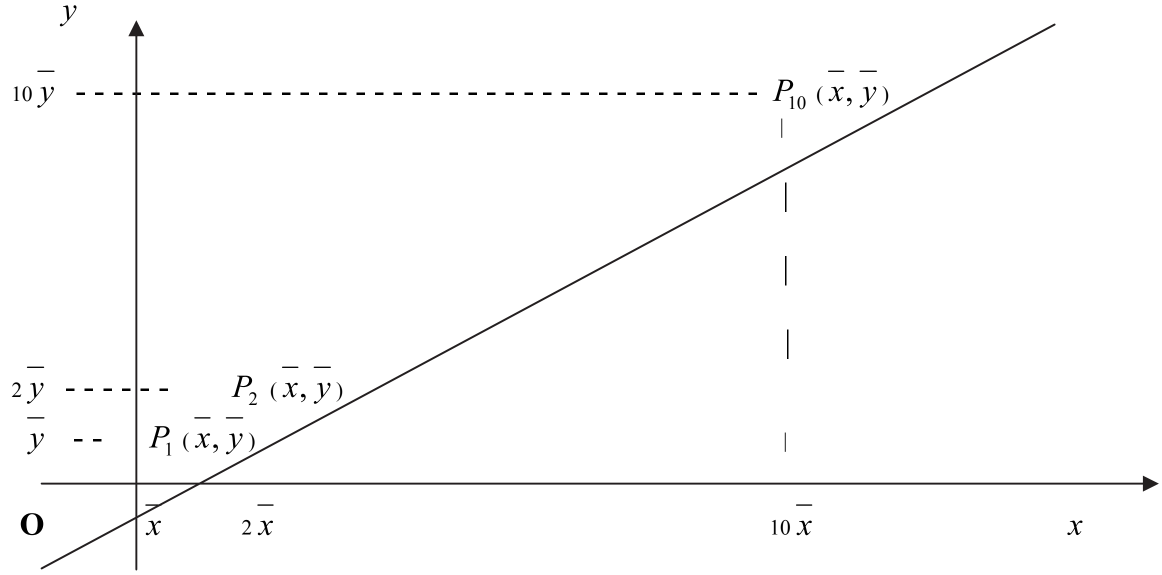


fig. 6.1

Let us consider the sequence of points:

$$P_1(\bar{x}, \bar{y}) \equiv \left(\frac{\bar{x}}{1}, \frac{\bar{y}}{1}\right); \quad P_2\ (2\bar{x}, 2\bar{y}) \equiv \left(\frac{\bar{x}}{\frac{1}{2}}, \frac{\bar{y}}{\frac{1}{2}}\right); \ \ldots\ldots\ldots\ P_{10}(10\bar{x}, 10\bar{y}) \equiv \left(\frac{\bar{x}}{\frac{1}{10}}, \frac{\bar{y}}{\frac{1}{10}}\right); \ldots\ldots$$

$$P_{1000}\ (1000\bar{x}, 1000\bar{y}) \equiv \left(\frac{\bar{x}}{\frac{1}{1000}}, \frac{\bar{y}}{\frac{1}{1000}}\right); \ldots\ldots\ldots$$

that, considering the denominator as a third coordinate, we can also write:

$$P_1(\bar{x}, \bar{y}, 1); \quad P_2\ (\bar{x},\ \bar{y},\ \frac{1}{2}); \ \ldots\ldots\ P_{10}(\bar{x},\ \bar{y}, \frac{1}{10},); \ldots\ldots\ldots\ldots\ P_{1000}\ (\bar{x},\ \bar{y}, \frac{1}{1000}); \ldots\ldots\ldots$$

The points of this sequence all belong to the line $r$ and with the increase of the index $k$, also increases the distance $\overline{OP_k}$ , i.e. when the denominator common to the two coordinates (or third coordinate) tends to zero, the point $P_k$ moves away indefinitely.
Given that:

$$x = \frac{x_1}{x_3} \qquad y = \frac{x_2}{x_3} \tag{6.1}$$

we can define the coordinates of improper points as the triples of real numbers ($x_1$, $x_2$, $x_3$) where is $x_3 = 0$.

So, the expression

$$x_3 = 0 \tag{6.2}$$

identifies *the locus of improper points of the plane, i.e. the improper line*.

**References**

[1] Records of the course "*Design and Mahtematics: Proposals for a didactics directed to the use of new technologies*", edited by di Cesare Cundari: Sorrento, 11-15 december 1990; Rome, 6-10 may, 8-12 december 1991. Ministry of Education - Representation and Survey Department - Faculty of Engineering of University "La Sapienza" of Rome.

[2] G. LORIA: "*History of Descriptive Geometry: from origins until our days*" - Hoepli Editore - Milano 1921.

[3] F. CASOLARO: *"The Erlangen Program and the geometric Transformations"*. Records of the course "*Design and Mathematics* "[1], pag.101-103; 220-231.

[4] F. Casolaro: "*Projectivity between planes. Duality law. Homografy and homology".* Records of the course "*Design and Mathematics*" [1], pag. 104 -118.

[5] F. CASOLARO - L. Cirillo: *"Homologic transformations".* Records of the National Convention of Mathesis "*Principles of Mathematics for its instruction and links with the contemporary society*". Verona, 28-30 november 1996.

[6] F. CASOLARO: *"Analitic geometry: historical notes and didactic function"*. Records of the course "Design and Mathematics " [1], pag. 91-100.

[7] F. CASOLARO: "*Homological transformations in the Art History, in the Didactics*" – Records of the International Convention "*Mathematics and Art: a surprising pair*" Vasto, 10-12 april 2003; pagg. 129-148.

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# MITHRAS: ART, NUMBERS AND SYMBOLS ANALYSIS OF THE FRESCO IN MARINO MITHRAEUM

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**Abstract.** The fresco of the Mithraeum (a place of worship for the followers of the mystery religion of Mithraism) in Marino, near Rome, is unique in its kind. At the bottom of a long rectangular underground room, in the front wall, the painting shows Mithras in its most classical iconography, which includes all the symbols related to the mystery cult. The peculiarity consists in the starry sky, depicted on the flap of the cloak of the God. Unlike other frescoes that are in fact similar and more or less contemporary - such as those of the Barberini Mithraeum in Rome and the Santa Maria Capua Vetere one in Campania (where the seven stars are not placed within a context, but are sparse in the mantle) - in the case of Marino the sky with the is highlighted and the stars are located in a structured way, four arranged in a square and, below the belt, three in a straight line. The interpretation we give provides several levels of interpretation: the symbolic/worship, symbolic/astronomical and symbolic/numeric.

**Key words.** Mithras, Astronomy, Numbers, Symbolism, Mythology

*Mathematics Subject Classification:* Primary 60A05, 08A72; Secondary 28E10

## 1 Mithras Cult

During the Imperial Age, in Rome, there were several cults. Coming in most cases by the various Eastern provinces, but also from the North, such as Druidism, these cults have taken root within the Empire in a more or less effective way.

An unusual story is that of the god Mithras, originally a mere servant, "soldier" of the great Persian god Ahura-Mazdah, then risen to the holder of a mystery cult in the late Empire. His name appears in the oldest sacred texts, such as the Iranian Avesta and Indian Vedas [2], as the third member of a triad of Heaven [1]. In the land of Mesopotamia the connotations of the cult were more

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* Sections 1 and 2 are due to Luciana De Rose, Section 4 is due to Marcella Giulia Lorenzi and Mauro Francaviglia, while Section 3 is inspired from [1] L. De Rose, F. Piperno, A. Zavaglia, *Il cielo nell'affresco del Mitreo di Marino tra immagine simbolica e conoscenze astronomiche* (in print); editing and translation by M.G. Lorenzi.

[1] See R. Iorio, *Mitra. Il mito della forza invincibile*, Marsilio, Venezia 1998, p. 12.

strictly cosmogonic, since Mithras was committed to provoke generating powers: he was the slayer of the Selenic Bull, by means of whose blood spreads the principle of life. At this point the young god assumed a completely independent status from the Indo-Iranian homonymous, paving the way to a long and successful own path, characterized by the power of the initiation rite.

This worship enters in the Urbe in the First century BC, during the campaigns of Pompey against the pirates of Cilicia. At first, the religion remains lukewarm prerogative of some minorities, but Mithras cult continues its slow journey, making a syncretism with other deities, including the best known one, the "*Sol Invictus*" ("Invincible Sun" was the official sun god of the later Roman empire).

The peace of Augustus had brought as a result the "*pax deorum*" (peace among the large number of deities), and the Empire received in its Pantheon a countless number of other gods, in addition to its own. In this context, the mystery cults, from the Hellenistic cult of Dionysus/Bacchus to the Egyptian ones of Isis, Osiris and Serapis, had a great diffusion. Alongside the cult of Mithras acquired new force, both for its aspect of mystery, than for the very meaning of the name, from the Persian "covenant relationship", well adapted to Roman religion's main characteristic [3], i.e. "*do ut des"* [2].

Fig. 1 – *The Fresco in Marino Mithraeum*

The cult spread especially into the category of the soldiers, in the areas at the borders of the Empire, or where the troops stationed. The decisive role in the dissemination and propagation of the cult is indeed due to the army. Following the promise of an afterlife cult practice, the soldier fighting to death in the name of Mithras - a god that never underwent neither death nor Resurrection - acquired a pass for the otherworldly dimension[3]. In addition, the initiate path was modeled on the

[2] See L. De Rose, *recensione a* R. Iorio, *Mitra. Il mito della forza invincibile*, Marsilio, Venezia 1998, "Ou. Riflessioni e Provocazioni", XI, 1/2001, pp. 120-121.

[3] See R. Iorio, *Mitra*, cit., p. 13 – ref. [2]

military hierarchy, ranging from the lower level "*miles*" to the highest officer. The initiation rites involved more than seven degrees of strength and courage and a martial footprint, typically a "male" [4]. Even the Emperors were fascinated by the Mithraic planetary experience: Nero was adored by the Armenian King Tiridates as *"Mithras"* [5]; from Commodus to Severus, the so-called "military monarchy", whose policy aimed at fostering polytheism in order to protect the "social unity" in a larger and larger state, favored the exotic deities that celebrated the military spirit, thus expanding the already plentiful Roman Pantheon.

In this period the cult of Mithras reached its climax. The fortunes of Mithraism grew and proceeded until it collided with the more and more powerful Christianism. Mithraism and Christianism are somewhat similar, but very different in substance, as Iorio declares: *"while the former is a sectarian and legendary cult, the second one is ecumenical and historical"*. Mithras and Christ have the same "*dies natalis*" (birthday), December 25, both births were announced by extraordinary events, both cults take account of a baptism, a "*Crismatio*" (chrismation), and even angels and devils are common in the eschatological vision, but the similarities stop here.

## 2 Marino Fresco: Animals and Symbols

Marino is an important archeological town in the area of Castelli Romani, near Rome, Italy. Marino Mithraeum came to light accidentally in 1962. It is located in a cellar near the railway station and it is one of the finest examples of Mithraic iconography (see Fig. 1). *Alessandro Bedetti*, Director of the Marino Civic Museum, writes [4]: *"at the sight of the discoverers there appeared a long gallery, at the end of which they glimpsed, on a frescoed wall, a painting of rectangular shape, depicting the god cutting the throat of a white bull"* [6]. The iconography is the classical representation of Mithras worship, with great detail. In particular, at the center stands the figure of the god of oriental origin, in the act of the divine service of killing the great bull with his dagger, symbol of the struggle against evil and its defeat. At the same time it is linked to the Greek and Magna Graecia traditional "taurine" image of the river-god, in the role of the bull able to fertilize the earth: in fact from the tail of the white dying bull appear some ears of wheat, confirming its role of regenerator. The bull is a classic symbol of the Mediterranean and Egypt areas: in Egypt people worshiped the Apis bull, while the goddess Hathor has the features of a cow. In the Cretan-Mycenaean culture, the bulls are part of the ludic-iconographic heritage; in classical mythology, starting from the white bull who kidnaps Europe, we move through the stories of the Minotaur in the labyrinth, to arrive to the goddess Hera, described as "cow-eyed" [5], and the representations of the river gods, conceived in the form of bulls, usually with human heads [7].

In Marino fresco there are also a dog and a snake licking the blood that flows from the dying bull: the snake, other animal linked to fertility and to the earth, is also a representation of the river; Achelous and Hercules fighting for the love of Deianira, assume the appearance respectfully of a snake and a bull [8] and from the narrative derives a flourishing iconography, especially visible in the coinage [7] of Magna Graecia and Sicily [9]. The dog was a creature of the god Ahura Mazda, the

[4] See R. Iorio, *Mitra*, cit., pp. 21-26 - – ref. [2]

[5] Dione Cassio., 1,23,1-7.

[6] A. Bedetti, *Il Mitreo di Marino*, Comune di Marino – Assessorato alla Cultura, Marino (Roma) 2003, p. 17, to which we refer the reader for History and a detailed description of the Mithraeum and its discovery.

[7] See L. De Rose, *Divinità fluviali in Magna Grecia*, in «Magna Graecia», XXXIV, fasc. 3-4/2000, pp. 25-28, here bibliography. Again see L. De Rose, *Le raffigurazioni zoomorfe degli Dei e semidei greci*, in «*spHera*», Osservatorio universitario, n. 4, 2011, pp. 76-77.

[8] Sophocles, Trachis, 9 et seq.

[9] See *Ibidem*; Cfr. G. Giannelli, *Culti e miti della Magna Grecia*, Firenze 1963, p. 255; S. Mirone, *Les divinités fluviales répresentées sur les monnaies antiques de la Sicilie*, in «Revue numismat.», XXI 1917-18, pp. 1-24.

symbol of good in the religion of the Vedas and the Avesta, but also present in the company of other allegorical figures representing the rivers in the coinage [8] of Magna Graecia [10]. In Pandosia, for example [9], is the image of the god Pan is present in the majority of its coins, usually accompanied by a dog, which is also a symbol of a river [11]. A dog often appears as a symbol of a river also in the coins of Sicily: for example, the river god Crimiso took the form of a dog to join Egesta or Segesta; on the back of another coin, the personification of the river Metauros near Medma is represented as a naked male figure, sitting on a large stone, at the foot of which there is a dog with its head turned back: the character of the river would be explicitly revealed by the presence of the dog with the head looking back [10], in the same way in which the bulls are usually depicted [12.]

The young God wears a red tunic, long sleeves, eastern-style, which lies above the red pants; on his head the classic Phrygian cap tipped forward and on his shoulders flutters a blue mantle, bordered with red. Inside the cloak a starry sky is painted. In this particular stands the uniqueness of the fresco. In fact in other similar frescoes, those of the Barberini Mithraeum in Rome and the one at Santa Maria Capua Vetere, there are seven stars, that symbolize in fact "the seven Planets", under the auspices of which the degrees of initiation to the mystery cult settles. In the Marino fresco the sky is painted in a different way. The internal surface of the cloak is dotted with small stars, and is divided into two parts by a central band, above which there are evident four bigger stars, arranged in a square, while below the band there are three other stars, placed on the same line of the former.

There is no doubt, within the cult, on the symbolic function of the number seven. The initiation to the mystery cult is divided into seven stages, grouped in three and four: in the first three phases some truths are revealed, in distinct phases. The third degree ("*Miles*"), submitted to the planet Mars, was mostly achieved; the last four degrees were more difficult to reach and they included the core of the deepest mysteries. There is a correspondence between each degree and a planet, in order: Mercury, Venus, Mars, Jupiter, Moon, Sun and Saturn.

In the Marino painting, a scorpion attacks the testicles of the bull [11], an obvious symbol of fertility[13]. On the contrary of the dog, the scorpion is sent by the Evil spirit, Ahriman. Even in the Bible the scorpion (like the snake) is related to the evil forces. In the Greek-Roman world the scorpion is an instrument of the goddess Artemis, who uses it to kill the giant Orion [14.]

On the sides of the fresco, at the bottom, there are the two *Dadofori* (torchbearers), Cautes, with the torch up, and Cautopates, with a pine torch down and off, while above there are the Sun and the Moon. Another animal next to Mithras is a black raven, sent by the Sun.

The young God wears a red tunic, long sleeves, eastern-style, which lies above the red pants; on his head the classic Phrygian cap tipped forward and on his shoulders flutters a blue mantle, bordered with red. Inside the cloak a starry sky is painted. In this particular stands the uniqueness of the fresco. In fact in other similar frescoes, those of the Barberini Mithraeum in Rome and the one at Santa Maria Capua Vetere, there are seven stars (see Fig. 2), that symbolize in fact "the seven Planets", under the auspices of which the degrees of initiation to the mystery cult stand. In the Marino fresco the sky is painted in a different way. The internal surface of the cloak is dotted with small stars, and is divided into two parts by a central band, above which there are evident four

[10] See J.H. Gardner, *Types of greek coins*, Cambridge 1883, p. 150; G. Giannelli, *cit.,* p. 158.

[11] See L. De Rose, *La leggendaria Pandosia*, in *Raccontiamoci la città. Cosenza tra storie, miti, leggende (parte terza)*, le nuvole, Cosenza 2005. pp. 15-27.

[12] L. De Rose, *L'età antica e medievale: da Stesicoro all'ascetismo monastico*, in *Gioia Tauro. Storia, economia cultura*, a c. di F. Mazza, Rubbettino, Soveria Mannelli 2004, pp. 23-38, 52-57.

[13] L. De Rose, *I simboli dell'Eros nel mondo classico*, in «*Hera*. Miti, civiltà scomparse, misteri archeologici», n. 127, agosto 2010, pp. 24-29; *Priapo e le falloforie*, in «*Hera*. Miti, civiltà scomparse, misteri archeologici», n. 127, agosto 2010, p. 28.

[14] Ig. Astr. Poet. 2.34; Erat. Cat 7, 32, Apd. Bibl. 1,4,2, ss.

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An interpretation of the seven stars and their position may therefore be the distinction between the first three degrees from the other four: the groups of stars are placed on the vertical same line (a kind of hierarchical progression) under and above the Zodiac band that divides the portion of sky shown in the mantle; so probably the square formed by the four stars represent the higher degrees.

But assumptions can be manifold, as well as the interpretations can be parallel. In fact, the same animals mentioned in the fresco: bull, dog, snake, scorpion and raven are also constellations, and all the mythological stories associated with them end with "catasterismi" [15]. The celestial representation could therefore exceed the mere decorative level, since the planetary/astronomic-type cult could be considered a sort of "*vademecum*" (handbook), with several levels of knowledge for its followers. In addition, it could demonstrate the knowledge of the sky by the unknown painter of the Third century A.D. In this regard, some very suggestive hypotheses have been formulated.

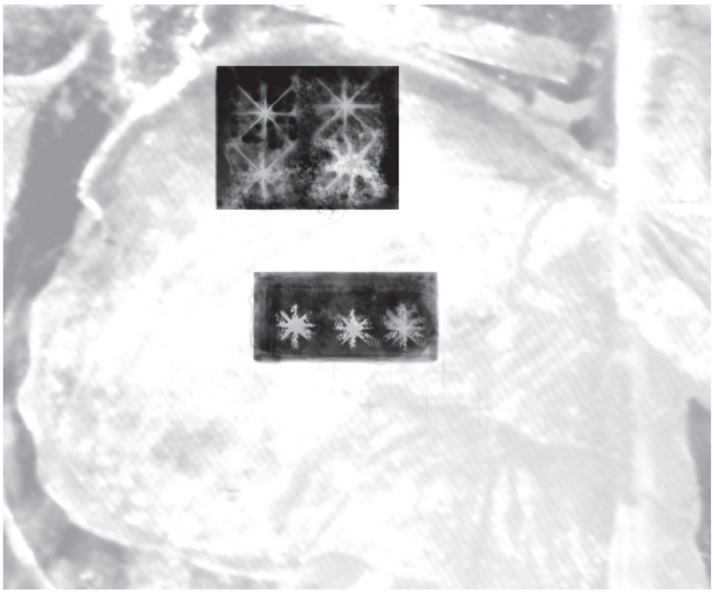

Fig. 2 – *Detail of the Fresco in Marino Mithraeum with the „Seven Stars"*

## 3 Astronomical Vision

With the help of the free planetarium software *"stellarium"*, with all due cautions, the sky of the night of December 25 has been reproduced - that is the day Mithras is said to be born, more or less corresponding to the winter solstice of AD 200/208. The resulting map of the sky has been

[15] Catasterismi ("placings among the stars") is an Alexandrian prose retelling of the mythic origins of stars and constellations, as they were interpreted in Hellenistic culture.

overlapped in transparency to that of the Marino Fresco and the images do have common characteristics. Piperno [1] proposed at first sight a suggestive interpretation: the top four stars could be labeled as the body of the Big Dipper (Plough or *Ursa Major*), the central area the celestial equator and two of the three stars in the Twins line [16].

So, according to this hypothesis, the central band represents the celestial equator. Parmenides of Elea (sixth century B.C.), was the first who divided the celestial sphere in five climate zones: two poles and two symmetrical tropical areas and at the center the equatorial band [17]. At the turn of the third and second centuries BC, Eratosthenes (276-196 B.C.) will add other two to the five Parmenidean temperate zones. Therefore, in the third century this division was well known and may have been inserted in the painting in order to mark the parts "above" and "below" with respect to the equatorial belt, representing the "middle". Nevertheless, the figurative image of the celestial equator is rather rare. Usually, the only band represented at that time was usually that of the Zodiac, in an oblique position.

The stars at the top may be part of the Big Dipper, reported in the four stars that fall within the numerical symbolism of the cult. In fact, the Big Dipper in Ursa Major was known as "*Septem Triones*", the seven oxen, image and personification of the North ("septem-trional"), constellation of the Polar Circle of the stars that never set.

The Big Dipper can therefore fall in the Mithraic religion, devoted to the eternal struggle of good over evil, light over darkness, both for the number seven, than for the oxen/bulls, but also because it is the symbol of winter, in which solstice Mithras was born.

The Twins could be referring to the Dadofori (torchbearers), or the myth of the Dioscuri (sons of Zeus). The latter story is well known: Zeus, in the form of a swan, loved Leda, the wife of Tindaro, and two pairs of twins, Castor and Pollux, and Helen and Clytemnestra, were born. Pollux and Helen where children of the God, while Castor and Clytemnestra were human off-springs [18]. The story, which is actually very complicated and symbolic, tells of the god Zeus, who wants to rape Nemesis, goddess and abstraction at the same time, as the daughter of Night. To avoid the meeting the goddess is transformed in a thousand ways, but when she turns into a goose, then the god takes the form of a swan. Nemesis gives then birth to an egg, which is abandoned, but some shepards bring it to Leda. In this version the brothers Helen and Pollux are totally divine [19] (see [12]).

The twins and the egg lead back in a circular way to Mithras, according to the conception of the cosmic egg. In Birds [20] Aristophanes shows the Orphic doctrine according to which the Night (already met in the Theogony in the earliest generations, passim), had laid a silver egg in the dark Erebo, in this egg there was contained the "Kosmos". The egg was fertilized from the North Wind (Borea), with his spirit. From the egg Eros (Love) would eventually be born [21].

In the Pelasgian myth the story is almost identical: Eurynome (the goddess of generation of the Titans, daughter of Oceanus and Thetis) would be "removed" from the primordial chaos, made

[16] L. De Rose, F. Piperno, A. Zavaglia, *Il cielo nell'affresco del Mitreo di Marino tra immagine simbolica e conoscenze astronomiche*, in «*Arcana* magazine», n. 2, in corso di stampa.
[17] Strabo II, Ezio 5, III, 15
[18] Apd. *Bibl.* 3,10,6 ss.; Luc. *Dial. d. dei* 20,14; Plut. *Tes.* 31 ss; Ov. *Met.* 8,300; Erat. *Cat.* 10.
[19] Apd. *Bibl.* 3,10,7; L. De Rose, *Dall'uovo cosmico al Big Bang. L'origine dell'universo nella Grecia classica*, in «*spHera*», n. 5, febbraio 2011, pp. 22-23, L. De Rose, M.G. Lorenzi, M. Francaviglia*, From the* "Cosmic Egg" *to the* "Big Bang"*: a Short Excursus on the Origin of the Universe between History, Mathematics and Art*, Aplimat, 10th International Conference on Applied Mathematics, Department of Mathematics Faculty of Mechanical Engineering, Slovak University of Technology – section: Mathematics & Art, Bratislava 2011, pp. 907-914; L. De Rose, M.G. Lorenzi, M. Francaviglia*, Dall'uovo cosmico al Big Bang. Breve* excursus *sull'origine dell'Universo,* in *Raccontiamoci la città (parte sesta). Cosenza tra storie, miti, leggende. Divulgare la scienza moderna attraverso l'antichità*, le nuvole, Cosenza 2011, pp. 131-141.
[20] 695 ss.
[21] Aristofane, *Uccelli*, 695 ss.

fertile by the snake Ophion; she would so generate the cosmic egg, which is wrapped by the snake coils in seven spires, including the expanding Universe (do not forget the resemblance with the female egg-shaped uterus-womb).

Fig. 2 – *Mithras in the Egg – source: Modena, Museo Civico Archeologico, inv. 2676*

This doctrine was inherited by the Romans: "*ab ovo"* (i.e., "at the beginning") is the phrase made famous by Horace and it assumed a universal connotation, perhaps luckily due to flourishing the in the Roman Empire the cult of Mithras (see [9]). Mithras (sometimes also named Phanes), a god of Eastern origin, assuming some characteristics similar to that of Christ, is frequently represented in the creative act of coming to birth from a golden egg.

A bronze statue of Mithras, emerging from an egg-shaped zodiac ring, found in a Mithraeum along the Hadrian's Wall, as well as an inscription in Rome, both suggest that Mithras may have been seen as the Orphic creator-god Phanes who emerged from the cosmic egg at the beginning of time, giving life to the Universe. This view is reinforced by a bas-relief at the Museum Este of Modena, representing Phanes coming out of an egg [22], surrounded by the twelve signs of the Zodiac (see [12]).

Following the same protocol and its overlapping with Stellarium, Angela Zavaglia has suggested a possible additional interpretative hypothesis. The central band in this case would be the Milky Way, the group of stars at the top should be identified with the constellation of Orion, and the stars at the bottom would be the Twins, in line with the planet Saturn.

According to a widespread opinion, probably deriving from Pythagoras, ideas coming from the colonies of Magna Graecia, and taken up by Heraclides Ponticus, the Milky Way was the place of the souls, which, after having spent their time on the Earth, went back to heaven after death along

[22] L. De Rose, M.G. Lorenzi, M. Francaviglia*, Dall'uovo cosmico al Big Bang…,* cit.

the same stellar route, so designating the galaxy as the kingdom of heavenly Hades, the Elisio. The deities related to the Milky Way as the seat of souls are Apollo and Hermes, both with a double meaning (celestial and underworld), both mythologically connected to the snake, to the Mantica, to Mithras and to constellations crossed by the Milky Way itself, and also to the hero Orpheus.

Hermes is the protagonist of the Eleusinian Mysteries, especially in Magna Graecia; Orpheus, son of Apollo, chairs the Orphic Mysteries; Hermes / Apollo / Orpheus are connected to the constellation of Lyra (near the Swan and the Serpent), so that in this case Mithras would become a connecting ring [23] (ref. [13]).

The cult in fact promised the victory of Good but also an eternal life beyond death, the glory of heaven. Just the promise of immortality made the cult particularly appealing and mainly soldiers attended the Mithraic school, through which the adepts were instructed to walk through the path of Initiation. The combination of Earth / Sky (Mithras is sent by the Sun, as "celestial soldier", but born out of a rock: his temples are underground) offered to the faithful the protection of Mithras on Earth in order to ascend to Heaven through the eschatological aspect of "mystery experience".

At the top of the fresco (actually inverting the image of the sky, according to the initiatic rule of "above and below"), Zavaglia recognizes the four stars of the Orion constellation. The mythological story of the Giant Orion is linked to Artemis, like that of Callisto / Bear. According to tradition the Earth gives birth to Orion, like other giants; the son of Poseidon, Orion had the gift of being able to walk on sea waters.

Extraordinarily beautiful and strong Orion did fall in love with Aurora, but he was killed by Artemis (perhaps having tried to use violence to her), who made him stung on the heel by a scorpion. Both were turned into constellations. And when the Scorpion rises in the Sky, Orion, forgetful of his courage, flies to the West and disappears under the horizon (Ig. Astr. Poet. 2.34; Erat. Cat 7.32; Apd. Bibl. 1,4,2) . Orion, then, can be traced back to the Mithraic cult for the birth of the Earth and the presence of a scorpion.

The presence of Saturn, that in that period of the Year is close to Gemini, fits perfectly in the discussion about mystery, identifying with the last degree of Mithraic initiation, the seventh (and Saturn would be just the seventh and last star) and being under the protection of this Planet. Those who had reached this initiatic level was entirely devoted to the worship and service of the brethren, without being allowed to perform other mundane tasks. The Father presided over the community, he was at the absolute top of all hierarchies and he performed periodical liturgical functions and regular daily services explicitly dedicated to the astronomical cult.

It is therefore not surprising that the Planet could be captured already at that time in an image with such a strong symbolic power.

## 4 Numbers and their Esoteric Meanings

As is well known, among the "theoretical disciplines", Mathematics is certainly the oldest. Since his earlier steps out of caverns, in fact, Mankind has wondered about structures in the World and has faced the practical problems of counting and measuring.

Out of these has slowly emerged what we call nowadays "Mathematics", whose developments belong, in a sense, to all antique civilizations, from those of the Eastern Indo-Arabic and Mesopotamic basins, to Egyptians, but also meso-Americans and Oceanians. Nevertheless, it was during the Greek civilization that the two big branches of Mathematics begun to be codified. On one side Arithmetic – conventionally attributed to Pythagoras, although due to a collective work that lasted for Centuries to be finally systematized by the "Pythagorean School"; on the other side,

[23] L. De Rose, *Il volo della tartaruga*, in "Atti X Convegno SIA", maggio 2011, in corso di stampa.

Geometry – attributed conventionally to Euclid, but again due to the work of Centuries and a number of scholars, to be eventually codified by Euclid in his celebrated "Elements of Geometry".

In older times the Integer Numbers have been studied and classified on the basis of their properties; the distinction between Even and Odd Numbers was recognized and investigated; Addition and Multiplication were given special emphasis, since these two operations take two Numbers and produce a third one. It was also developed and understood the meaning of "Exact Quotient" and out of this the notion of Prime Number eventually emerged. According to definitions, a Prime Number is a Number that can be divided only by itself and One, being **One** excluded a priori from their list. The first Prime Number is thence **Two** and no other Even Number is Prime. All other Prime Numbers are odd and the fascinating problem of finding all Prime Numbers – a problem tat was investigated by many antique thinkers [24] is still open [14]. Their list is endless, but no complete classification has been yet achieved and a "generating algorithm" is still missing.

Greek Arithmetic had nevertheless reached a high level of perfection. Following the Pythagorean School a Number is a "Perfect Number" if it is the sum of its divisors. As an example the Number Twenty-Eight is "perfect"; Plato speaks of it in his Timaeus. Pythagoreans relate also Numbers to all "regular geometrical shapes" (triangles, squares, pentagons, etc….) and Numbers are classified accordingly in accordance with their "geometrical identification" (see [15]). There are, therefore, Triangular Numbers (1,3,6,10, and so on), Quadratic Numbers (1,4,9,16,25, and so on –those that are "perfect squares"), Pentagonal ones (1,5,12,22, and so on), etcetera. One belongs to all families, of course [25].

Numbers have therefore always fascinated Mankind. Artists and scientists have always investigated them. Why…? As we said, since the beginning of Civilization Humanity had understood the periodicity of some fundamental cycles in nature, related with the Earth, the Moon and the Sun (the two most important "Planets", according to the old terminology – that are present in the Cult of Mithras and in the fresco we are discussing). This would had later led Mankind to recognize in "cosmic signs" the presence of several Integer Numbers, to be eventually assigned special "esoteric" meanings [15]: in particular the Numbers **Three**, **Four**, **Seven** and **Twelve** [26]. Still far from a complete understanding of "Irrational Numbers" – those upon which at least part of the so called "Pythagorean Mystery" was based, after the discovery of "incommensurable magnitudes" that were even more mysterious being them related to simple geometrical figures, like the Diagonal of a Unit Square or the diagonal of a regular Pentagon or the length of a Circle – Antique thinkers tried to reduce the Cosmic Harmony to Ratios between Integer Numbers or even pure Integers. What was not following this kind of "Rational Exactness" was interpreted in various ways and even considered as being related with the material impossibility to make "infinitely sharp measurements" (something that, after all, has been recently revived by Quantum Mechanics) – so that Rational Numbers [16],[17] were, at that time, considered the main regulators of Cosmic Beauty [27].

The fundamental cycles of nature are three: the Day Cycle (its unit is the Day, that nowadays we know to be related to the period of rotation of Earth around its axis but, in the oldest geocentric view, was considered to be a byproduct of the rotation of the Sun around the Earth); the Lunar Cycle (that defines the "Lunar Month", i.e. the period of revolution of the Moon around the Earth): the Annual Cycle (the Year, i.e. the periodicity of Seasons, that we know to be related with

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[24] D. J. Boorstin, *The Seekers. The Story of Man's Continuing Quest to Understand his World*, New York, Vintage Books, Random House Inc.,1999.

[25] A. Reghini, *I Numeri Sacri nella Tradizione Pitagorica Massonica*, Roma, Atànor,1980.

[26] A. Reghini, *I Numeri Sacri nella Tradizione Pitagorica Massonica*, cit.

[27] M. Francaviglia, M.G. Lorenzi, *Dal Cosmo al Numero ed alla Geometria Euclidea*, Technai **1** (1), 23-37 (2009) - Proceedings del Convegno "Scienza e Tecnica nell'Antichità Greca e Romana," CNR Roma, 3-4 Giugno 2008 ISSN 2036-8097

the period of the rotational motion of the earth around the Sun) [28]. Out of these the first Integer Number to spring up is the Number **Four**, since it is easily seen that all "cyclic phenomena" are subdivided into exactly "Four Phases". From the "beginning" to the highest point (the "Zenith"), from the Zenith to a new point placed at a position comparable with beginning point, from this one to a lowest point (the "Nadir", opposed to the Zenith) and, finally, from Nadir to a new point identical with the beginning. Interpreted as a "New Beginning" for the cycle. This Number Four was recognized into the four Seasons, in the continuous alternation between Daylight and Night, with Zenith at Noon and Nadir ad Midnight. An approximate (although inexact) calculation showed that within an Annual Cycle one encounter **Twelve** Lunar Cycles (they are, in fact, around Thirteen…) and during an Annual Cycle there are about **Twenty-Eight** Daily Cycles (they are in fact 29, but Twenty-Eight was chosen since it was exactly divisible by Four…!). Out of these observations Lunar Calendars eventually emerged, with months of 28 days (still preserved in the month of February). On the other hand, the periodicity of the Lunar Phases induced, by dividing Twenty-Eight into Four identical period of Time, to a division into Weeks, i.e. periods of **Seven** Days; as well as the Twelve Months correspond to the division of the Celestial Sphere (i.e. the Zodiac) into as many parts, called Months (or Zodiacal Signs, if counted differently).

Even more inprecise is the calculation that leads to divide the Year into **360** Days and to use this to measure Angles in a Circle by a subdivision into as many "Grades". According to this a Grade in the 360° Subdivision should correspond to the space that the Sun wipes in the Celestial Sphere in one Day. Of course it was not so and the correct result was about 365, but again the 360 was chosen for practical and esoteric reasons, being again divisible by Four (but also by **Five**, the Number of our Fingers, by Twelve, the Number of Signs in the Zodiac and by Sixty, that was the basis of Numeration in Babylon). The first Solar Calendars (in Egypt as in Central America) were in fact of 360 days and Five more had to be added, with fantastic mythological interpretations. Out of many re-handling the actual subdivision into twelve months of 30 or 31 days was eventually reached, without provoking discussions and diatribes.

But let us come back to these "Fundamental Numbers" (or "Sacred Numbers") in Nature: **Three**, **Four**, **Seven** and **Twelve**. Which was their Pythagorean interpretation…?

**Three** is the first "Magic Number", It evokes the "creative power", Number of the Sky and of the Soul. In a sense it is the "whole", since it includes the three phases of Life and the three "material components" of Matter (Air, Water and Earth). Because of these the Number Three is at the bases of several Religions (in Polytheism gods and goddesses or "natural forces" are often grouped in "triads" and also Christian tradition has a remnant when it speaks of the "Holy Trinity"). It is, in fact, the first Prime Number among Odd (Masculine) Numbers. It is naturally associated to the shape of the (Equilateral) Triangle and is, in fact, the first of (non-trivial) Triangular Numbers.

**Four** follows: it is the first "Square Number". It evokes Space, since upon it one constructs the first three-dimensional geometric figure (the Tetrahedron, the simplest among Platonic Solids – a Pyramid with Triangular Basis and Triangular sides, all identical within each other). The Number Fours is also the Number of Earth and of the Body. It adds, in fact, to the first three material elements the fourth constituent of Kosmos, the one that has an "energetic nature": Fire. Three and Four do form, in fact, the fundamental bricks of the two most "symbolically important" Numbers: **Seven** and **Twelve** (being $3 + 4 = 7$ e $3 \times 4 = 12$). Four was the Number of the "Pythagorean Oath", that was given in front of the **Tetractys**, a triangular shape containing the first Integers: **One**, **Two, Three** and **Four**. Their sum reproduces the next Triangular Number, **Ten**, who has and still has a special meaning (especially now, when we use it as a basis for all calculations). It ends up the *Decade.*

[28] L. Russo, *La Rivoluzione Dimenticata. Il Pensiero Scientifico Greco e la Scienza Moderna*, Milano, Feltrinelli, 1997.

**Seven** – as we said – is the Number of the Weekly Cycle. But Pythagoreans did knew vey well that it does not exist a Right-Angled Triangle with Integer Sides having Hypotenuse Seven, neither one exists with Integer Sides and having Seven as the Square of the Hypotenuse. Moreover they also knew that it is not possible to split a Circle into exactly Seven parts by just using the elementary rules of Geometry (Square and Compass). The Number Seven was therefore the only Number into the Decade "not having a father and mother and not having children" (neither generated nor generator). For symbolic combination it was associated with Athena, the goddess of knowledge. Because of this the Number Seven has always been considered as the "number of wisdom": who does not know the "Seven Wises" or the "Seven Liberal Arts"…? Ziqqurats in Babylon had Seven Floors. Among the "descendants" of Seven and Four we encounter the already mentioned Number Twenty-Eight: the "perfect" Number related to the (approximate) length of the Lunar Month, the period of exactly Four Weeks of Seven Days each.

The further number of Astronomical Cycles to which we shall pay attention for our "Mithraic Reading" is the Number **Twelve**. Intimately related with the Ratio between Year and Month, it was adopted in many places (e.g., in the Mesopotamian basis as well as in other Eastern civilizations) as a fundamental unit, together with its multiple **Sixty**. Still in our actual age "numbering by dozens" has remained (e.g., when we buy eggs…). Israel had twelve tribes, as many the Apostles of Christ; as many are the Signs of Zodiac. The Number Twelve exists, therefore, in practically all esoteric traditions, the Mithraic one included (which is, in turn, based on the earlier traditions of Zorastrism and Vedic culture, as we said).

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**References**

[1] L. De ROSE, F. Piperno, A. Zavaglia, *Il cielo nell'affresco del Mitreo di Marino tra immagine simbolica e conoscenze astronomiche,* Arcana Magazine 2 (in Italian - in print)
[2] R. IORIO, *Mitra. Il mito della forza invincibile*, Marsilio (Venezia 1998) – in Italian
[3] L. De ROSE, *recensione a* R. Iorio, *Mitra. Il mito della forza invincibile*, Marsilio, Venezia 1998, "Ou. Riflessioni e Provocazioni", XI, 1/2001, pp. 120-121 (in Italian)
[4] A. BEDETTI, *Il Mitreo di Marino*, Comune di Marino – Assessorato alla Cultura (Marino, Roma, 2003) – in Italian
[5] L. De ROSE, *Divinità fluviali in Magna Grecia*, in «Magna Graecia», XXXIV, fasc. 3-4/2000 – in Italian
[6] L. De ROSE, *Le raffigurazioni zoomorfe degli Dei e semidei greci*, in «*spHera*», Osservatorio universitario, n. 4, 2011, pp. 76-77 – in Italian
[7] G. GIANNELLI, *Culti e miti della Magna Grecia*, Firenze 1963, p. 255; S. Mirone, *Les divinités fluviales répresentées sur les monnaies antiques de la Sicilie*, in *« Revue numismat. »*, XXI 1917-18, pp. 1-24
[8] J.H. GARDNER, *Types of greek coins*, (Cambridge, 1883)
[9] L. De ROSE, *La leggendaria Pandosia*, in *Raccontiamoci la città. Cosenza tra storie, miti, leggende (parte terza)*, Le Nuvole (Cosenza, 2005), pp. 15-27 – in Italian

[10] L. De ROSE, *L'età antica e medievale: da Stesicoro all'ascetismo monastico*, in *Gioia Tauro. Storia, economia cultura*, cured by *F. Mazza*, Rubbettino (Soveria Mannelli, 2004), pp. 23-38, 52-57

[11] *Priapo e le falloforie*, in „*Hera. Miti, civiltà scomparse, misteri archeologici*", n. 127, agosto 2010, p. 28

[12] L. De Rose, *Dall'uovo cosmico al Big Bang. L'origine dell'universo nella Grecia classica*, in «*spHera*», n. 5, febbraio 2011, pp. 22-23;

[13] L. De ROSE, M.G. LORENZI, M. FRANCAVIGLIA*, From the* "Cosmic Egg" *to the* "Big Bang"*: a Short Excursus on the Origin of the Universe between History, Mathematics and Art*, Aplimat, 10th International Conference on Applied Mathematics, Department of Mathematics Faculty of Mechanical Engineering, Slovak University of Technology – section: Mathematics & Art, Bratislava 2011, pp. 907-914;

[14] L. De ROSE, M.G. LORENZI, M. FRANCAVIGLIA*, Dall'uovo cosmico al Big Bang. Breve* excursus *sull'origine dell'Universo,* in: *Raccontiamoci la città (parte sesta). Cosenza tra storie, miti, leggende. Divulgare la scienza moderna attraverso l'antichità*, Le Nuvole (Cosenza, 2011), pp. 131-141 (in Italian)

[15] L. De ROSE, *Il volo della tartaruga*, in "Atti X Convegno SIA", may 2011, in print

[16] D. J. BOORSTIN, *The Seekers. The Story of Man's Continuing Quest to Understand his World*, Vintage Books, Random House Inc. (New York, 1999)

[17] A. REGHINI, *I Numeri Sacri nella Tradizione Pitagorica Massonica*, Atànor, (Roma, 1980) - in Italian

[18] M. FRANCAVIGLIA, M.G. LORENZI, *Dal Cosmo al Numero ed alla Geometria Euclidea*, Technai 1 (1), 23-37 (2009) - Proceedings del Convegno "Scienza e Tecnica nell'Antichità Greca e Romana," CNR Roma, 3-4 Giugno 2008 ISSN 2036-8097 (in Italian)

[19] L. RUSSO, *La Rivoluzione Dimenticata. Il Pensiero Scientifico Greco e la Scienza Moderna*, Feltrinelli (Milano, 1997) – in Italian



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# "ADOTTA ARTE E SCIENZA NELLA TUA CLASSE" A PROJECT OF MATHEMATICS AND PHYSICS POPULARISATION IN THE ITALIAN MIDDLE AND HIGH SCHOOLS

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**Abstract.** Art and mathematics, art and physics, are closely related in the masterpieces of all time and constitute one of the more effective canons of science communication to the general public. This paper describes a popularisation project conceived by *Esplica-no profit,* working for the promotion of culture, with particular reference to formation and popularisation in the field of scientific culture. Through the "Banco Culturale", it offers an educational-divulgative programme to the Italian secondary schools, based on collaborative actions for the dissemination of concepts of physics and mathematics to students. Starting from 2012, "Adotta Scienza e Arte nella tua classe" – *Adopt Science and Art in your class* – will propose to secondary middle and high school students to choose among a hundred quotes on physics and mathematics by physicists, mathematicians, scientists, writers, and to realize a piece of artwork inspired by the interpretation of the selected quote, accompanied by an original short sentence.

**Key words.** Science popularisation, Physics popularisation, Mathematics popularisation, Art, mathematics, physics, secondary school, Esplica-no profit.

## 1 Introduction

*"An ancient link binds art and science, especially mathematics and physics. From the idea of harmony in the Greek architecture, based on the Golden Section, to the link between music and mathematics defined by Pythagoras, and when, centuries later, at the beginning of the Scientific Revolution, art offers the tools for observation"* [1]. At the opposite side of the timeline, modern and contemporary artists like Seurat, Dalí, Escher, the Futurists, Lichtenstein, the digital artists and others have exploited in many different ways these bonds in their artworks. Art and mathematics, art and physics, are closely related not only in field of the masterpieces of all the times, but also because art constitutes one of the more effective canons of science communication to the general public. If it is true that "Organize perception is what art is all about", as said Roy Lichtenstein [2], it is also true that physics and mathematics popularisation have often used this connection to make several aspects of science perceptible to the non-specialists. Paraphrasing Lichtenstein, "Organize

communication is what art is also about". This paper describes the popularisation project "Adotta Scienza e Arte nella tua classe" – *Adopt Science and Art in your class* - addressed to secondary middle and high school students in Italy, a three-year project starting in 2012, organized by *Esplica-no profit* under the "Banco Culturale" programme [3]. *Esplica-no profit, Laboratorio per la divulgazione culturale e scientifica nell'era digitale*, works for the promotion of culture, with a particular reference to formation and the popularisation of scientific culture. The "Banco Culturale" is a programme of activities offered to schools, concerning projects based on collaborative actions, aimed at disseminating the concepts of physics and mathematics to students. "Adotta" will propose the students to choose among a hundred science quotes, mainly by physicists and mathematicians, and to realize a graphic artwork inspired by the selected sentence. It must be accompanied by each participating student's original short sentence associated to the quote chosen and to the work done. Here we present the "Adotta" project at the end of its conceptual stage, we discuss its more relevant creative and operational aspects, the possible timetable and several ancillary activities .

## 2 The unbreakable connection of art and science: a claim against the *two cultures* gap

The idea that art and science are two separate cultural spheres is an exclusively modern prejudice. It starts during the eighteenth century and polarises through the following centuries with the change of the anthropological, social and cultural models of the scientist and the artist. In former times, especially during the classical times and the Renaissance, their roles had often been culturally indistinct. Then they differentiate. The artist emphasises his individuality, he exhibits his secular works – once restricted to the nobility and their courts – to the general public, he participates to exhibitions and expositions, he enters the commercial art circuit, settling in the would-be cultural areas of the bourgeois society. The scientist reduces his public activity and lessens his individual action, he joins research groups and scientific societies – complex structures gathering the intellectual élite, closed, not communicating with the bourgeois society at large. This is the first, and no doubt the most important, process that will give origin to the Western model in the following centuries, separating the artistic culture from the scientific and technological culture.
Towards the end of the nineteenth century, the specificity of the social and public behaviour of the *scientist* and the *artist* represents a total break with the classical and the Renaissance experience, that united the two figures in one single profile. It originates a social perception of art and science as two separate domains, producing the bias that science and art are intrinsically different, alternative, opposed or clashing with each other. An unnatural bias indeed, since it forgets the ties that had connected art and science since the ancient times. Besides, it has been generally ignored in the practices and the contents, though at the beginning of the twentieth century it was embedded by the Crocian idealism [4]. There is no doubt that the inseparable union of science and art is still in practise today, though clumsily disputed by the circle of the *two cultures* [5].

### 2.1 The unnatural prejudice - the ancient bond: art&science conspiracy

The relationship between scientific discovery and the pursuit of beauty has existed for centuries. In the earliest years of science these two concepts were inseparable. In the ancient Greece the relationship between art and the natural sciences referred both to the visual and the non-visual arts. The discussion on Pythagoras' conception of music and on the golden ratio, started in the Greek classical art, goes on through the European Middle Ages and the Renaissance. Leonardo creates artworks, machines and detailed drawings of natural elements, the models and basis for the scientific observation developing from the sixteenth century [1]. The German painter Albrecht

Dürer visits Italy in 1494 and 1505 to learn perspective, he publishes a *Treatise on Mensuration* and one on the proportions of the human body. In his fundamental history on the birth of modern science in Europe P. Rossi remarks: "The collaboration of the artists had revolutionary effects on the descriptive sciences" [6]. The alliance between art and science is inextricable in the western world, its roots are in the ancient Greece and it goes on through the centuries until the Renaissance outburst in architecture, sculpture, painting. In Rembrandt's *Anatomy Lesson* (1632) the representation of death, far from the medieval "dance of the death", is rather the occasion to depict a subject of scientific observation and of curiosity (as well as to celebrate a social group).

### 2.2 The not applicable prejudice - The modern practices: art&science contamination

Starting from the end of the eighteenth century, the socio-cultural evolution, of the work organisation, and the economic, structural, individual implications have divided the atelier and the laboratory, the artist and the scientist, creating two cultural domains, totally separated in their methods and procedures. In spite of this, the best artistic practices have gone on with their contaminating dialogue with science, with physics and mathematics especially. On the other hand, the scientific process shows many similarities – as we shall analyse later – with the artistic creative work. In the first half of the twentieth century, the *Cahiers* of the poet Paul Valéry are full of notations and references to the links between art and science, mostly astromony [7]. In spite of bias, art uses more and more consciously and constantly of the discoveries of optics, chemistry, for example in the case of the Impressionists and the Pointillists [1]. It draws inspiration from mathematical assumptions and aspects of the material world created, emphasized or aggravated by the Industrial Revolution – the speed intriguing the Futurists, or the noises produced by machines transformed into elements for musical compositions [8]. Through the whole century science is so naturally talking with art that many scientist find a spontaneous way to canalize scientific concepts using artworks originated by the creativity of their author, often in contexts and situations far from conscious scientific thinking. The works of M. C. Escher have been used as illustrations for science magazines, physics, chemistry and mineralogy handbooks. The lithograph *Day and Night*, where they are represented as two mirror images, with white birds melting to form daylight on the left, and black birds on the right to form the night sky, has been used to recall the CP simmetry typical of the physical theories of the elementary particles. For the same reason, the image *Knights* was personally selected by the Nobel prize Chen Ning Yang as the cover image for its famous book *Elementary Particles*. The litograph *Reptils* has been printed in chemistry books and in the *Scientific American* magazine [9]. In the twentieth century science and its applications are both tools and ispiration for the artwork. M. Duchamp reproduces on canvas the moving figure of the cinema and the cartoons, Salvador Dalì takes inspiration from the physics-mathematical theories of the multi-dimensions [1]. The fractal theory and stochastics inspire painting and music [8]. The scientific images of knowledge visualisation – with ancient roots: polyhedrons, forms from the natural world, artistic sketches by Leonardo, Ernest Haeckel's disputed representations of embryos in the eighteenth century [10] – give origin to new aesthetics, more and more frequently unconscious to the scientist. This shows how natural this contamination is, it is in the facts before than in any cultural layer, as the very foundation of a young, luxuriant *Wissenschaftsästhetik* [11], [12]. In this context, the recent essays edited by P. Galison and C. A. Jones [13] are important for their enhancement of the implicit artistic content of the graphic representations of science. E. P. Fischer also shows how many discoveries, from Bacon, to Watson and Crick are led by a creative instinct and a deep aesthetic sense [14]. He explores concepts like the golden ratio, evolution, symmetry in nature, imaginary and irrational numbers, as proofs of beauty in science. So strong is the scientist's adhesion to the aesthetic criterion that it sometimes results counterproductive and a

cause of lag in getting over a by then obsolete paradigm. Quoting J. W. McAllister: “Neither Copernicus nor Einstein brought about a true revolution in astronomy or physics, respectively, because each retained the established aesthetics of theory choice. In sharp contrast, introducing ellipses over circles into mathematical astronomy and indeterminism over determinism in quantum physics represent two genuine revolutionary episodes in the history of science” [15].
A settled example of the use of art to communicate scientific concepts is represented by the NASA photo galleries featuring images received by the space probes together with artistic interpretations of astronomical object, both equally beautiful and explanatory [16].
A popular way to get people interested in the results of the LHC experiments is the CERN project started in 2010 to transform the data concerning the particle interations into sounds with a technique called “sonification”. Besides “musical compositions”, other developments will be an app for the iPhone and mobile ringtones. [17].
The link between the scientific and artistic fields is stressed by the Aplimat Conference itself, that has been presenting a section of papers on Mathematics and Art for some years now.
Yet, in spite of all this, the prejudice started in the eighteenth century with a social transformation that distinguished between the craftsman and the technician, the artist and the scientist is accepted in the common culture even today. It does not seem that the separation between *culture* and *scientific culture* has been overcome yet. Nothing seems more appropriate than closing this paragraph with a quotation from the mathematician Paul Lockart, engaged in the identification of efficient teaching methods, who more than any other philosopher of science or art critic, though enlightened, expresses our thought thoroughly: “The first thing to understand is that mathematics [and physics] is an art. The difference between maths and the other arts, such as music and painting, is that our culture does not recognize it as such” [18].

### 2.3 The Feyerabend legacy - Science as an art

The philosopher of science Paul Feyerabend gives a representation of science advancing and changing not only through the application of the Popperian validation method, but also as a consequence of casual events, of redundant interpretations, of unexpected results of mistaken theories. According to Feyerabend, science has in itself several paradigms that can be compared to those of art [19]. Though not sharing Feyerabend’s savage criticism to Popper’s epistemology, in our opinion there is no doubt that imagination, creativity and the cleverness to project beyond the apparent are positive abilities of the modern scientist. They unite him to the artist, who applies these skill to the human universe, while the scientist practises them on nature. It is only after this phase that art and science go along similar courses, though not identical. The artist ends his job here, leaving it to the favour or rejection of the artistic-art critics-art gallery managers community of reference. On the contrary, the scientist’s job is not limited to its creative phase, he validates (always only temporarily) or disproves his discovery, his theory in accordance with the prescriptions of the scientific method. He does not leave it to the peer community or to the public, but to nature. Unlike Popper, who places all the value of science in the second part of the course, Feyerabend revalues the initial creative phase and maintains its intercultural potential. The research of the Max Planck Institute for the History of Science on the intercultural aspects of art and mathematics, art and physics, art and biology is exemplary [20]. In particular, we recall the works of K. Krauthausen on the *Cahiers* by Paul Valéry and of M. Wegener on Valéry, K. Goldstein e J. Lacan [21], [22]. Among the experiences of common work between scientists and artists, an exemplary instance of collaborative research is Sci-Art [23], developing for several years now in the field of biomedicine. It aims at the identification of the effective canons of science comunication for different target publics. Certainly Feyerabend’s aspiration went in the direction of a more organic collaboration for the formation of new methodological and epistemological criteria, yet extended communication is the first step to validate again a link never missing in the practises, though denied by the university and formal education.

**2.4 - Bridging the communication gap - Art as a powerful tool for science popularisation**

Today the necessity to inform the tax-payers that science improves their lives, the possibility to disseminate the results of scientific research among the general public and better the social perception of science, which in the end promotes new investments, make science popularisation one of the most important cultural and organizational issues related to "making science".
Art is a powerful tool for science popularisation, a bridge between the citizens who demand information and the scientists, a way to vercome the separation between humanism and science.
The web 2.0 has opened new domains to science popularisation, besides the traditional magazines, lectures, permanent exhibitions, playing a primary role in science communication today, for data recording, sharing, discussion, comprising the active involvement of the non-scientists in the citizen science projects [24]. As a whole, these tools constitute *learning in the wild*, the informal learning that forms "most of what the general public knows about science" today [25]. An interesting musical project is *Symphony of Science*, using a combination of quotations and images of famous scientists, who "sing" lyrics ispired to the various aspects of scientific knowledge [26], while in the visual arts the examples are countless. In the twenty-first century the digital artists create their works in the cyberspace handling the graphic tools produced by mathematics. Digital art is especially developed in the metaverses, online immersive environments that offer the users special 3D graphic programmes, through which physics-mathematical objects and artworks can be built. Science popularisation and formation can make full use of their creative aspects, with the "building" of scientific interactive objects and simulations of experiments. In 2010 SLArt and Second Physics organized the contest *Scien&Art* in Second Life, centered on the artistic representation of science [27]: the artists were invited to create works inspired by scientific concepts, while the scientists were asked to look for creative ways to communicate scientific ideas. Second Life has hosted the "E8 polytope". An *n*-polytope is a geometric entity with plane faces in *n*-dimensions. The term *polytope* is used for a number of mathematical concepts. The physicist Garrett Lisi maintains that the symmetries of the E8 polytope make up the framework accounting for the behaviour and interaction of the particles forming the universe [1], [28]. An example of a Real Life exhibition in the metaverse is *Imaginary*, the virtual reproduction of the Museum of Mathematics and Mineralogy at Oberwolfach-Walke, Germany, displaying geometrical forms in two, three or four dimensions [29]. EsplicaSL has been experiencing the possibilities of the virtual worlds for some years now, proposing itself as a "bridge between the creativity of the metaverse and the creativity of the physical world". So has Second Physics, group for science and physics popularisation in the Italian communities in the metaverse, with lectures, lessons, the participation virtual conferences like the VWBPE on the best practices in education [30]. The potential of the virtual worlds as a teaching laboratory has also been studied by the European project *Avatar* [31].
The *Global Particle Physics Photowalk* competition, giving the amateur photographers access to the experiments and staff of CERN, DESY, Fermilab, KEK and TRIUMF, originates from a reflection on the visual arts for physics popularisation and on the artistic value of the representations of scientific machines and apparatuses [32].
"Adotta" is part of these activities not only under the aspect of the combination of practises in the physical world with communication and publicising on the web, but also because the cyberspace appears effective for the production of ideas, the promotion of creative activities of great impact and for the aspects related to the visual arts and for those regarding the scientific concepts.

## 3 The "Adotta" project - The early creative stage

"Adotta Scienza e Arte nella tua classe" is a project for the popularisation of Physics, Mathematics and the link Science-Art in the Italian middle and high schools.
The aim of "Adotta" is making the students experience the link science-art and offering the teachers the occasion to elaborate contents for a natural reflection on the connection between artistic creativity and scientific production. The project communicates to the students the message on the art-science connection as condensed in chapter 2. The students approach science through artistic creativity, with the interpretation and representation of science with a graphic work inspired from quotes on the subject, a challenge that becomes a practise where science and art are both medium and content of the message of the inextricable tie between science and art. The teacher's role is fundamental to provide the students the cultural and educational support necessary to catch the twofold message: the content of the challenge and of the quotation, and to trigger their interpretative and creative abilities. For an example of product from a similar project go to the link [33]. "Adotta" will develop in the schools for its educational aspect, and on the web for those related to communication - among the students, the teachers adhering to the project, the presentation of the artworks and their evaluation by the general public and the experts, communication in the media at large. This fulfils another of the project aims: connecting to the young with their congenial media - social networks, blogs, free photo sharing platforms, instant messaging, Twitter. In the age of the deconstructed and shared communication, this is an event in itself. The "Adotta" project of Esplica no-profit, whose early creative stage we are going to describe, is open to the collaboration of other partners in all of its phases – operational, planning, production.

### 3.1 Esplica-no profit

*Esplica-no profit, Laboratorio per la divulgazione culturale e scientifica nell'era digitale*, is a cultural non-profit making association engaged in activities of communication, popularisation and formation of scientific culture. The mission of Esplica-no profit originates from the firm belief that the separation between "culture" and "scientific culture", the existence of *two cultures* is a legacy from the past. Esplica wants to recompose the division in the social perception of science and art, operating in the cultural, scientific, technological and political fields. It proposes innovative communicative experiences for the popularisation of science and culture, with projects addressed to the young and the school, convinced of the projective strength that the actions in these environments will have on the whole society. Among the activities carried out by Esplica-no profit we recall "Reporter di Scienza", a "cross-universe" initiative, between Real Life and Second Life: the reporters to the "Festival della Scienza" (Genova 2010) and "Infinitamente" (Verona 2011) gave accounts on the news and comments to the Italian communities in Second Life. Esplica-no profit has participated to the ebook Festival in Fosdinovo, coordinating and selecting the publishers active in several virtual platforms, and to the annual conference *Scientix* in Bruxelles on the role of science education. In Second Life Esplica-no profit sponsors projects of scientific-cultural popularisation and formation like *Adventure in Second Life*, *Beyond the Third Dimension*, *Scienza on the Road*, *Serate Immersive*. Esplica-no profit has recently defined the several years' programme "Banco Culturale", an offer of activities and practices addressed to schools, clubs, associations, consisting of lessons from experts, lectures and seminars on different scientific and technological subjects of popular or formative character. "Adotta" is the leader project in the programme of the Banco Culturale for the years 2012 to 2014.

### 3.2 The concept project – The general description of “Adotta”

The cultural reference of the project is based on the analysis of the great figures of the Renaissance, round characters of artists and scientists, represented in our age by painters like Seurat, who recognized explicitly the scientific dimension of his work [34], or Dalí, who found the strength to express feelings and elevated aspiration in the visions of science. Or the Copernicus and the Einsteins ever searching for the elegance and the beauty of symmetries in their theories.
The purpose of “Adotta” will be of making the students aware to the scientific themes sometimes hidden but always present in the general cultural context and of art in particular. The students will be invited to join a global event where they will have the possibility to participate actively in the dispaly of their creative ability inspired by a scientific theme. Starting from a quote from a scientist or writer on scientific subjects, they will have to create an artwork inspired to it considering that science determines our future, and of the young to a greater extent.

### 3.3 The players involved

• Students – the privileged actors in the project, the focus of its local and global phases, the first addressees of all communication, each of them the creator of his artwork, in competition with himself and with the other competitors.
• Teachers – the references of the format are large and versatile, so teachers of all subjects can be involved. The teachers will have the task of better defining the local configuration of “Adotta” according to the type of school, adjusting it to the programme and the schedule. The teachers are the only artificers of the local management of the project, introducing it to the students in the cultural scenario that they consider the most appropriate.
• Esplica-no profit – the referee, provide the first information to the schools, creates and maintains the database and looks after all the global organizational aspects:

- the preparation and distribution of the material necessary to undertake the local activities that the teachers integrate with their professional competence;
- the gathering on a national scale of the works done locally, to make them accessible to the public and to the expert jury, the realization of the web spaces for communication and exhibition, the organization of the final event.

### 3.4 The timetable

• First communication to the schools from the project proposers, from April 2012.
• Local phase in the single schools, from October 2012.
• Global phase- gathering of the works, exhibition, selection of the artworks- from December 2012.
• Final event – public exhibition of the works selected by the expert jury, announcement of the winner, announcement of the winners of the web contest.

### 3.5 The communication tools: mail, email, FB, Twitter, Google+, website, tri-fold poster

• The informative phase to the schools will be done mainly through the web and with the direct contact with the teachers and the schools. They will be contacted through postal mail first, then through email. The materials will be provided in digital form, freely downloadable from the Esplica-no profit website. They will comprise: a brochure illustrating the project for the students and teachers, an explanatory poster to display in the schools, a booklet describing the project “Adotta“, and the hundred quotes, with an explanatory note and short biographies if their authors. Yet these first phases of information and “recruitment“ are part of a larger communication strategy aiming at keeping the public attention to the project high for all of its duration.

• The local phase will be carried out in whatever period of the duration of the project, adjusting it to the organizational, cultural and educational exigencies of the teachers and the school. The project “Adotta” can also be a tool for the teachers to deal with the theme of art and science in the curricular classes, as a characterising feature of contemporary culture and supporting the progress and innovation of the civil society. In these phases the Esplica-no profit website will host a section for the teachers community, to share opinions and ideas on the collateral best practices employed. The Banco Culturale of Esplica-no profit will offer seminars on the themes of art and science as a possibile support to the project.

• The students’ graphic works will be gathered accompanied by each participant’s original short sentence associated to the quote chosen and to his work and displayed on the web from January 2013 through December 2013, depending on each school local schedule. The works displayed on the web will be anonymous and the webnauts will vote them through the device “I like”.

• The works displayed will also be voted by an expert jury composed of teachers, scientists, artists, which will choose the best for the final event.

### 3.6 Ancillary activities

The project features described so far - its global sequence, the web display, the public and the expert jury selection – though important, are but a part of the project, since it wants to encourage teachers and students to engage in in-depth study starting from the subject of the artworks, where art becomes a vehicule for the diffusion of science and mathematics. Parallel to the initiatives of the proposers, the schools or the single classes will be motivated to organise activities auxiliary to the project - exhibitions, discussions, seminars on the quotes, on the artworks, etc. The publication of these initiatives in the web spaces offered by Esplica-no profit will make the project visible and will be of stimulus to the other participant schools for the realization of similar or new initiatives, so starting a virtuous competition mechanism.

### 3.7 The communication messages to the students

The student is the true protagonist of the project, with his individuality, the ability to catch instinctively the challenge of the communication message, so we will pay particular attention to its realization in the contents and the pratices. To that end we will prepare social networking tools to follow the students’ activity and project it beyond the local limit of the school.

### 3.8 The “bidibi bodibi bu” quotes

The following are examples of the quotes we will propose. Each sentence presented to the students will be accompanied by a short interpretation and a biographical card of the author, highlighting the aspects not immediately concerning science and the historical references, to make the scientist’s figure more human and define the age where he lived. Yet, here we limit the presentation to the interpretation for want of space.

• *As I have said so many times, God doesn't play dice with the world*. Albert Einstein, conversation with William Hermanns in 1943, quoted in Hermanns' book *Einstein and the Poet*.
The concept that Albert Einstein, one of the fathers of modern science, wants to express with this famous sentence is that the world seems to work in accordance with (scientific) laws not chaotic, but rather supported by universal general principles. Actually Einstein was critising the quantum physics that was becoming of importance in those years and the concept of uncertainty (physical probabilistic phenomena determined by radioactive decay, Heisenberg's principle, etc.). Indeed Einstein never totally accepted the quantum physics that today is the very basis of all modern physics.
• *Numbers rule the world*. Attributed to Pythagoras by Iamblichus, 4th century, *Life of Pythagoras*. Other famous sentences by several scientists refer to the order of the universe and to mathematics as a tool for its interpretation and modelling. The great Italian scientist Galileo Galilei said similarly that "Mathematics is the language of nature". Today the science cannot say whether mathematics is the language through which the universe talks to the man or it is only a logical model to interpret the universe. See also the famous article by Eugene Wigner *The Unreasonable Effectiveness of Mathematics in the Natural Sciences* and the simple discussion in Scienzapertutti [35].
• *Knowledge is power*. This sentence is generally attributed to Sir Francis Bacon, but it is rather the editors' summary of Bacon's idea. Bacon's most similar sentence is "Human knowledge and human power meet in one". The Latin version is in Thomas Fowler (ed.), *Bacon's Novum Organum*. 2nd ed., 1878, 188, later published in F. Bacon and J. Spedding (translator), *The Works of Francis Bacon* (1864), Vol. 8, 67. The term *power* in this case must not be meant as "military power", but in the broader sense of "ability to advance". The future of man and civilization comes from science. Knowledge has a moral and political value inspiring healthy doubts and makes us refuse the apparent.

## 4 Conclusions

The project "Adotta" is described in its preliminary stage. The final configuration may be, of course, subject to adaptations and modifications.

## References

[1] FLANAGAN, T., DELPHIN, G., FARGIS, M., LEXINGTON, C., *When Art Meets Science in Second Life*. Paper presented at the 10th International Conference Aplimat 2011, Bratislava.
[2] Quoted in SHLAIN, L.: *Art and Physics. Parallel Visions in Space, Time and Light*. Viking, New York, 1993.
[3] www.esplica.it
[4] http://plato.stanford.edu/entries/croce-aesthetics/
[5] SNOW, C. P.: The Two Cultures. Cambridge University Press, 1993 (1959).
[6] ROSSI, P.: *La nascita della scienza moderna in Europa*. Laterza, Roma-Bari, 1997, p. 60.
[7] VALÉRY, P.: *Cahiers/Notebooks*. Edited by B. Stimpson. Peter Lang Verlag.
[8] FLANAGAN, T., DELPHIN, G., FARGIS, M., LEXINGTON, C.: *Art, Mathematics and Physics in Second Life*. In Journal of Applied Mathematics, Aplimat, volume III (2010), number I.
[9] FLANAGAN, T., DELPHIN, G.: *M.C. Escher*. Book published in Second Life, 2011.
[10] HAECKEL, E.: *Kunstformen der Natur*. Prestel 2005 (1899-1904).
[11] GABOR, P.: *What is beautiful? Aesthetics and realization*. Peppering castle, 2003. ISBN of 3826024257.

[12] TAUBER, A. I. (Hrsg.): *The elusive synthesis. Aesthetics and science*. Kluwer, Dordrecht, 1996.
[13] *Picturing Science, Producing Art*. Edited by P. Galison and C. A. Jones. Routledge, New York-London, 1998.
[14] FISCHER, E. P.: *Beauty and the Beast. The Aesthetic Moments in Science*. Munich, 2005.
[15] MCALLISTER, J. W.: *Beauty and Revolution in Science*. Cornell University Press, Ithaca (NY), 1999.
[16] www.nasa.org
[17] *It sounds good!* In CERN Bulletin, Nos 23 & 24 – 9 & 16 June 2010. http://user.web.cern.ch/user/Bulletin/23-24-E-web.pdf
[18] LOCKHART, P.: *Lockhart's lament*, 2002. http://www.maa.org/devlin/devlin_03_08.html
[19] FEYERABEND, P.: *Against Method: Outline of an Anarchistic Theory of Knowledge*. London, 1975.
[20] http://www.mpiwg-berlin.mpg.de/en/index.html
[21] KRAUTHAUSEN, K.: *Valéry's Cahiers (1894-1945). Drawing and Writing as a Practice of Thought*, www.mpiwg-berlin.mpg.de/en/institute/search.html?volltext=Karin+Krauthausen
[22] WEGENER, M.: *Three Undiscovered Epistemologists: P. Valéry, K. Goldstein, J. Lacan*. http://www.mpiwg-berlin.mpg.de/en/institute/institute/search.html?volltext=Mai+Wegener
[23] https://sites.google.com/site/sciart1/home
[24] BOCCARDI, B., FRAGONA, M., PAROLINI, G., *Popularisation of Physics in the Wild*. Paper presentato al 12° Simposio Internazionale Frontiere della Fisica Fondamentale, Udine, 2011.
[25] *Learning in the wild*, editoriale del numero 464 di Nature, International weekly journal of science, 8 aprile 2010: http://www.nature.com/nature/journal/v464/n7290/full/464813b.html
[26] http://www.symphonyofscience.com/
[27] https://sites.google.com/site/sciart1/home
[28] LISI, G. A.: *An Exceptionally Simple Theory of Everything*. 2007. http://arxiv.org/PS_cache/arxiv/pdf/0711/0711.0770v1.pdf
[29] http://www.mima.museum/museum.php
[30] VWBPE, http://ejournal.urockcliffe.com/index.php/JOVS/article/viewFile/20/3
[31] http://www.avatarproject.eu/avatar/
[32] http://cdsweb.cern.ch/journal/CERNBulletin/2011/07/News%20Articles/1327589?ln=en
[33] http://www.esplica.it/images/stories/EXT/aplimat_05.jpg
[34] SEURAT, G. (1859-1891) said: "Some see poetry in my paintings; I see only science".
[35] http://scienzapertutti.lnf.infn.it/risposte/ris121.html

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# FIVE CONSTANTS FROM AN ACHEULIAN COMPOUND LINE

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**Abstract.** The origin of advanced mathematics is always spoken of in Babylonian, Egyptian, Greek or Indian terms. In the case of Paleolithic mathematics we usually hear that the evidence represents little more than simple tallying or, at most, calendars. However, limited appraisals such as these are based less on rigorous study of the actual evidence than on a core assumption carried over from evolutionary anthropology, the belief that early peoples were incapable of mathematics. In this paper, I offer more evidence that by the time of Bilzingsleben—a Lower Paleolithic archaeological site 320,000–412,000 years old—*Homo erectus* people already had a long history of mathematical thought including in geometry, trigonometry and fractals; and whether by deliberation or intuition, an awareness of mathematical constants. Scientific techniques not usually applied to ancient artifact studies such as the elimination of variables are used to isolate the constants proposed. Although interpreting Paleolithic artifacts certainly entails a degree of subjectivity, over 15 years time the author has done countless tests similar to these, often to exacting tolerances of 4-5 decimals which collectively demonstrate the same basic mathematical capabilities. The artifacts from Bilzingsleben have the potential for precision studies such as these because they most certainly were engraved with the aid of a straight edge and feature a very high quality of line and angle.

**Key words.** Mathematical constants, Bilzingsleben, *Homo erectus*, Paleolithic mathematics

*Mathematics Subject Classification:* Primary 01A10, 00A30; Secondary 00A65

## 1 Definintions

**Constant** – A number that arises ‘naturally’ in mathematics and remains the same independent of physical measurement (e.g., **Fig. 1**).

**Acheulian -** Designation for a “pre-modern” human culture dating c. 1.8 million to c. 100,000 years ago. It is associated with *Homo erectus*—long regarded as ‘ape-men’ by the science community.

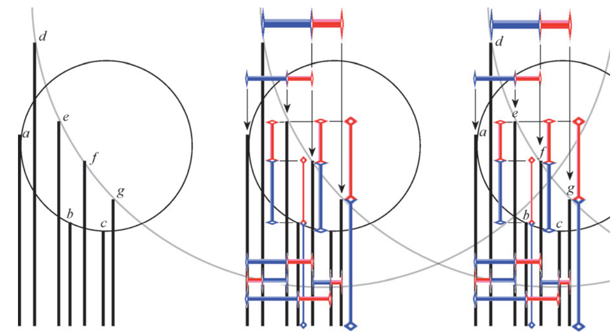

**Fig. 1.** Left: *How the compound line suggested the two circles toward the 5 constants*. Middle and right: *The Golden ratio* (Feliks 2008). See Fig. 5 for how the line itself was constructed.

**Compound line -** A musical term related to the musical style known as counterpoint (two or more independent melodies played together). Specifically, compound line is when one melody line has a zigzag quality where the outer notes of the zigzag can be read as separate melodies such as in the J. S. Bach *Lute Prelude in C Minor* in **Fig. 2** below (here in A minor). Melody *a g a g a f a e* in the upper right of the measure can be read as "Lower" *a g a a a* and "Upper" *g f e*.


**Fig. 2.** Example of a compound line from Bach's *Lute Prelude in C Minor*. The stems-up melody *a g a g a f a e* in the upper right of the measure's last two beats can be read as "Lower" *a g a a a* and "Upper" *g f e*.

## 2 Geometric evidence

In several earlier papers (1, 3, 4, 5) beginning with *Phi in the Acheulian*, I demonstrated that one useful way to explore the unwritten knowledge of early peoples, even with evidence as limited as a single artifact, was by noting the presence of mathematical constants whose deliberate or intuitive representation within an artifact might be confirmed by way of their repetition. The evidence was presented first in relation to the *Golden ratio* (1.618) or *Phi* (e.g., **Fig. 1**). **Figs. 3 & 4** demonstrate in a precision manner how four more apparently related constants were found to exist without any alterations except the removal of two variables—line angle and horizontal positioning—as detailed in **Fig. 5**.

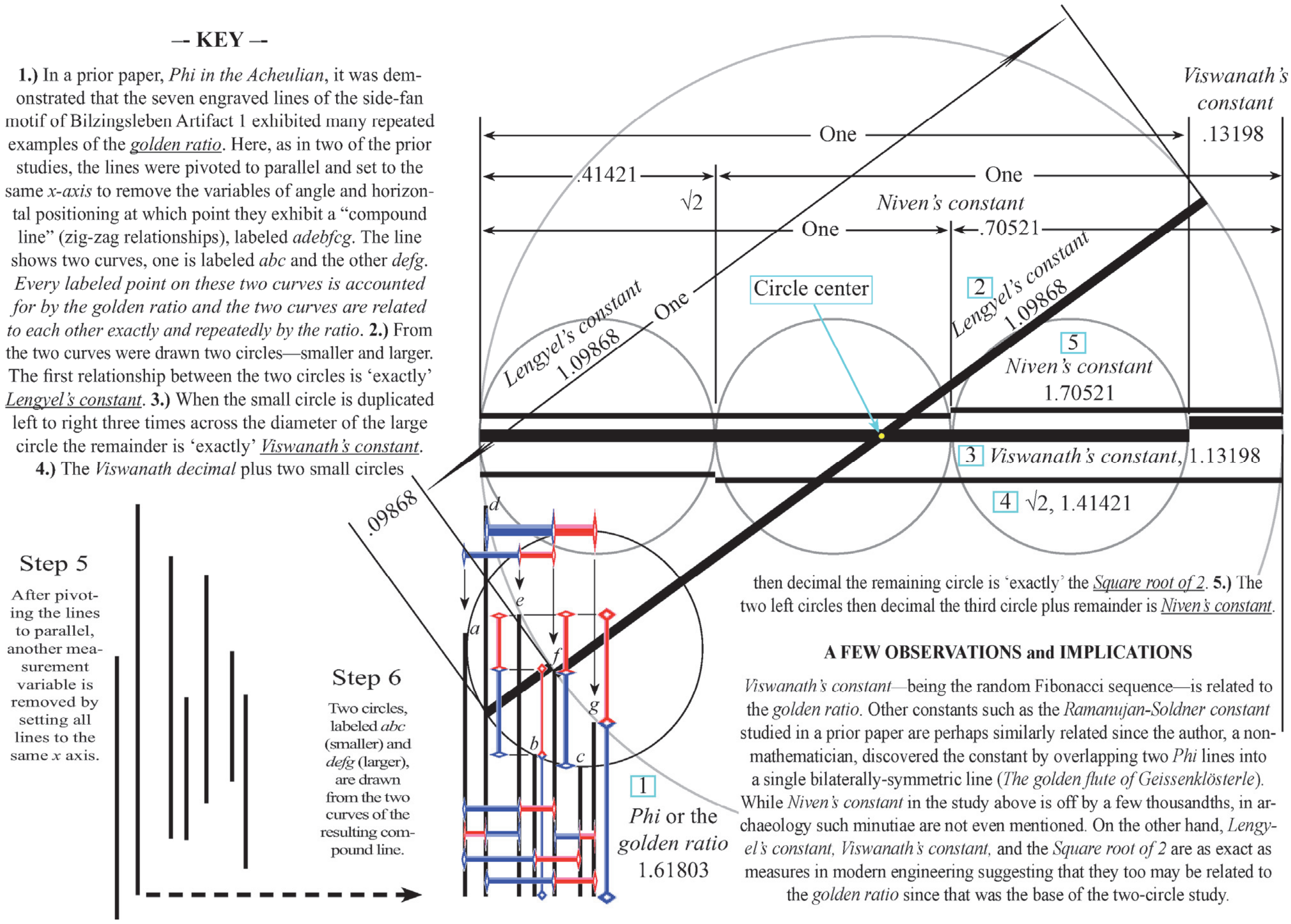


**Fig. 3.** *Five constants from an Acheulian compound line: Overview*. The constants demonstrated are **1.)** the *Golden ratio* in the motif and circle, **2.)** *Lengyel's constant*, **3.)** *Viswanath's constant*, **4.)** the *Square root of 2*, and **5.)** *Niven's constant*; each to a precision of 4-5 decimals. Details are explained in the figure's key above.

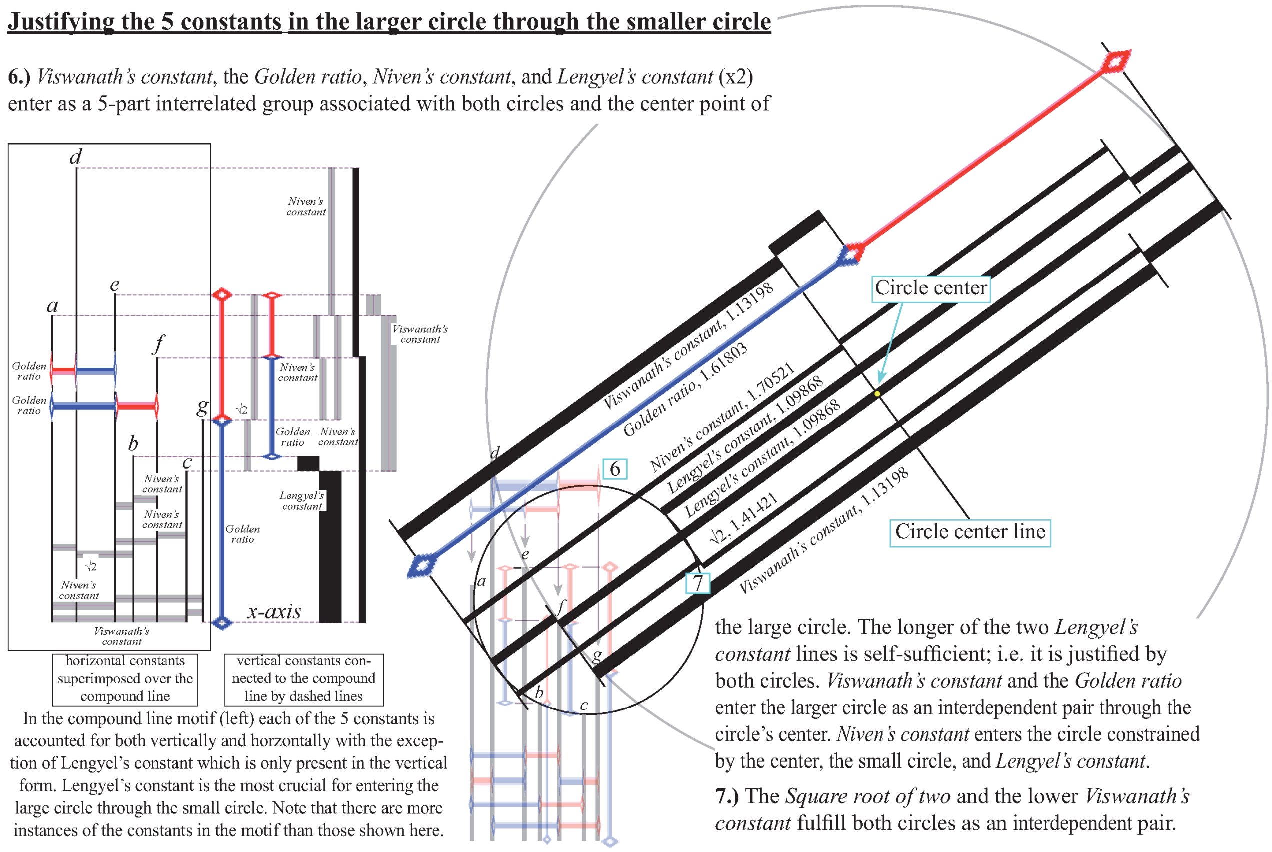


**Fig. 4.** *Justifying the constants in the large circle*. Each of the constants which are named and laid out to high precision in the compound line (left) are represented to equal precision within the two circles in the present figure (right) and in the prior, Fig. 3. The tolerances applied can be seen by zooming into each of the figures. This present study is an attempt to explain at least geometrically how it is that the constants in the compound line are somehow transferred through the small circle into the large circle as if by a sort of *encoding and decoding* process. While such a process was one of the author's initial inclinations, there is also the possibility that the idea of transfer is overly simplistic. It may instead be that just like the demonstrated interdependent relationship between *Viswanath's constant* and the *Golden ratio* (see Point #6 in the figure) as well as the interdependent *Viswanath's constant* and the *Square root of 2* (Point #7) the results imply that all three geometric elements—the compound line, the small circle and the large circle—are themselves interdependent and that none of the observed relationships would exist outside of this unique configuration. While already an enigma with just these five, there are surely additional constants within the configuration suggesting not only that the constants are related but that it is possible 'scholars' at the time of Bilzingsleben 400,000 years ago were *working with constants*. Obviously such a possibility extends beyond the already challenging idea of Mania and Mania that Bilzingsleben was a permanent long-term settlement with a complex culture (10), and even that idea conflicts with the standard science view that *Homo erectus* people were not even intelligent enough to have developed language let alone higher mathematics. I believe that the question will ultimately be one for mathematicians. This is because anthropology for the past 15 years has repeatedly demonstrated its unwillingness and inability to deal with such evidence objectively—censoring these and similar scientific observations from proper discourse and publication even in proceedings volumes where publication was promised in advance. Apart from the agendas of competitive researchers as peer reviewers controlling publication, this problem of censorship must be viewed in light of the fact that the evidence does not support the standard ape-man paradigm which the anthropology community has committed to by faith and enforces with an iron hand. Finally, while the author is open to the possibility that these and other constants which have made appearances in the Bilzingsleben artifacts may be related to the engravers' intuition, the precision demonstrated along with clear indication that the lines were engraved with the aid of a straight edge (2, 6, 7) makes it more reasonable to approach the subject from the perspective of mathematics and that there was likely great skill involved.

As mentioned in the Aplimat 2011 paper, *The golden flute of Geissenklösterle*, the idea that humans became gradually more and more intelligent over time is uncritically accepted in anthropology. It is presently known as 'cognitive evolution' and was first advanced as an aspect of evolutionary theory by Charles Darwin in his 1859 book, *On the Origin of Species*, where Darwin not only presents the idea as a way to explain the psychology of humans but also as the way to interpret human history:

> *"Psychology will be based on a new foundation, that of the necessary acquirement of each mental power and capacity by gradation. Light will be thrown on the origin of man and his history." – Charles Darwin, 1859: 488*

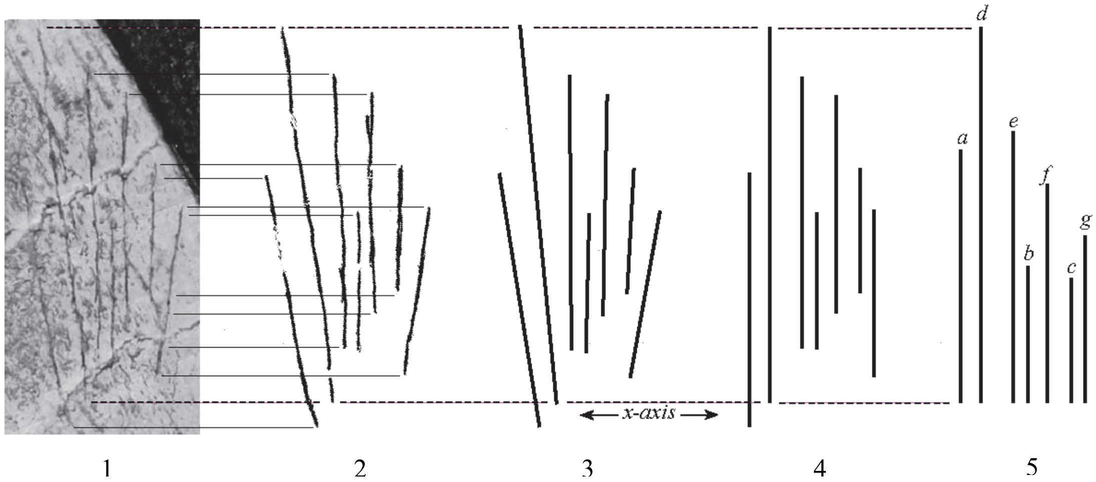


**Fig. 5.** *Steps performed to isolate the compound line by removing two variables*. As can be seen, there is a degree of subjectivity involved; however, similar results to those presented in this paper occur even with slightly different interpretations of the engraved lines. As background, the author has done systematic studies of the Bilzingsleben engravings and many other artifacts since 1993 both without alterations and with alterations to eliminate mathematical variables such as in this presentation. Although results may vary, the differences between the two are minor and many of the same mathematical constants tend to show up somewhere even with slightly different line lengths or spacing applied. 1.) 400,000-year old extinct elephant bone engraving from Bilzingsleben, Germany, discovered by Dietrich and Ursula Mania (9, 10, 11); Photo by R. Bednarik cropped with permission (6). 2.) Original line redraws using light table showing line length interpretations from 2004 (7). 3.) Lines redrawn digitally for geometric tests [Feliks 2008, *Phi in the Acheulian*; XV UISPP Congress, Lisbon 2006 (5)]. Unlike in normal proceedings volumes, immediately after presentation, the Congress attempted to block the *Phi* paper from publication on the grounds that it was highly problematic and would ruin the presenter's credibility if published, and did block the requested Part 1 paper, *The Graphics of Bilzingsleben* (6), for 5 years after a long battle which began within one week of the congress, relegating it finally to an obscure miscellanea volume. The facts of censorship have become a primary and intertwined part of the story of this evidence. The latter mentioned paper was also circulated in anonymous peer review and then, not unexpectedly, blocked from publication by the evolutionary community. 4.) Angle variable is removed by pivoting the lines to parallel on their center points. 5.) Horizontal position variable is removed by setting all lines to the same x-axis. The resulting zigzag is what is studied as a compound line read as two curved lines labeled *abc* & *defg*.

So, once a premise such as cognitive evolution is accepted uncritically as it has been not only by the anthropology community but by the entire modern science community as though the level of evidence in this area is similar to that of claims in other sciences, the motivation for adherents is of only one kind—find evidence which supports the premise. If there is no evidence supporting the premise, then whatever evidence is available, pro or con, must be made to fit into the premise regardless of what it may actually suggest. The only other option for strict adherents since changing the paradigm is not one of them is to censor challenging evidence and ignore it as if it simply were not there. This is the main difference between the three fields which have committed themselves to the evolutionary template—paleontology, biology, anthropology—and fields which follow a more normal scientific course and go wherever the evidence leads. For this reason I propose that the true nature of early human intelligence study needs to be taken out of the hands of anthropology and taken up by other fields such as mathematics or the arts as anthropology has continued to demonstrate that it can no longer be depended upon to provide objective assessments as it withholds evidence from the public in order to prevent challenging material from being seen. If one adheres to the paradigm uncritically, as most scientists do today, then one has no choice but to interpret cultural evidence in a way that supports the paradigm regardless of the intelligence indicated by such as ancient precision engravings or even collected and curated objects such as fossils (8).

## 3 Conclusion

It was not the author's original intent in 2006 to discover constants but only to continue studying engraved artifacts by geometric means. However, the *golden ratio* appeared repeatedly in three of the artifacts (5) which eventually led to the other constants suggesting that these also may have been accessible to Acheulian people. The constants are in the context of artifacts which exhibit very meticulous work and are confirmed by radiometric dating to be 320,000–412,000 years old (9, 10, and 11). In other words, the studies support the proposition that the origins of advanced mathematics does not begin with the Babylonians or the Egyptians but much farther back in time, at least as far back as *Homo erectus* at Bilzingsleben. Cognitive evolution is a faith-based paradigm which has been so persistently promoted in science as to create an impression in the public's mind that it is obvious and that any proposals such as those put forth in this paper and others are absurd. However, the weaknesses of the evolutionary paradigm become immediately apparent as soon as one begins to look at the evidence objectively outside the influence of the consensus scientific community. Instead of intelligence gradually increasing from ape-man levels then culminating in modern man, the evidence shows that there has been a verifiable continuity of human intelligence from earliest times. This opens a new door, one to genuinely ancient mathematics and philosophy.

## About the author

John Feliks has specialized in the study of early human cognition for nearly twenty years using an approach based primarily on geometry and techniques of drafting. Feliks is not a mathematician; however, he uses the mathematics of ancient artifacts to show that human cognition does not evolve and that early people living hundreds of thousands of years ago were just as intelligent as anyone living today. He is founder of the Pleistocene Coalition, a group challenging mainstream science and its peer review which prevents evidence not adhering to the evolutionary template from being published creating the false impression that there is no challenging evidence. He is also layout editor for the group's newsletter, *Pleistocene Coalition News*. One aspect of Feliks' experience that has given unique perspective on the mathematical or symbolic qualities of ancient artifacts is his background in music; he is a long-time composer in a Bach-like tradition as well as a songwriter in the acoustic-rock tradition and taught computer music including MIDI, digital audio editing, and music notation in a college music lab for 11 years. This musical background was the inspiration to study more closely the Acheulian compound line and many other aspects of the artifacts from Bilzingsleben.

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## References


[1] FELIKS, J.: *Base grids of a suppressed* Homo erectus *knowledge system*. Pleistocene Coalition News Vol. 3 (6), pp. 12-14. 2011.

[2] FELIKS, J.: *Straight edge use by* Homo erectus. Pleistocene Coalition News Vol. 3 (4), pp. 14-16. 2011.

[3] FELIKS, J.: *The golden flute of Geissenklösterle: Mathematical evidence for a continuity of human intelligence as opposed to evolutionary change through time*. Aplimat – Journal of Applied Mathematics, Vol. 4. 2011.

[4] FELIKS, J.: *Phi-based conceptual units: Pushing math origins back to the Acheulian age*. Published online, SCIENAR website. 2010.

[5] FELIKS, J.: *Phi in the Acheulian: Lower Palaeolithic intuition and the natural origins of analogy*. In: Pleistocene palaeoart of the world, Proceedings of XV UISPP World Congress (Lisbon, 4-9 September 2006). British Archaeological Reports International Series, Vol. 1804, pp. 11-31. Oxford. 2008.

[6] FELIKS, J.: *The graphics of Bilzingsleben: Sophistication and subtlety in the mind of Homo erectus.* XV UISPP World Congress (Lisbon, 4-9 September 2006). British Archaeological Reports International Series, Vol. 2224, pp. 71-91. 2011.

[7] FELIKS, J.: *Musings on the Palaeolithic fan motif*. In: Exploring the mind of ancient man, pp. 249-66. Research India Press, New Delhi. 2006.

[8] FELIKS, J.: *The impact of fossils on the development of visual representation*. Rock Art Research 15 (2), pp. 109-34. 1998.

[9] MANIA, D., and U. MANIA.: *Deliberate engravings on bone artifacts of* Homo erectus. Rock Art Research, Vol. 5, pp. 91-107. 1988.

[10] MANIA, D., and U. MANIA.: *The natural and sociocultural environment of* Homo erectus *at Bilzingsleben*, Germany. In: C. Gamble and M. Porr (eds.), The Hominid Individual in Context: Archaeological investigations of Lower and Middle Palaeolithic landscapes, locales and artifacts, pp. 98-114. Routledge, New York. 2005.

[11] MANIA, D., and U. MANIA.: *Bilzingsleben* – Homo erectus*, his culture and his environment. The most important results of research*. In: J. M. Burdukiewicz and A. Ronen (eds.), *Lower Palaeolithic small tools in Europe and the Levant.* British Archaeological Reports, Vol. S1115, pp. 29-48. 2003.


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# ASTRONOMY AND PERSPECTIVE IN THE CITIES FOUNDED BY ALEXANDER THE GREAT

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**Abstract.** During his life Alexander the Great revived, in a original way, the Greek tradition of town's foundations. Scores of cities are attributed to him, although only few have been documented archaeologically. The paradigm of Alexander's towns is Alexandria, founded in 331 BC. In the present paper we examine the topography and the astronomical orientation of the original urban system, and show that the choice of the venue was mainly due to religious and symbolic reasons. Alexandria was actually the prototype of a series of Hellenistic towns designed as "king's towns" and aimed to convey the ideology of the divine power of their founder; these towns all had a spectacular axis of perspective. The prototype of such axes is the Alexandria Canopic road, which turns out to be orientated to the rising sun on the day of birth of Alexander the Great and to the "king's star" Regulus. A second family of towns, related also to military criteria and orientated close to the cardinal points, is discussed as well.

**Key words and phrases.** Ancient town planning, Ancient Astronomy, Alexander the Great, Alexandria

*Mathematics Subject Classification:* Primary 01A16, 51-03

## 1 Introduction

The foundation – or also re-foundation – of towns was certainly one of the characteristics of Alexander the Great' conquest policy. Some were simply military settlements aimed to settle part of the army and control the territory, but others were founded with the aim of being "true" Greek colonies, with civilization purposes. The tradition of town foundation was then inherited by the Hellenistic Monarchs.

Alexander is the pioneer of a new concept of civilisation, based not upon the conventional dichotomy of Greek and non-Greek but upon a universal equality of status and understanding. Diodorus the Sicilian, compiling in the first century BC and supported perhaps by implication in passages of Strabo, Plutarch and Arrian, cites a *memorandum* by Alexander which professes to show that he was beginning to think in international terms. Whether literally historical or not, the *memorandum* has a certain ring of truth about it: he was planning to create "cities with mixed populations, to transplant people from Asia to Greece (Europe) and in the opposite direction from Greece to Asia, and so to establish the greatest continents in common unity and friendly kingship by

intermarriages and domestic ties”. On such showing, it can be claimed for him that he was the first true internationalist [1,2].
The prototype of Alexander's towns is certainly Alexandria. The foundation of Alexandria was a truly symbolic act, inspired by “religious” criteria and aiming at the celebration of Alexander’s power and divine nature [3]. As a consequence, the ideology of power can be seen reflected in the city design. We report here on recent work aimed at understanding the way in which all this was achieved with the use of astronomy an perspective [4]. Further, we extend our approach here to the whole series of “related” towns and show that they actually divide into two groups: a group of “king's cities” clearly inspired by Alexandria, which in most cases exhibit a similar “perspective” plan based on a longitudinal axis and similar astronomical alignments as well, and a second group of “military” towns which are mostly oriented to the cardinal points as were the corresponding Roman's “castra” (fortified military camps).

## 2 Geometry and Astronomy in the urban plan of Alexandria

Alexandria is the most famous and important city founded by Alexander the Great. The town, founded in 331 BC [4], can be considered as the outcome of a long debate on the idea of “ideal town”: indeed both Socrates and Plato repeatedly prefigured the birth of the ideal city. The inspiring principles are based on harmony as related to the laws and the divine, reflected in the mathematical rigour of the design of the “Hippodamean” city plan [5,6]. Foundation of Alexandria can be seen as the beginning of a series of new towns, those of the Seleucids, which will repeat its inspiring principles [2] . The city becomes an explicit representation of the power of its divine founder, the rigorous order of its plan being a reflection of the “cosmic” order, in compliance with the “orthogonal grid” principles. The orthogonal grid of Alexandria can still be perceived, and forms the basis for an ongoing project of reassessment of antiquities into a coherent architectural scheme of fruition [7,8,9]. The original matrix route was conceived on the basis of a main longitudinal axis, later called Canopic Road. This road played the role of an “extended center”, a wide, longitudinal open space, with the main buildings distributed along it, thus avoiding the idea of a “central point” as the focus of the urban plan. The first to put in evidence such a “longitudinal” character in the original project of Alexandria was 19th century astronomer Mahmud Bey Al-Falaki (1861). Later excavations along the modern street showed that the Canopic Road was actually deeply etched in the rock subsoil. The axis is thus a peculiar characteristic, a sort of icon in the foundation of the city, and as such it is an independent architectural unity [10,11,12,13,14].

In spite of what is emphatically reported by Plutarch in his Life of Alexander (26, 2-3) and by Diodorus Siculus (17, 52) the site where the newly founded town was built did not have special characteristics of suitability. In particular, the city was planned in a strip enclosed between the sea to the north and west, the marshy lands of the Canopus mouth of the Nile to the east, and Mareotis Lake to the south, in contrast with many of the healthy criteria of Alexander's tutor, Aristotle [15]. Another characteristic (to be discussed later on) which clearly conflicts with utilitarian principles is that the orthogonal grid seems to be not conformal to the characteristics of the landscape. As mentioned above, the rectangular grid of Alexandria was based on the so called Canopic road, a spectacular axis which crossed the city leading to the Canopic mouth of the Nile and the Canopus (today Abukir) bay. At the opposite ends of the street two main gates were located; at least since the work of Achilles Tatius (early second century AD) the east and west gates were called Gate of the Sun and Gate of the Moon respectively [16]. The Canopic road bears an azimuth of 65°15’ ± 30’ (measured directly and validated with existing sources), and the horizon to the east extends towards the Abukir bay and was therefore flat in ancient times; the same holds to the west.


Tav.1 Alexandria, reconstruction scheme of the original town plan

Tav.2 Alexandria, a early 19$^{th}$ century photograph showing the Canopic road looking west.

In 331 BC the azimuth of the rising sun at the summer solstice - occurring on June 28 (all the dates in this paper are Julian) - was 62° 20’ (today it is slightly displaced due to the variation in the obliquity of the Ecliptic). It can of course be said, therefore, that the orientation of Alexandria axis at 65° 15' is “solar” in that the sun was (and is) rising along this direction twice a year. The dates are July 24 and the symmetric date in relation to the summer solstice, June 2. The range - one degree wide - centered on azimuth 65° 15', was spanned by the rising sun in a period of a few days before and after this date, respectively. We have recently proposed that the city was orientated to the rising sun on the day of the birth of Alexander the Great. Alexander was indeed born on July 20, 356 BC, and in the 4th century BC the sun was rising at Alexandria on that day at an azimuth 64° 30', only 45' less than our best estimate for the azimuth of the Canopic Road (of course, the Julian date of birth of Alexander has nothing to do with the calendar in use in that earlier period; for a complete discussion about the issue see [4]).
Due to the waving of the lunar calendars and to differences between different local calendars, the Greeks elaborated astronomical methods to act as harbingers for relevant festivities [17,18,19]. The Alexandria alignment was operational also in this sense. Surprisingly indeed, the “King's Star” Regulus (alpha-Leonis), was at that time rising at the very same azimuth (65° 20' at altitude of one degree, appropriate for the visibility of a first magnitude star) and had heliacal rising very near to July 20. Association of Regulus with the king was already extremely old (it is documented in ancient Babylonian sources such as the Mul-Apin) but it is certainly a fortunate circumstance for Alexander to be born in Leo, and association of Regulus with kingship will be strengthened during the Hellenism. In particular, it has been recently recovered in the funerary monument of Antiochos I, King of Commagene, at Mount Nemrud [20]. The monument, constructed in the first half of the last century BC, is well known for the famous “lion horoscope”, depicting Mars, Mercury, Jupiter and the crescent moon in Leo, the constellation of Regulus [21]. Belmonte and Garcia have shown that the two terraces of the monument are orientated to the solstices, but also that the huge plinths holding the colossal statues in the eastern terrace point to sunrise around 23 July and the heliacal rising of Regulus, the date of the celebration of Antiochos’s ascent to the throne mentioned in the inscriptions of the monument. The coincidence with Alexandria is really striking, considering that Antiochos makes explicit reference to Alexander the Great as one of his ancestor.

## 3 Alexander’s and Seleucid’s towns

As recalled above, Alexander the Great founded many cities, most of which are scattered in a west-to-east progression following the route of his conquests. Further, many cities were founded by the Seleucids, and it is known that Alexandria has been considered as a sort of model town during Hellenism. In particular in the Farther East a rather extraordinary experiment occurred, being the scene of the interaction between the Greek civilisation and those of Babylonia, Iran and India. The Greek empire of Bactria and India was an Hellenistic state and so must be treated. The Seleucid Empire was not centralized and no imperial citizenship, because the governor of a Eparchy had an organisation ready to his hand, even to a *basileion* or palace residence. Following the idea of Alexander, the Greek settlement of Asia under the Seleucids is a complex of contiguous and quasi-autonomous city states, the whole under a quasi-divine king who managed policy and order [22].
In Table 1 we present the available data on those cities which have been documented archaeologically, either because they are still inhabited or due to excavations. All such cities are based on an orthogonal grid. In the table, the orientation of that axis which lies in the first quadrant of the compass is reported. Data have been extracted from maps and compared with high-definition satellite images to align to true north; the expected error in this procedure – mainly due to flatness distortion effects - does not exceed 1°.

| NAME | FOUND. | AZIMUTH | SS/WS |
|---|---|---|---|
| ALEXANDRIA | 331 BC | 65° 20' | 62° 20'/117° 40' |
| ALEXANDRIA of ARACHOSIA (KANDAHAR) | IV B.C | 28° | 61°24' / 119°16' |
| ALEXANDRIA in ARIA (HERAT) | 330 B.C | 88° | 60°23' / 120°17' |
| ALEXANDRIA ESCHATE (CHUJAND) | 329 B.C | 65°40' | 57°36' / 123°4' |
| BALKH (ALEXANDRIA OF BACTRIA) | IV B.C | 33° | 59°21' / 121°19' |
| ALEXANDRIA in CAUCASUM (BEGRAM) | IV B.C | 5° | 60°7' / 119°3' |
| ALEXANDRIA of MARGIANA (MERV) alias Antioch in Margiana | IV B.C | 89°30' | 58°57' / 121°43' |
| AI-KHANUM (Greek name never found, sometimes supposed as Alexandria on the Oxus) | IV B.C | 34°30' | 59°9' / 120°01' |
| ALEXANDRIA of SUSIANA alias Antioch, later Charax Spasinou | 324 B.C | 86° | 61°38' / 119°02' |
| ALEPPO | 333 B.C | 89° | 59°35' / 121°05' |
| ANTIOCHIA on the ORONTE | 300 B.C | 46°70' / 39°40' | 59°36' / 121°04' |
| APAMEA on the ORONTE | 300 BC | 89° | 59°56' / 120°44' |
| DAMASCO | 332 B.C | 86°10' | 60°42' / 119°58' |
| DURA EUROPOS | 303 B.C | 65° | 60°13' / 120°09' |
| LAODICEA at the sea | IV B.C | 2° | 59°54' / 120°06' |
| SELEUCIA on the TIGRIS | IV sec. B.C | 25° | 60°52' / 119°48' |
| TAXILA | IV B.C | 2°10' | 60°36' / 120°4' |

Tab. 1 The towns founded by Alexander and by the Seleucids and documented archaeologically.
For each town ancient and modern name, approximate date of foundation, azimuth of the axis in the first quadrant, and solar azimuths at the solstices are given.

In such cities, two families can be immediately singled out.

### 3.1 The cardinal family

In this family, containing 10 cities out of 17, the axes are all found in an interval of ±5° with respect to true north. These cities were probably inspired by military camps and forts. The idea of military colony goes back to Alexander, but some clues seem to show that Alexander learned a lot about the organization and logistics of the army from the ancient Egyptians, inspiring himself to another “great” whose campaigns he certainly saw represented in the Luxor temple, Ramesses II. The Egyptian camps were indeed oriented to cardinal points as well as the Greeks; the army of the king of Epirus is also documented.

Traditionally Alexander founded 70 cities; but comparatively few can be identified (Alexandria Egypt, Alexandria in Aria, Alexandria of Arachosia, Alexandria in Bactria, Alexandria Eschate, Alexandria in Caucasum); the same is true of many of the cities attributed to Seleucus. The first

Antigonus continued Alexander's system, as did the early Seleucids, but it is only rarely that we know under which king any particular colony was founded. A military colony was settled either with time-expired troops, sometimes mercenaries, or with men able and willing to serve; normally it was located at or beside a native village, and it was usually founded by provincial governor upon the king's order; the king had to provide the land and the money required, but he did delegate the actual work to a subordinate. The purpose of the military colony was primarily defence: those in Bactria-Sodgiana, started by Alexander, were to safeguard the frontier against nomads.
It was a planned foundation; perhaps a stereotyped form existed. In the Mediterranean countries the great majority of settlers were Greek and Macedonian, but east of Euphrates this elements tended to become thinner. Greek was always the official language.
The aim of every military colony was to become a full polis, which in the East meant a city, not necessarily of Greek nationality, but of Greek organisation and civic forms; there was a steady up ward growth of the colony into the *polis*, and it was this which before the end of the second century BC had filled Asia with Greek cities [1,22].

### 3.2 The Alexandria-like family

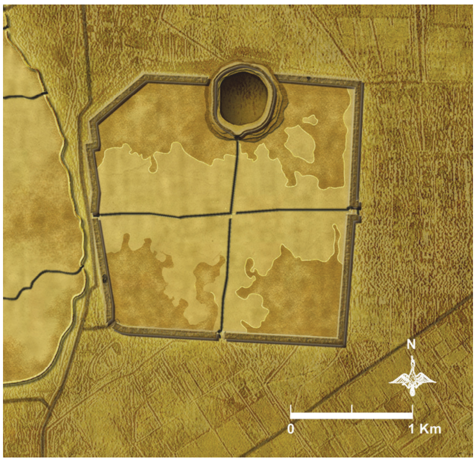
In this family either the axis considered is orientated very close to that of Alexandria (Alex. Eschate, Dura), or the orthogonal axis points to the symmetric direction, that is, to sunset instead of sunrise. This is the case of the most important of Seleucids towns, Seleucia on the Tigris, which was founded in 300 BC by Seleucus I Nikator (305-281 BC) [23] . The site is not far from Babylon, where Alexander died on June 10, 323 BC and where the first residence of the Seleucids was established. The topography of Seleucia is manifestly inspired by that of Alexandria, with a main longitudinal road and a regular urban matrix nested on such an axis [24,25,26,27]. The "Canopic" axis of Seleucia is not the axis of the first quadrant (azimuth 25°) but the orthogonal one at 115° . Therefore, it points 25° north of west which is specular with respect to the meridian to the Canopic road of Alexandria. Would Alexandria and Seleucia be at the same latitude, the sun with a flat horizon would thus set in alignment with the longitudinal axis at Seleucia the very same days as at rising in Alexandria. Due to the slight difference in latitude the sun was actually setting along this direction on the days around July 27, with a slight displacement, but in any case still very close to the date of birth of Alexander. As well as in Alexandria, there is also a close concordance with Regulus, whose setting occurred approximately at azimuth 294° 40' . It is therefore tempting to speculate that also Seleucia, being inspired by Alexandria both symbolically and practically, but being also close to the place of death of the revered king, was orientated towards the same astronomical targets, but at their settings.

The remaining three towns have an orientation which requires further studies, but is likely topographical.

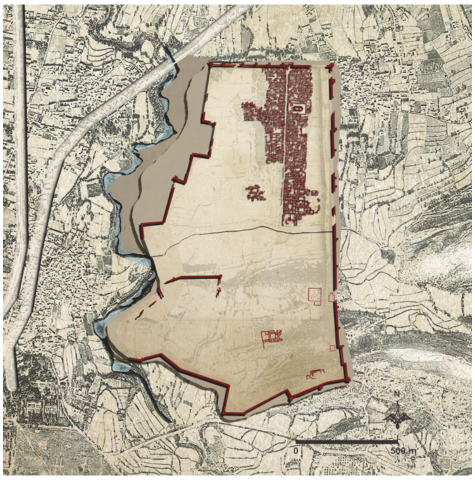
## 4. Conclusions

Among the possible symbolic aspects related with foundation and to be considered in the analysis of a town's project are, of course, geometry and orientation. We presented here the first results of a project aimed at understanding the way in which the ideology of divine kingship was embodied in the towns founded by Alexander the Great and his successors. Two main characteristics come out from such an analysis: the first is the idea of the "main axis" which opens up the perspective of the town. Such a main axis breaks, in a sense, the monotony of the orthogonal grid, introducing an

"extended point of attraction". Second, the astronomical orientation of such axis, related to the sun and the stars on the birthday of the founder. Both such features appear to be established for the first time with the foundation of Alexandria. A second family of towns also exists, were the axes are orientated cardinally. Several influences can be hypothesized here, including Egyptian and Assyrian camps but also the Buddhist foundation ritual.
All in all, Alexander the Great confirms himself, once again, as "marking a major devide in the broad history and archaeology ideas" as Sir Mortirmer Wheeler once said.




**ALEXANDRIA of MARGIANA (Merv)**



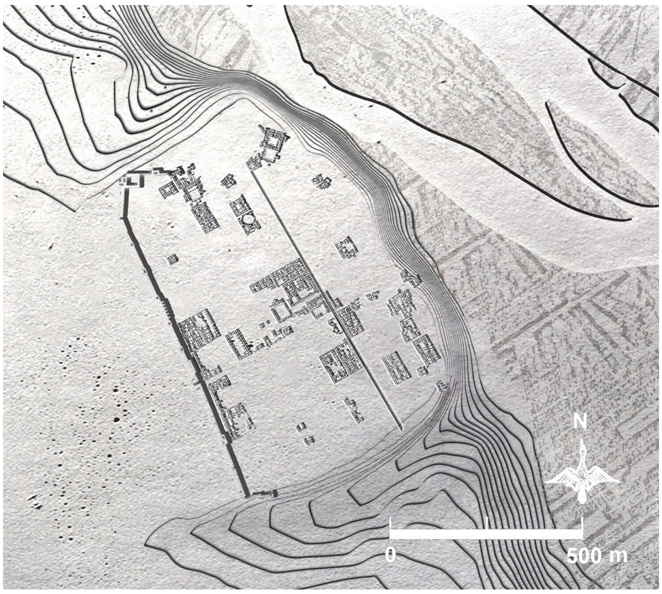
**TAXILA**

**APAMEA on the ORONTE**

**Tav.3 Three examples of the cardinal family of towns**


ALEXANDRIA ESCHATE (Chujand)


SELEUCIA on the TIGRIS




DURA EUROPOS

**Tav.4 Three examples of the Alexandria family**

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**References**

[1] TARN, W.W. 1966. The Greeks in Bactria and India. Cambridge University Press.

[2] WHEELER, M. 1968. *Flames over Persepolis. Turning-point in History*. London: Weidenfeld and Nicolson.

[3] ROSS TAYLOR, L. *The Cult of Alexander at Alexandria* Classical Philology, Vol. 22: 62-169

[4] FERRO L. AND MAGLI G. 2011 *The astronomical orientation of the urban plan of Alexandria* http://arxiv.org/abs/1103.0939

[5] BAGNALL, R.S. 1979. *The date of the foundation of Alexandria*. American Journal of Ancient History IV: 46-9.

[6] CASTAGNOLI, F. 1971. *Orthogonal Town Planning in Antiquity*. Cambridge, MA: MIT press

[7] SHIPLEY, G. 2005. *Little Boxes on the Hillside: Greek Town Planning, Hippodamos, and Polis Ideology* in M. H. Hansen (ed) *The imaginary polis : symposium*: 335-403. Historisk-filosofiske meddelelser, 91; Acts of the Copenhagen Polis Centre, Det Kongelige Danske videnskabernes selskab,.

[8] FERRO L. & PALLINI C. (ed) 2009. *Alessandria d'Egitto oltre il mito. Alexandria Beyond the Myth. Architecture Archaeology Urban Change.* Boves-Cuneo: Araba Fenice.

[9] TORRICELLI, A. 2010. *Aree archeologiche e progetto per le città*. Ananke 61: 110-115.

[10] TORRICELLI, A. 2010. *La ricerca progettuale come interrogazione del tempo*: 11-14 in *Riprogettare l'archeologia*. Milano: RotarArch.

[11] BRECCIA, E. 1914. *Alexandrea ad Aegyptum. Guide de la Ville ancienne et moderne et du Musée Gréco-Romain*. Bergamo: Istituto Italiano d'arti grafiche.

[12] MUMFORD, L. 1967. *The City in History*. Milano: ed. it Bompiani Edizioni.

[13] CARUSO, G. 1993. *Alcuni aspetti dell'urbanistica di Alessandria in età ellenistica: il piano di progettazione*, in *Alessandria e il mondo ellenistico-romano. Studi in onore di Achille Adriani*. Roma: "L'Erma" di Bretschneider.

[14] FERRO, L. 2010. *Alessandria d'Egitto. Un progetto per la città*. Ananke 61:117-125.

[15] BERNAND, A. 1995. *Alexandrie des Ptolomées*. Paris: CNRS Editions.

[16] HAAS, C. & MCDONALD, A. 1997. *Alexandria in Late Antiquity: Topography and Social Conflict*. Baltimore: The Johns Hopkins University Press.

[17] HANNAH, R. 2005. *Greek and Roman Calendars: Constructions of Time in the Classical World*. Duckworth: London.

[18] HANNAH, R. 2009. *Time in Antiquity*. London: Routledge.

[19] SALT, A. BOUTSIKAS, E. 2005. *Knowing when to consult the oracle at Delphi.* Antiquity 79: 564-572.

[20] BELMONTE, J.A., and GARCÍA, C.G. 2010. *Antiochos's hierothesion at Nemrud Dag revisited: adjusting the date in the light of astronomical evidence*. JHA 41, 469-481.

[21] NEUGEBAUER O. and VAN HOESSEN, H. 1959. *Greek horoscopes*. Memoirs of the American Philosophical Society 48, 14–16.

[22] FRASER, P.M. 1996. *Cities of Alexander the Great*. Oxford: Clarendon Press.

[23] HADLEY, R. A. 1978. *The Foundation Date of Seleucia-on-the-Tigris*: 228-230 in Historia: Zeitschrift für Alte Geschichte, Bd.27, H. 1. Verlag.

[24] FERRO, L. 2010. *Archeologia e progetto di architettura*: 35-38 in *Riprogettare l'archeologia*. Milano: RotarArch.

[25] GULLINI, G. 1967. *First Report of the Results of the First Excavation at Seleucia and Ctesiphon*: 136. Mesopotamia II.

[26] HOPKINS, C. 1972. *A Bird's-eye View of Opis and Seleucia*: 440-448. Antiquity 52, vol.13.

[27] MESSINA, V. 2007. *Seleucia al Tigri*: 107-115 in *Sulla via di Alessandro da Seleucia al Gandhara*. Milano: Silvana Editoriale.

[28] BARBER, P. 2001. *Mito, religione e conoscenza: la mappa del mondo medievale* in *Segni e sogni della terra. Il disegno del mondo dal mito di Atlante alla geografia delle reti*. Milano: Istituto Geografico de Agostini


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# THE GEOMETRY OF POLYTOPES

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**Abstract.** Among Geometries that have mainly fascinated and captured the interest of artists and the genius of researchers surely there are the non-Euclidean ones, and especially *Hyperspaces*. These have in fact had a major role. After the birth of *"Kaleidoscope"* up to more complex architectures, such as *Cyberspace*, mathematicians, physicists and artists, each one according to his own scientific inclinations, have addressed their studies towards this themes, completing each other, not only with the purpose of visually representing a multidimensional system but also in order to find methods of calculation allowing one to describe forms that were more and more complex than ever before.

**Key words.** Algebraic Curves, Geometry, Art

*Mathematics Subject Classification:* AMS_01A99

## 1 Introduction

It is well known that *Platonic Solids,* meant as the smallest *"modular forms"* that organize visible and invisible Matter, provide a primitive although important and refined model for ***Polytopes***, as being composed of *Polyhedra* (i.e., regular 3-dimensional Polytopes) as well as Polytopes endowed with faces having a Polygonal shape (i.e., 2-dimensional regular Polytopes).

Their first known description is found in the *Timaeus* of *Plato*, dating back to 424 B.C. – 347 B.C., even if one cannot a priori exclude that some properties of these solids were already known well before this.

Besides being quoted in the XII Book of *Euclid*'s Elements (300 B.C.) and later more deeply investigated by *Archimedes* (287-212 B.C.) with his **Thirteen** *semi-Regular Solids*, during Renaissance they become the object of an intensive investigation, mainly because of their metric properties, both by artists and by mathematicians as *Piero della Francesca* (in his treatise *De Corporibus Regularibus*) and mainly by *Luca Pacioli*, who in 1519 (in his *De Divina Proportione*) makes even use of the famous drawings on Polyhedra executed by *Leonardo da Vinci* (whom they were commissioned to). Later on, the neo-Platonic mathematician and astronomer *Johannes Kepler* in his *Mysterium Cosmografaphicum (1619)* revisits them, even if mainly under an esoteric viewpoint. It is also fundamental to recall what *Bernhard Riemann* said in his Doctoral Thesis *"On the Hypotheses that Lie at the Bases of Geometry"* (1856), where – starting from the notion of "distance" – he defines the notion of ***multi-dimensional manifold*** with a variable curvature, so developing those concepts that, as he himself admitted, were already present in the earlier works done by *Friedrich Gauss* on curved surfaces. A new model of *Geometry of Space*, therefore, destined to become also a model for investigating the propagation of physical interactions and forces in *Albert Einstein*'s ***Theory of (General) Relativity***.

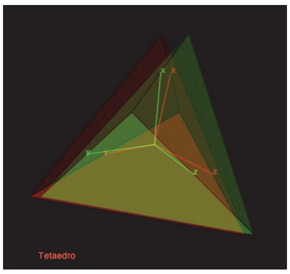
Therefore we have a fundamental mathematical tool in which some diagrams of Polytopes in $R^n$, of an arbitrary dimension *n*, such as those of *G. Schlegel* and of *Gale*, become essential elements to investigate not only the *"Chronotope"* (i.e., ***Spacetime***) but also the phenomena of modern Physics that are related with *Micro- and Macro-Cosmic scales*. The analogues of the five ***Platonic Polyhedra*** of $R^3$ are the **six** Regular Polytopes in $R^4$ called, respectively, the *5-cell*, having five tetrahedra as faces, the *Hypercube*, with eight cubic faces, the *16-cell*, delimited by 16 tetrahedra, the *24-cell*, with 24 octahedra, the *120-cell*, with 120 dodecahedra and the *600-cell*, with 600 faces formed by as many tetrahedra.

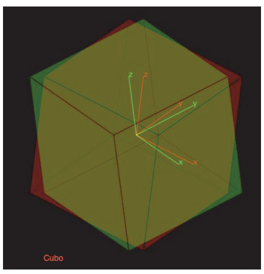
Among the six Polytopes listed above, an historically privileged position is occupied by the Hypercube. Its notion [1] belongs to the Geometry of Hyperspaces and, in its turn, it belongs also to those mathematical objects the form of which is "multidimensional". These hyperspaces are described within the context of *n-hypersurfaces* and, as such, they are generated by the intersection of a number of subspaces of dimension ($n$ – 1) such that no half-line belongs to the intersection itself. These hyperspaces are usually called *Convex Polytopes*. The very notion of *Polytope of Degree n* is defined as a closed and bounded (and therefore compact) subset of $R^n$ where the cells are ($n$ – 1)-Polytopes forming the boundary, the edges and the vertices. In $R^4$ the Hypercube is a geometric figure formed by 3-dimensional cells, i.e. polyhedra, by 2-dimensional (cubic) cells called "faces", by 1-dimensional cells (squares) called "edges" and by 0-dimensional cells (i.e., points) that are the vertices. A Polytope is recursively said to be *Regular* if and only if all cells and all figures (also called "stars" or "cusps") at all vertices are regular ($n$ – 1)-Polytopes. The Hypercube is a Regular Polytope.

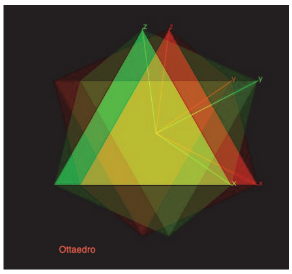
To each Regular Polytope one associates an hypersphere, circumscribed to it so that all vertices are on it. The fundamental relation isin the so-called *Euler-Poincarè Theorem*, that states that in a convex Polytope $S_n$ of dimension *n* the following holds true

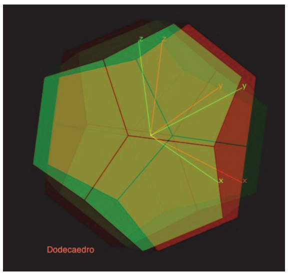
$$N_0 - N_1 + N_2 - \ldots.. + (-1)^{n-1} N_{n-1} = 1 - (-1)^n .$$

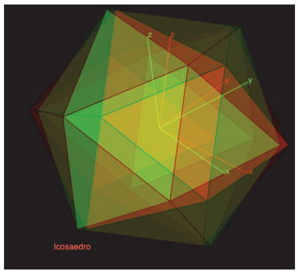
where $N_i$ denotes the number of cells of dimension *i*, for each dimension. It was introduced in the XVIII Century by *Euler* for the study of surfaces ($n$ = 2) and later generalized by *H. Poincaré* within the context of *Topology*, a new branch of Mathematics that was being developed at the end of the XIX Century. With the aid of this equation, in dimension 2, one can easily show that the regular 2-Polytopes are just **five**, the aforementioned Platonic Solids, wiz.: the *Tetrahedron*, the *Cube*, the *Octahedron*, the *Dodecahedron* and the *Icosahedron*.








*Anaglific Representation of the five Platonic Solids – © Vincenzo Iorfida*

More specifically, in the case of a Polytope in $R^4$, such as the Hypercube, according to what we said above we have the following equation

$$N_0 - N_1 + N_2 - N_3 = 0$$

where $N_0$, $N_1$, $N_2$ and $N_3$ denote, respectively, the number of vertices, edges, faces and 3-polhyedra in the boundary.
This topic provoked the interest of A. Cayley who, in his 1843 "*Chapters in the Analytical Geometry of n-dimensions*" and later in 1846 in his "*Sur quelques théorèmes de la géométrie de position*", until the publication in 1870 of "*Memory on Abstract Geometry*", remarks how the Geometry of *n*-dimensional spaces might be investigated without making use of coordinates, until establishing the general principles of "n-dimensional geometry".

A fundamental contribution in this direction was given by the Italian mathematician *G. Veronese* who, in 1882, published his "*Fondamenti di geometria a più dimensioni*", where he gives a synthetic and elementary treatment of 4-dimensional Geometry and also the *n*-dimensional one, explicitly claiming that the method used by him was "mainly synthetic and deductive" (in Italian, "*principalmente sintetico e intuitivo*").

The idea that Geometry has to be strictly connected with the physical world implied the existence of just one kind of Geometry (to be "dynamically" chosen according to experiments, as Riemann himself claimed in his 1856 prolusion quoted above). For this reason mathematicians have been reluctant for long time to fully accept the Geometry of Hyperspaces. If for Poincaré *n*-dimensional Geometry offers a "useful geometric language", that for him is however unable to "speak to senses", the Italian geometer Corrado Segre claims, on the contrary, that when using the Geometry of Hyperspaces it is not at all necessary to bother with the problem of their "effective existence". It is just a *"secundary question",* Segre claims, *"that takes nothing away from the logical rigour used to develop such a Geometry"*. In fact, it has revealed to have a fundamental importance in the developments of modern Mathematical Physics.

Investigations on Hyperspaces has allowed to better understand Group Theory, non-Euclidean and Projective Geometries, Algebraic Curves, Topology and Differential Geometry. They have contributed in an essential way to all extraordinary developments that modern Mathematics and Physics had in the XX Century.

Among the scientists that mostly have contributed to make "*visible*" the notion of Hyperspace – also for "large public" – we should of course mention *Albert Einstein*. His *Theories of Special and General Relativity,* by introducing the Spacetime variables (*x, y, z, t*) he argues that gravity is a geometric effect generated by Matter in the surrounding space (as predicted by *Riemann*). Spacetime has "corrugations" in presence of Galaxies, Stars and Planets, in much the same way a billiard ball makes curve an elastic canvas over which it rolls freely (this hyperbole is in fact due to Sir *Arthur Eddington*). Therefore, physical space becomes "non Euclidean" and closes upon itself as a surface, but with one dimension more. One might even think of spaces that wrap around in such a way that‘ like in a sphere, moving in one direction one can eventually come back to the starting point (although “causality“ seems to exclude such a possibility, that would imply “travel backward in Time“).

Physics provides thence a deeper and deeper and more unitary description by means of hyperspaces with *4, 5, ..., n* dimensions and non-Euclidean geometries. In the fascinating new theory known as “String Theory“ the Universe is assumed to have **ten** dimensions, into which strings (or “superstrings“) move and vibrate generating physical effects; six of these dimensions are in fact wrapped around to very low sizes and include the effects of particles and sub-particles (such as electrons, quarks, and so on). It is therefore intriguing to think about the fact that in order to understand the “shape of the earth“ and its position in Kosmos one has, in fact, to move farther apart from it.

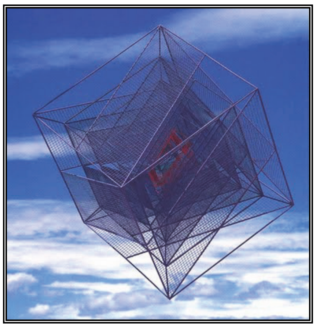
## 2 Polytopes in Art and Mathematics

Several artists have shown interest in the structures of Polytopes together with complex *n*-dimensional mathematical models, later used also to make experiments by *H. S. M. Coxeter*, with his celebrated study on Kaleidoscope, with a Greek *“viewer of beautiful images“,* up to *M.C. Escher* who dedicated many of his works to hints suggested by the Polytopes investigated by Coxeter, such as *Circle Limit* that one can see in his engravings, to *I. Xenakis* (Braila 1922 - Paris 2001), a French-Greek musician and engineer who is considered, for his works and theories, among the most outstanding composers of the second half of XX Century.

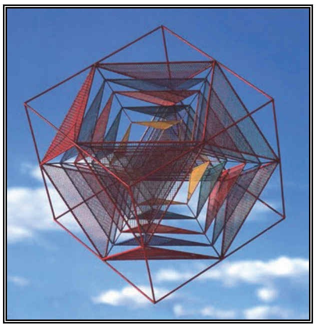
In his musical work “the Polytopes” (*Polytope*), these are “moving performances of light and sound that have the purpose of merging the spatiality of sound to the spatiality of visual arts“. Also notable is the work of *A. Pierelli*, who – already referring in 1980 to the properties of Hyperspaces with his *Cube-Octahedron* [2] - in 1983 realizes a work in which the architectonical model, inspired to the geometry of Hyperspaces, is proposed for the planning of a church.

The artist himself claims to have discovered only later, after having realized the model, the existence of a deep correlation between his “geometrical intuition“ and the description of the “New Jerusalem“ given in the *Apocalypsis of John*.

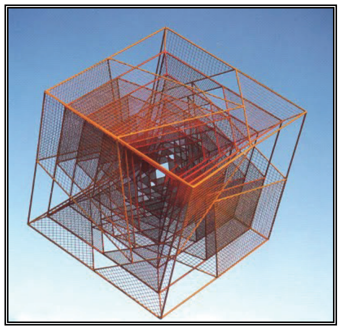
In 1994 *E. Maldonado*, who was particularly fascinated by geometrical problems and by the discovery of new dimensions, qualifies his artistic path with the studies about geometry and on Polytopes. The Spacetime of relativistic theories and in particular the hyperspace introduced and investigated by Poincaré had therefore become the subject of some of his works. He organized an anthological exhibition in Santo Domingo, where along with his drawings also his last creations, called *Hypercubes*, have been exhibited [3], [4], [5], [6]..



Tav. 121. Estuardo Maldonado
Hipercubo Hawking, 1993



Tav. 124. Estuardo Maldonado
*Hipercubo Leonardo*, 1994



Tav. 122. Estuardo Maldonado
*Hipercubo Einstein, 1994*

## 3 Conclusions

The studies on Polytopes continue to generate a great interest both from a geometrical and an algebraic viewpoint. Noteworthy for their importance are the applications: besides those in Numerical Analysis, to investigate mathematical models, also in Computer Graphics, to manipulate, through adequate algorithms, images suitably realized by means of an algebraic representation.

Even if our mind, with the obvious human limits, can reach the stage of understanding notions so abstract, the complexity related with the task of making higher-dimensional Polytopes to be "visible" has led scientists to investigate them also through the use of specific tools, necessary to define sharp rules to codify the single objects. In this way Polytopes are made more understandable on a conceptual basis, also thanks to non-static (and therefore "non reductive") geometric forms (we mean computer "graphically assisted" constructions); the dynamical constructions, on the contrary, make part of a kind of "sub-nuclear structure", something that refers to a reality still unknown to us and difficult to interpret.

In many fields of research the "symmetry groups" of Regular Polytopes are used and their combinatorial structure is used to make self-evident the great potentiality of Computer Graphics, also relying of movies that can better support these new complex mathematical (and physical) models. It should also be remarked that nowadays a great attention is paid also to investigations on the "discretional complexity" of formal systems in Cryptography and in Combinatorial and Computational Analysis, where Physics, Mathematics and Art merge together to become a unique instrument for communications more and more secure.

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**References**

[1] M. FRANCAVIGLIA, M.G. LORENZI, V. IORFIDA, *Algebraic Curves in the Art of XIX Century: a Rational and Correct Definition of the Concept of Hyperspace Art,* APLIMAT - Journal of Applied Mathematics, 4 (2011) – ISSN 1337-6365, in: "Proceedings 10th International Conference APLIMAT 2011" (Bratislava, February 1-4, 2011); M. Kovácová Ed.; Slovak University of Technology (Bratislava, 2011), pp. 945-951

[2] Marcel JORAY, Giselle MATTEWS *“Pierelli e il problema dell'iperspazio”*- A. Pierelli, *Verso gli iperspazi,* Odràdek, (Roma – Venezia, 1980)
[3] Lenin OÑA, *Visión Integral de Estuardo Maldonado in Estuardo Maldonado. Antología 1945 – 2002*, Pontificia Universidad Católica de Ecuador (Quito, 2002), p. 13
[4] Monica ESPINEL, *cat. mostra Exposición Antologica Dibujos* 1948 – 1994, Ambasciata ecuadoriana, Casa de Bastidas, (Santo Domingo, 1994)
[5] Francesca PIETRACCI, Estuardo MALDONADO, *Arte, Dimensionlismo e Hiperspacio*, Documento Final, (Quito, 2006)
[6] Gino SEVERINI, *La peinture d’avant-garde* (1917) published in *“Le Mercure de France“*, 1917 – later reprinted in Maria Drudi Gambillo, Teresa Fiori, *Archivi del Futurismo*, vol. I, De Luca Editore (Roma, 1958), p. 220.

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# RANDOM WALKING IN GENERATIVE ARTS, THOUGHTS AND VIEWS

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**Abstract.** Generative Art refers to any art practice where the artist uses a system, such as a set of natural language rules a computer program, a machine, or other procedural invention, which is then set into motion with some degree of autonomy to or resulting in a complex work of art (Philip Galanter). Also the basic definition tell us "Generative art refers to art that has been generated, composed, or constructed in an algorithmic manner through the use of systems defined by computer software algorithms, or similar mathematical or mechanical or randomized autonomous processes." So what is really generative art and is generative art the piece of art? These questions are discussed in this paper. Some web*Mathematica* applications are presented as a part of our piece of generative arts.

**Keywords and phrases.** generative art, random processes, art computer program, webMathematica

*Mathematics Subject Classification:* 00A66

## 1. Generative art

The basic definition tells us: "Generative art refers to art that has been generated, composed, or constructed in an algorithmic manner through the use of systems defined by computer software algorithms, or similar mathematical or mechanical or randomized autonomous processes."

But what is really generative art and how should be used web*Mathematica* techniques in this process was presented in this article.
First of all let us to collect the definitions of Generative art. There are very different and look for several different tasks on generative art techniques and proceses.

**Philip Galanter** tells

> *"Generative Art refers to any art practice where the artist uses a system, such as a set of natural language rules a computer program, a machine, or other procedural invention,*

*which is then set into motion with some degree of autonomy to or resulting in a complex work of art."*

The next definition comes from **Celestino Soddu.**

**Generative Art is the idea realized as genetic code of artificial objects.** *The generative project is a concept-software that works producing three-dimensional unique and non-repeatable events as possible and manifold expressions of the generating idea identified by the designer as a visionary world. This Idea / human creative act renders explicit and realizes an unpredictable, amazing and endless expansion of human creativity. Computers are simply the tools for its storage in memory and execution. This approach opens a new era in design and industrial production: the challenge of a new naturalness of the industrial object as a mirror of Nature. Once more man emulates Nature, as in the act of making Art. This approach suddenly opened the possibility to rediscover possible fields of human creativity that would be unthinkable without computer tools. If these tools, at the beginning of the computer era, seemed to extinguish the human creativity, today, with generative tools, directly operate on codes of Harmony. They become tools that open new fields and enhance our understanding of creativity as an indissoluble synthesis between art and science. After two hundred years of the old industrial era of necessarily cloned objects, the one-of-a-kind object becomes an essential answer to emergent aestethical needs.*

Also **Wikipedia** give us another point of view.

*Generative Art is art or design generated, composed, or constructed through computer software algorithms, or similar mathematical or mechanical autonomous processes. The most common forms of generative art are graphics that visually represent complex processes, music, or language-based compositions like poetry. Other applications include architectural design, models for understanding sciences such as evolution, and artificial intelligence systems.*

To define generative art as artwork which uses mathematical algorithm was proposed by **Carlo Zanni.**

*Generative software art, as it is usually understood today, is artwork which uses mathematical algorithms to automatically or semi-automatically generate expressions in more conventional artistic forms. For example, a generative program might produce poems, or images, or melodies, or animated visuals. Usually, the objective of such a program is to create different results each time it is executed. And generally, it is hoped that these results have aesthetic merit in their own right, and that they are distinguishable from each other, in interesting ways. Some generative art operates completely autonomously, while some generative artworks also incorporate inputs from a user, or from the environment.*

The next two definitions were found in literature. Both of them define process of producing an artwork as a automated process by the use of computer,

**Adrian Ward**

*Generative art is a term given to work which stems from concentrating on the processes involved in producing an artwork, usually (although not strictly) automated by the use of a machine or computer, or by using mathematic or pragmatic instructions to define the rules by which such artworks are executed.*

**Vera Sylvia Bighetti**

*System usage is identified initially as a key element in generative art. This leads to the adoption of complexity, order and disorder as efficacious organizing principles in the* *comparison of several generative systems of art. The trace of definition of generative art is the preference the artist establishes in a system, that can generate a number of possible forms, and better than a single terminated form. The artist's role is to build, begin or merely select the frame of procedures to generate possible expressions and, for this, the visual aspect may or may not be determining.*

So how can we define and view generative art. What are the main principles and main keywords for deciding about the truth. The important thing is that **generative art describes a strategy for artistic practice, not a style or genre of work.** The artist describes a rule-based system external to him/herself that either produces works of art or is itself a work of art. We agree with Philip Galanter that work with generative qualities can be found throughout art history, but I typically use the term to describe computer-based work created from the 1960s to today.
We consider much of the work in abstract painting and sculpture done in the 1960s as essential for the understanding of generative art. For the term generative art to have any meaning when applied to a given work, **the aspect of generatively must be dominant in the work**. **Many computer-based art projects have generative elements, but are not concerned with generative systems as an end result.**

In these days generative art is typically connected with software-based abstractions. We think the popularity of the term is due to an emerging group of younger artists and designers concerning themselves with code as an aesthetic material. This naturally leads to explorations of the ways code affects both the artistic process and the end result, including a materiality of algorithms etc.
Generative art is a contested term but for our purposes refer to artwork that is broadly rule-based, a further understanding of which has been informed by the co-curation of touring exhibition Generator. The exhibition title "**generator**" describes the person, operating system or thing that generates the artwork, shifting attention to the interaction not separation of these productive processes. Significantly, once the rules have been set, the process of production is unsupervised, and appears self-organising, though only if knowledge of other aspects is suspended. As a result, although generative art might appear autonomous and out of control, our argument is that control is exerted through a complex and collaborative interrelation of producer/s, hardware and software.

The relations of production within generative artwork are thus seen to be decidedly complex (its operations not open-ended or closed, as complexity theory and dialectics would verify).Like the programmer, the code that lies behind a generative artwork remains relatively hidden and consequently difficult to interpret.

Also Tjark Ihmels, Julia Riedel Even give us the interesting point of view for this term.

*Wolfgang Amadeus Mozart developed a "musical game of dice" that contained most of the elements that today are associated with generative tools. The piece carries the explanatory title "Composing waltzes with two dices without knowing music or understanding anything about composing". Using this historical example, the methodology of generative art can be appropriately described as the rigorous application of predefined principles of action for the intentional exclusion of, or substitution for, individual aesthetical decisions that set in motion the generation of new artistic content out of material provided for that purpose. To describe this method, musicologists introduced the concept of "aleatoric music". The name is derived from the Latin "aleator" (the dice player), and could not be more appropriate for the above example. In aleatoric music, the principles of chance enter into the composition process. There is no standard artistic position connected with the concept of "generative", but rather, a method of artistic work, which was and is employed with the most diverse motives. At the same time, it is interesting to observe that this way of working appears not*

> *only in connection with a certain genre, but has in fact established itself in nearly every area of artistic practice as music, literature and fine arts.*

Until 100 years ago every musical event was unique: music was ephemeral and unrepeatable, and even classical scoring couldn't guarantee precise duplication. Then came the gramophone record, which captured particular performances and made it possible to hear them identically over and over again. But Koan and other recent experiments like it are the beginning of something new. From now on there are three alternatives**: live music, recorded music and generative music.** Generative music enjoys some of the benefits of both its ancestors. Like live music, it is always different. Like recorded music, it is free of time-and-place limitations- you can hear it when you want and where you want. And it confers one of the other great advantages of the recorded form: it can be composed empirically. By this I mean that you can hear it as you work it out- it doesn't suffer from the long feedback loop characteristic of scored-and-performed music.

Generative art was also practiced among others by Neagu, Eduardo McEntyre and Miguel Ángel Vidal [1928- ] in the Argentine."

> ***"A form of geometrical abstraction in which a basic element is made to ' generate' other forms by rotation, etc. of the initial form in such a way as to give rise to an intricate design as the new forms touch each other, overlap, recede or advance with complicated variations.*** A lecture on 'Generative Art Forms' was given at the Queen's University, Belfast Festival in 1972 by the Romanian sculptor Neagu, who also founded a Generatiave Art Group.

Generative Art: Process by which a computer creates unique works from fixed parameters defined by the artist. The result can range from an engaging screensaver to a jazz solo to a lush virtual world. The visual application of generative art is newer, however. In the mid-1970s British abstract painter Harold Cohen plugged in his palette and designed AARON, a computer artist that produces original work. Since then, generative techniques have been used to grow artificial life based on genetic algorithms and massively complex virtual worlds that take infinitely longer than seven days to create by hand. But whatever the output, there is always a human behind the high tech curtain. "The computer is actually generating the art in partnership with the artist/programmer, who defines the fields of possibilities," says Holtzman, who has been experimenting with generative music for more than 20 years. "People live with this romantic notion that an artist gets struck with a thunderbolt of inspiration and runs to the piano or canvas and expresses an idea. The reality is that art has a formal underpinning, and computers are a perfect tool because they're perfect for manipulating formal structure."

## 2. Generative Art Is Not … An Artistic Style

**While Genenerative Art is almost always abstract in nature,**
**it cannot be defined by the style of the work.**
**The common factor of generative artworks is the methodology of its production,**
**not the style of the end result.**

There are many varieties of production methods, but to be able to call a methodology “generative” our first hard and fast rule needs to be that there has to be autonomy involved.
The artist creates ground rules and formulae, usually including random or semi-random elements, then kick off an autonomous process to create the artwork. **The system cannot be entirely under**

**the control of the artist or the only generative element will be the artist herself.** Our second hard and fast rule therefore is there has to be a degree of unpredictability. It must be possible for the artist to be as surprised by the outcome as anyone else.
**Creating a generative artwork will always be collaboration, even if the artist works alone.** Part-authorship of any generative work must belong in part to the tools the artist uses; the system that has generated it. Fortunately anonymous autonomous systems are not usually too bothered if their unscrupulous artistic partners decide to steal all the credit.
There are many types of autonomous systems we might choose to collaborate with. Our focus has only ever been generating visual artwork using a programming language, but Generative Art may also be the product of mechanical systems, games of chance, natural phenomenon or subconscious human behavior too.

**Software as material is always liquid, potentially intelligent, interactive and constantly changing. The only way to approach such a medium is as a sum of processes and interactions.** Generative art and design describes a process-based practice, where the artist enters into a collaboration with the machine, describing aesthetic qualities in terms of rules and instructions. Random factors are allowed for in order to produce organic behavior. **By combining rational/scientific principles with subjective/aesthetic choices, new and unexpected products are created.** The results are dynamic forms and processes through which we gain a new understanding of the world around us, as well as a new and dazzling source of aesthetic experience.

## 3. Back to SCIENAR

During the 2008 – 2010 years were realized the project Scienar supported by EU Culture Programme. The Project SCIENAR takes into account the links existing between Science and Art; it used the innovative possibilities that new media and ICT offer for a better Visualization and Communication. Visualizing and Communicating theoretical achievements of present Culture are by no means simple; moreover, presenting them in an innovative way is a fascinating challenge dictated by new trends of Society. ICT allow us to explore and represent these fruitful relationships in a way unthinkable before.
The project Scienar which takes into account the common European cultural heritage and was profoundly based on the links existing between Science & Art, from the very beginning of Greek Culture, through Renaissance, until present time, built an interdisciplinary approach using mainly digital technologies. **It aims were to explore the present day interactions of these two facets of our culture, bringing together both communities of "Human Culture" and "Scientific Culture".**

The starting and leading idea is that these two facets are NOT separate entities – as they frequently happen to be considered – but just two facets of the SAME Culture, which encompasses all human endeavors to understand, represent and transcend the whole of our know-ledge of "reality" in which we live. Two facets are share a common past, a common present and a common future; two facets share the mutual interrelationships of which are profound and extremely important.

As it was stated in the declaration of SCIENAR, **Mathematics has evolved along with our way of conceiving, perceiving, experimenting and representing "reality"; while Art develops the means to harmonize, describe, represent aesthetically, transcend and transfigure the World of our sensations and perception.** A completely analogous pathway is recognizable in other forms of Visual Art, in Architecture, in Music, in all forms of "modern and contemporary Art", from Photography to Film, up to Digital Art.

**The project SCIENAR explicitly aims in particular to create an interactive environment for both Scientist and Artists, where Scientists can explore the role that Mathematics plays in understanding and making Art, as well as produce mathematical objects that are useful in Art; while Artists can found mathematical structures and forms that they can directly use, without needing the subtleties of Mathematics, to inspire and produce their artworks.**

In this paper we will refer on one part of general concept on the border between science and Art – random walking, especially about random number generators and the possibility to transform their potential to the graphic form and inter alia on the results created by artists based on the random walking mathematical concept. We will refer also on web-*Mathematica* techniques suitable for fulfilling the previous main goals and ideas.

## 4. webMathematica visualization techniques

*WebMATHEMATICA* is a new web technology that allows the generation of dynamic web content with *Mathematica*. It integrates *Mathematica* with a web server. *webMATHEMATICA* harnesses the full range of *Mathematica* technology to build sophisticated web applications, especially in creating dynamical web art objects, or the graphical objects for teaching. *WebMATHEMATICA* provide immediate access to the technical computing software with very firm abilities especially in *Mathematica* graphics from any web browser. It allows incorporate also dynamical possibilities to creating graphics objects, so the graphics are live, interactive and responsive to user needs. In Scienar project there are these special webMathematica abilities used as a tool to produce new artistic works.

***WebMATHEMATICA* allows a site to deliver HTML pages that are enhanced by the addition of *Mathematica* commands**. When a request is made for one of these pages, the *Mathematica* commands are evaluated and the computed result is inserted into the page and delivered to the client browser. This is done with JavaServer Pages (JSP), a standard Java technology, making use of custom tags. After the initial setup, all that you need to write *webMATHEMATICA* application is a basic knowledge of HTML and *Mathematica*. *webMATHEMATICA* is based on two standard Java technologies: Java Servlet and JSP. Servlets are special Java programs that run in a Java-enabled web server.

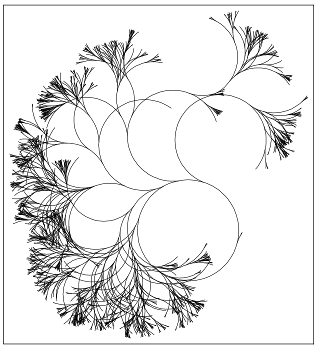
**webMathematica allows a site to deliver HTML pages that are enhanced by the addition of Mathematica commands.** When a request is made for one of these pages, the Mathematica commands are evaluated and the computed result is inserted into the page. This is done with a standard Java technology, JSP, making use of custom tags.

We will show in this section not only the programming capabilities of Mathematica, but also the technical capabilities of webMathematica. We will present one of jsp files (available on Scienar website) for generation Penrose tiling's and several results produced by these techniques. Here is shown an implementation.

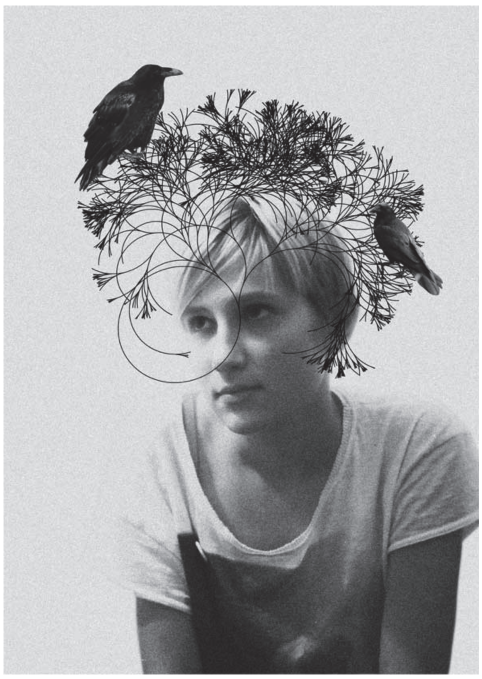
## 5. webMathematica likes generative art

The next examples were taken a curve given in the form of line and reflect parts of this curve on some randomly selected segments of list of points. Both of these objects were created randomly with webMathematica pseudo-random generator.

```
multipleReflector[pointList_, pp_(* how often *),pk_(* how many *)] :=
   Graphics[{Thickness[0.001], Map[Line,
      NestList[Function[l, Function[l, If[Length[l] < 3, {},
       Function[{p, n}, (* mirroring on the first line segment *)
         ((p + 2n(n.#) - #)& /@ (0.9 (# - p)& /@ Rest[l]))][
          l[[2]], #/Sqrt[#.#]&[l[[2]] - l[[1]]]]]]] /@
       DeleteCases[Flatten[(Function[l, If[l === {}, {},
       (* generate pk random parts *) Drop[l, #]& /@
          Union[Table[Random[Integer, {1, Length[l]}],
             {pk}]]]] /@ l), 1], {}]], {pointList}, pp], {-3}]},
        AspectRatio -> Automatic, PlotRange -> All,
        Frame -> True, FrameTicks -> None];
```



These objects were used also for art inspiration look as follows. Young artists from School of applied arts J. Vydru from Bratislava created several photo-montages based on webMathematica random concept during the Scienar meeting in Kremnica.



## Conclusion

In our days Science in general, and Mathematics in particular, play a direct and explicit role in several forms of Art (visual, plastic and musical). Very well-known is, for example, the existence of methods to generate Art and Music by means of computers and electronic devices. It follows that Mathematics is not only an essential tool for Science and Technology, but also for Humanities and, in particular, for Art.

More webMathematica applications should reader find on the project web-pages – http://www.webmathematica.eu/Scienar/index.php

## References

[1] KOVÁČOVÁ M.: Dynamical MATHEMATICA, STU Press, Bratislava, 2008, ISBN: 978-80-89313-11-2, pp. 346.

[2] KOVÁČOVÁ M.: webMathematica – Dynamical Mathematica Web Pages and their Bene-fits for Teachers, In: XXVII International Colloquium on the Management of Educational Process, Brno, 2009, 9 pages, on CD-ROM, ISBN: 978-80-7231-650-2

[3] KOVÁČOVÁ M. Jančo R.: Dynamical webart, In: Aplimat - Journal of Applied Mathematics. - ISSN 1337-6365. - Vol. 2, No. 1 (2009), s. 203-214

[4] WICKHAM-JONES T.: webMathematica 3, A User Guide, Wolfram Research Inc.

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# THE ART OF COLLAGE AND AUGMENTED REALITY 2D/3D TECHNIQUES

**STEFAN Livia, (RO)**

**Abstract.** This paper makes a parallel analysis of some of the techniques used in the modern visual arts and similarly, in modern advanced software applications.
Augmented Reality (AR) is a modern IT paradigm and a range of technologies that encompass several computer techniques in order to enhance the perception of the physical reality. It is a process of mediated reality by means of human senses and behaviour, through computer sensors and devices, in real time and in a synchronous manner. Advanced computer techniques were adapted or developed in order to put the AR concept into practice, that is to align the real image with presented images and information and produce a relevant visual, sensorial and cognitive impact. On the other hand, the AR applications have an intrinsic artistic potential, which is proved by a proliferation of AR usage in movies, public art installations, media and advertising or education, where it employs the art of collage.
The collage is one of the artistic techniques of the Modernity, emerged during the Cubist period, and artistically exploited by the avant-gardes in the 60's and 70's. A collage is a formal art creation, characterized by a combination of different media, producing a new vision of the whole. The AR technology in the contemporary pop culture represents a generalized usage of the collage techniques by means of an advanced IT technology. It can be concluded that one of the modern IT technologies is inspired by an artistic concept of the 20th century.

**Key words.** collage, augmented reality, mixed media, composite, 3D modelling, image align, image registration, computer vision, image processing.

*Mathematics Subject Classification:* Primary 68U05, 68U10; Secondary 68U35.

## 1 Introduction

Making connections between arts, science and technologies, this paper discusses analogies between some concepts and techniques used in modern visual arts and those used by Augmented Reality technologies, which make use of visual and sensorial techniques. The drawing line is made through 2D and 3D images and graphics. The paper goes deeper with augmented reality concepts, techniques and applications.

Connections between the arts and the sciences are historically proven but are strongly demonstrated ever since the modernist period.

Artists are expressing themselves through a variety of media and techniques. Visual arts, motion pictures, music and dance are using specific materials to create artworks. Within the visual arts, for exemple, charcoal, oil and canvas, slay, steel and even ice have been utilized to convey artistic ideas. Each artist seek the proper medium which best suits his message while other try to extend media in different and unexplored ways. Through such explorations, artists have helped discover new uses for technology or lead others to see beyond those originally intended and imagined. Art "helps to drive public perception of new medium" [11].

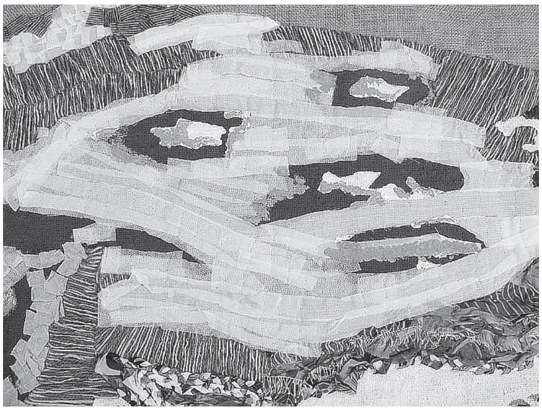
The collage is a modern technique through which different media are combined on a flat plan in order to obtain a new whole image and a significant message.
Augmented Reality term embraces a whole range of technologies used to emulate a hyper-realistic experience. This is done by superimposing computer graphics on top of a real-time view of the world, and where the real-time view of the world is usually seen through a video camera. The researches concern is to make disappear the differences between the real and artificial worlds and to obtain an augmented vision of the reality.

Augmented Reality is known for about 20 years, during which was mainly used in academia and industry for very practical functionality, i.e for simulation, training, design. The big hype of AR researches was stimulated by important accomplishments with virtual reality applications. The first consumer-accessible version of AR appeared in late 2008. Few years ago it became feasible to do AR applications on mobile and handheld devices, when these had enough processing capabilities, camera and sensors. At present with the wide spread of affordable mobile devices the public awareness of AR applications is identifying itself with the "mobile AR" applications.

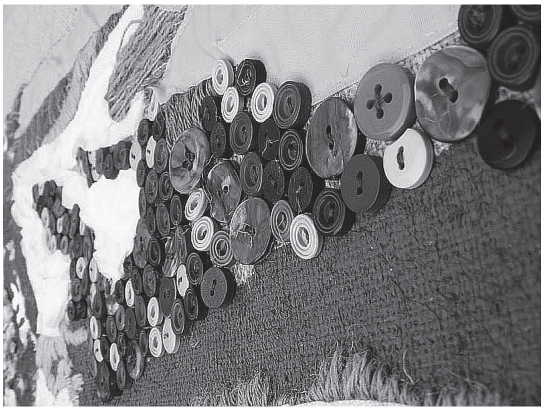
## 2 The collage art techniques

The **modernism** [12] is "an art movement characterized by the deliberate departure from tradition and the use of innovative forms of expression". Modernism is manifested by: "new types of paints and other materials; expressing feelings, ideas, fantasies, and dreams instead of the visual world we otherwise see; creating abstractions, rather than representing what is real; a rejection of naturalistic color, a use of choppy, clearly visible brushstrokes; the acceptance of line, form, color, and process as valid subject matter by themselves, a requirement that the audience take a more active role as interpreter. Each viewer must observe carefully, and get information about the artist's intentions and environment, before forming judgments about the work".

The **collage** is one of the artistic techniques of the Modernity, emerged during the Cubism period and artistically exploited by the avant-gardes in the 60's and 70's. A collage is a formal art creation, mainly used in visual arts, characterized by a combination of different media and artistic creations, producing a new vision of the whole such as painting and photo illustration, graphics or different materials.
In visual arts the collage is a picture or design created by adhering flat elements as newspaper, wallpaper, printed text and illustration, cloth etc. to a flat surface. The result gains a *third dimension*, and could also be called a "relief sculpture / construction / assemblage". Introduced by the Cubist artists, this process was widely used by artists who followed, and is a familiar technique in contemporary art.



"*Textile Collages*" – Courtesy of the Romanian artist Alexandra Andreea Rusu, 2011



In **Cubism** the subject matter is "broken up, analyzed, and reassembled in an abstracted form" [12]. Picasso and Braque initiated the movement when they followed the advice of Paul Cézanne, who in 1904 said artists should treat nature "in terms of the cylinder, the sphere and the cone." [12], i.e. using *geometric primitives*. Braque and Picasso brought recognizable illusionistic features back into their paintings. They used letters, fragments of words, musical notes, then significant material elements, and tend to make the picture more physically an object.

The patches which Braque and Picasso added to their canvases offered a new perspective on painting when the patches "collided with the surface plane of the painting" [13]. In this perspective, collage was "part of a methodical reexamination of the relation between painting (as *2D view*) and sculpture (as a *3D vision*), and these new works gave each medium some of the characteristics of the other," according to the Guggenheim Museum's online art glossary. Furthermore, these fragments refers to external meanings, such as current political and social events. This juxtaposition is fundamental to the objectives of the collage art, which is "to emphasize concepts and processes over the end product", transforming diverse and dissonant elements into a meaningful message.

**Surrealism** is a cultural movement that began in the early 1920s and is best known for the visual artworks and writings of the group members. Surrealist works feature the element of surprise, unexpected juxtapositions. Surrealist artists have made extensive use of collage. For the Surrealists, collage served as a "surrogate for the subconscious".
**Cubomania** is a surrealist method of making collages in which a picture or image is cut into squares and the squares are then reassembled automatically or at random to create an entirely new work. Robert Hirsch has seemed to imply that this process can be done with digital photography. At

least one cubomania has been made with triangular shapes. Cubomania was invented by the Romanian surrealist Gherasim Luca.

**Froissage** is a method of collage developed by Ladislav Novák in which the lines made by crumpling up a piece of paper are used to create a drawing. One major exponent is Jiří Kolář.

The **Futurists** and the **Dadaists** employed collage to protest against well-established values, while the artists of the Russian avant-garde used photomontage as another kind of collage, to demonstrate support for a "progressive world order".

**Pop artists** incorporate elements of popular culture into their work. Robert Rauschenberg expanded collage in his own way by creating Combines, assemblages of paintings and "found objects" that were intended, he said, to act in the gap between art and life. Pop artists have focused attention upon familiar images of the popular culture such as billboards, comic strips, magazine advertisements, and supermarket products. Leading exponents are Richard Hamilton, Andy Warhol and Roy Lichtenstein.

**Wood collage** art is considerably smaller in scale, framed and hung as a painting would be. It usually features pieces of wood, wood shavings, or scraps, assembled on a canvas (if there is painting involved), or on a wooden board. Such framed, picture-like, wood-relief collages offer the artist an opportunity to explore 'the qualities of depth, natural color, and textural variety inherent in the material", while drawing on and taking advantage of the language and conventions that arise from the tradition of creating pictures. The technique of wood collage is also sometimes combined with painting and other media in a single work of art.

The collage brings recognizable "signifiers" together in a kind of "semiotic collision". A truncated wooden chair can also be considered a potential element of collage in the same sense: it had some original, culturally determined context. Unaltered, natural wood has no such disruptive context associated with the collage idea, as it was originated with Braque and Picasso.

**Decoupage** is a type of collage usually defined as a craft. It is the process of placing a picture into an object for decoration. Decoupage can involve "adding multiple copies of the same image, cut and layered to add apparent depth". In the early part of the 20th century, decoupage, like many other art methods, began experimenting with a more abstract style. 20th century artists who produced decoupage works include Pablo Picasso and Henri Matisse. Most famous are Matisse's Blue Nude II. There are many varieties on the traditional technique involving purpose made 'glue' requiring fewer layers (5 or 20). Cutouts are also applied under glass or raised to give a three dimensional appearance. Currently decoupage is a popular handicraft.

Collage made from photographs or parts of photographs is called **photomontage**. Photomontage is the process (and result) of making a composite photograph by cutting and joining a number of other photographs. The composite picture was sometimes photographed so that the final image is converted back into a seamless photographic print. The same method is accomplished today using image-editing software. The technique is referred to by professionals as *compositing*.

David Hockney is known for a photo-collage named "Portrait of the Artist's Mother". This is called a photo-collage rather than a photomontage, because it is more three-dimensional. Hockney explored this process of collaging prints taken with a 35 mm camera as relating to the Cubist sense of "multiple angles and of movement". These "multiples" as he said, convey a strong sense of movement, in that the viewer must keep readjusting his imagined viewpoint as his gaze travels from print to print. By this the viewer builds up a single image that is many times wider in angle of view than the camera lens. The viewing angle of a standard 55mm lens for a 35mm format camera is about 45 degrees. Wide angle lenses increase the angle of view to about 75 degrees without obvious distortion, but the human angle of view, with eye movement, is about 180 degrees.

The 19th century tradition of physically joining multiple images into a composite and photographing the results was used in press photography and offset lithography until the widespread use of digital image editing.

**Digital collage** is the technique of using computer tools in collage creation to encourage associations of disparate visual elements and the subsequent transformation of the visual results through the use of electronic media. It is commonly used in the creation of digital art.
The concept of collage has crossed the boundaries of visual arts. In music, with the advances on recording technology, avant-garde artists started experimenting with cutting and pasting since the middle of the twentieth century. Collage novels are books with images selected from other publications and collaged together following a theme or narrative. A collage in literary terms may refer to layering of ideas or images.

**Collage film** is defined as “a film that juxtaposes fictional scenes with footage taken from disparate sources, such as newsreels”. Collage film can also refer to the physical collaging of materials onto filmstrips. The idea of combining film from various sources was first suggested by surrealist artist André Breton. Historical and archival footage is usually used in documentary films, for a more comprehensive understanding of the subject matter. Director and cinematographer Ken Burns is famous for using inclusion of archival footage. Often fiction films borrow this style in order to increase the artistic message. **Found footag**e is a filmmaking term which describes a method of “compiling films partly or entirely of footage which has not been created by the filmmaker, and changing its meaning by placing it in a new context”. The term refers to the "found object" (objet trouvé) of art history.

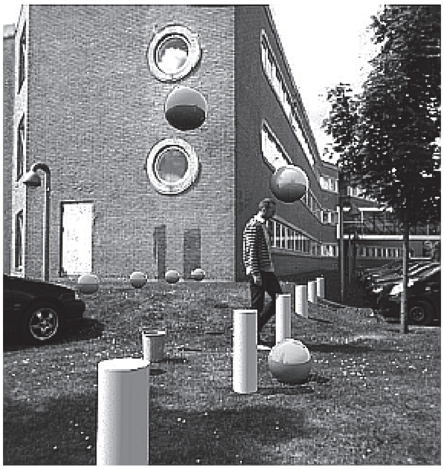
To conclude, a collage is a “medium of materiality, a record of our civilization, a document of the timely and the transitory”[12].

## 3 Augmented reality key concepts and techniques

The AR concept was first time introduced by Ivan Sutherland in 1960 and was demonstrated by means of a “see-through” displays that overlaid visual, auditory, and later on haptic elements on the user’s real life view. Azuma R. [1] provided reference definitions and explanations of the Augmented Reality processes.
Initial researches in augmented reality addressed well defined areas of applications, respectively in product manufactory, industrial training and design, medical training and surgery.

At present AR is embraced in many domains which previously experienced virtual reality. These application domains can be grouped in: a) gaming and 3D experience; b) advertising, media and marketing; c) educational and training. Furthermore, with the availability of high-end mobile devices having access to high-resolution digital cameras, displays, graphical capabilities and broadband connectivity, the AR applications are most effective implemented on mobile and handheld devices.

An Augmented Reality (AR) process is defined as “overlay of computer information atop of a view from a real-time camera feed”[8]. Computer information refers to texts, audio, video, 2d photos but especially 3D computer graphics. Some authors [8] consider that information like audio, or other forms of feedback are not augmentation, which correctly means aligning the graphics with the camera view.



Augmented Reality with simple graphic information [4]


Augmented Reality with adnotated information
[After www.wired.com ]

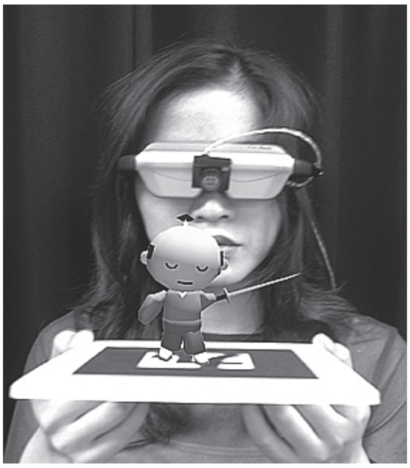
Augmented Reality on Smartphones
[After 5magazine.wordpress.com]

Augmented Reality is a component of the “Reality-Virtuality Continuum” [2] which spans from the “pure” real environments and ends in “pure” virtual environments. Even they are sharing some basic concepts, Augmented Reality is totally different from the *Virtual Reality*, in which the participant is either observing or immersing in a completely artificial world, which may or may not borrow the properties of a real-world environment (that is observe the gravity and physical laws). Virtual reality exists in the limit of the user’s display. Even if the user is manipulating a 3D object or he is walking about as an avatar, there is no connection to the real world. There are only figurative representations, but no real-time video capture of what is happening outside the computer. If we had this, then we were in an Augmented Virtuality environment. Augmented reality is “a real-time visual merge”.

To have an AR process is necessary that the juxtaposition of the perception of the reality and the received virtual information be made in real time and in a synchronous manner. The ideal for an AR process is to clear the differences between the real world and the created world. More explicitly, mixed reality means that the placement of the artificial object atop of the real world be made “in perspective and in dimension” [8]. The placement of the digital information is not a trivial process as it has to be in relationship with the video flow [8], that is to take into account the camera pose.

User’s actions/behaviour have an important role in an AR process “equation”, because the user ultimately defines it’s AR experience. That is why AR can also be described as a process of mediated reality by means of human senses and behaviour, through computer devices and sensors, respectively by means of an **AR tracking process**. This is performed by digital camera, GPS receiver, inertial sensors (accelerometer, gyroscope /G-sensor), Assistive GPS, NFC (Near Field Communication) sensors, wireless sensors. Tracking the user’s hand is made by means of the haptic interfaces (touch or multi-touch screens). The display devices can be: monitor, handheld display or head-mounted display.

A Head Mounted Display (HMD) is an external equipment, similar to a pair of glasses, that can implement a video “see-through” or an optical “see-through” view. Video see-through HMD have the advantages of a wide field of view and that real and virtual view delays can be matched, but are more complex. These equipments are producing an immersive AR experience.



A 3D immersive Augmented Reality
[After http://blog.pathoftheblueeye.com ]

A GPS receiver provide geo-referencing of the location whereas the inertial sensors provide instantaneous 3D orientation information of the camera (also called camera pose). These sensors together provide 6 DoF (degrees of freedom) camera pose at interactive rates. The accuracy, sensitivity and resolution provided by these sensors determines the performance of an AR process. To look realistic, the image alignement precision must be better than 0,1 mm.

The camera performs perspective projections of 3D world onto 2D image plane using the focal length, lens distortion, position, and pose of the device to determine exactly what is projected onto the image plane. Virtual objects is modeled in an object reference frame and generated by a standard computer graphics system. Graphics system requires information about the imaging of the real scene so that it can correctly render these objects. Sophisticated software algorithms must derive real world coordinates, independent from the camera images. That process is called **image registration**.

From the platform point of view there are desktop AR applications, which utilize a web cam (also called indoor AR) and mobile AR applications (outdoor AR).

From the application functionality point of view a possible classification of AR aplications is:

•Maping applications or location aware applications – which make use of GPS for location and of mobile inertial sensors to identify real objects or objectives to be augmented with point of interest information or superimposed 3D models.

•Vision-based applications or location agnostic applications – which performs image and object recognition on a frame by frame basis, so it can align augmented information. This kind of application is recently made possible with development of computer vision algorithms and with the high-end processor, capable to performantly execute the computer vision algorithms;

•Spatial AR application (SAR) – which makes use of digital projectors to display graphical information onto physical objects. The display is separated from the users of the system. Because of this characteristics, SAR can scale up to groups of users, allowing for a collaborative AR.

As for the object identification techniques, there is *marker-based* object tracking and *markerless* or **natural feature recognition**. The markers can be infrared (IR) markers which are tracked by IR cameras or black and white (B/W) and grayscale visual markers, tracked via optical cameras.

Not the least, AR also means a new user interface paradigm and technology which is a design concern. The user has to deal with a new type of content and the interface must be designed to facilitate a naturally interaction with the augmented reality environment.

## 4 The augmented reality research challenges

### 4.1 Image registration

Image registration uses different methods of computer vision, related to video tracking and image processing. These methods consist of two parts. First refers to detection of interest points, fiduciary markers, or (optical) camera flow. First stage can use feature detection methods like corner detection, blob detection, edge detection or thresholding. The second stage restores a real world coordinate system from the data obtained in the first stage. Some methods assume objects with known geometry (or fiduciary markers) present in the scene. In some of those cases the scene 3D structure should be precalculated beforehand. If part of the scene is unknown simultaneous localization and mapping (SLAM) can map relative positions. Mathematical methods used in the second stage include projective geometry, geometric algebra, rotation representation with exponential map, kalman and particle filters, nonlinear optimization, robust statistics.

For indoor environments, existing systems demonstrate excellent registration. Visual tracking generally relies on modifying the environment with fiducial markers placed in the environment at known locations. The markers can vary in size to improve tracking range and the computer-vision techniques that track by using fiducials can update at 30 Hz.

In outdoor and mobile AR applications, it generally it is not practical to cover the environment with markers. A hybrid compass/gyroscope tracker provides motion-stabilized orientation measurements at several outdoor locations. With the addition of video tracking (not in real time), the system produces nearly pixel-accurate results on known landmark features. Either the GPS or dead reckoning techniques usually track the real-time position outdoors, although both have significant limitations (for example, GPS requires a clear view of the sky). Ultimately, tracking in unprepared environments may rely heavily on tracking visible natural features (such as objects that already exist in the environment,without modification).

System delays are often the largest source of registration errors. Predicting motion is one way to reduce the effects of delays. Researchers have attempted to model motion more accurately and switch between multiple models. Video correction technicques can compensate for delays in 6D motion (both translation and rotation).

To make the graphical image appear in the proper place, for example as a wireframe outline superimposed on top of a real-world object, it is necessary to know as accurate as possible where that real-world object is and what its orientation and dimensions are. This is called a **2D image align** in a 3-D real scene. That is why it is necessary to perform an modeling of the real environment. A geometric model of the physical environment is usualy performed. Important concerns are to augment the vision by overlaying hidden structures or to perform occlusion detection with respect to user's point of view or for model-based vision tracking approaches. Effective AR requires knowledge of the user's location and the position of all other objects of interest in the environment. For example, it needs a depth map of the real scene to support occlusion when rendering.

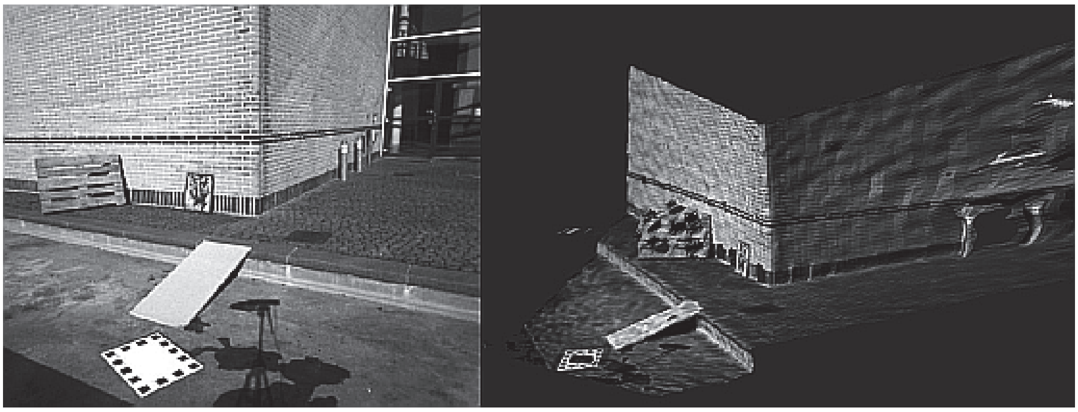
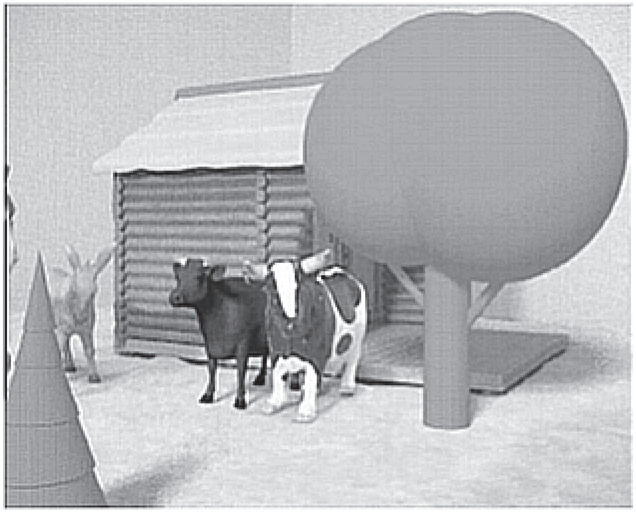


Image of the real world and the 3D modeled counterpart [14]

Creating 3D model of large environments is a research challenge. Models can be photorealistic or simple 3D point clouds. Complexity of the problem depends on the details that need to be modeled. A complete modelling of large urban area is complex and time consuming. Fully automatic, semiautomatic and manual modeling techniques are often employed depending on the required accuracy. The challenge is to develop methods to register (associate) the two sets of images (real and virtual) and keep them registered in real-time as real environments are dynamic.

Failures in registration are due to: a) Noise - Position and pose of camera with respect to the real scene; Fluctuations of values while the system is running; b) Time delays - In calculating the camera position; in calculating the correct alignment of the graphics camera.




Augmented Reality images illustrating occlusion between real and virtual images [4]

**4.2 Video Composition**

Video Composition for video see-through HMD: a) *Chroma-keying* - for special effects; background of computer graphics images is set to a specific color; combining step replaces all colored areas with corresponding parts from video; b) *Depth Information* - Combine real and virtual images by a pixel-by-pixel depth comparison.

**4.3 Interactive AR**

Interactive AR is performed through ***tangible interfaces*** which support direct interaction with the physical objects by emphasizing the use of real, physical objects and tools.

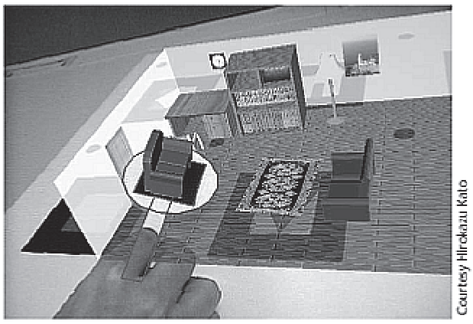
**4.4 Photorealistic AR rendering**

A key requirement for improving the rendering quality of virtual objects in AR applications is the ability to automatically capture the environmental illumination and reflectance information.
Researchers have different approaches that uses *ellipsoidal models* to estimate illumination parameters, *photometric image-based rendering*, and high dynamic *range illumination capturing*.

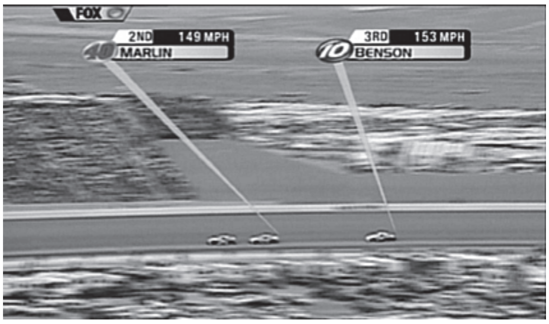
**4.5 Collaborative AR applications**

Many AR applications can benefit from having multiple people simultaneously view, discuss, and interact with the virtual 3D models.
A significant problem with collocated, collaborative AR systems [4] is ensuring that the users can establish a shared understanding of the virtual space, analogous to their understanding of the physical space. Because the graphics are overlaid independently on each user's view of the world, it's difficult to ensure that each user clearly understands what other users are pointing or referring to. Another form of collaborative AR is in entertainment applications. Researchers have

demonstrated [4] a number of AR games, including AR air hockey collaborative combat against virtual enemies and an AR-enhanced pool game.




Interactive AR [4]




AR in video broadcasting [4]

AR has been used for real-time augmentation of broadcast video, primarily with the purpose to enhance sporting events and to insert or replace advertisements in a scene. In these systems, the environments are carefully modeled ahead of time, and the cameras are calibrated and precisely tracked. For some applications, augmentations are added solely through real-time video tracking.

## 5. Conclusions

The objective of using 2D/3D image collage both in arts and in Augmented Reality applications is to enhance the user's perception and performance. It can be concluded that one of the most advanced IT technologies is inspired by an artistic concept of the 20th century.
In present creations/implementations the boundaries between art, science and technology are not very easy to discern.
The development of digital computer enables both artists and engineers to create new art forms and user applications, such as interactive art, public art installations, interactive story telling, experiential, situated, real-virtual collaborative computing.
Augmented Reality is not yet a mature technology. The future AR paradigme is "augmenting anything/anytime". The algorithms for computer vision, 3-D object reconition, face recognition, artificial intelligence to support markerless immersive and interactive AR experience are expected to be improved and made public available from academia for practical implementations.

## Acknowledgement

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## References

[1] AZUMA, R. T.: *A Survey of Augmented Reality*, Hughes Research Laboratories, 1997
[2] MILGRAM, P., TAKEMURA, H., UTSUMI, A., KISHINO F.: *Augmented Reality: A class of displays on the reality-virtuality continuum*, ATR Communication Systems Research Laboratories, Kyoto, Japan, 1994

[3] STEVEN, F., BLAIR, M., TOBIAS, H.,: *A Touring Machine: Prototyping 3D Mobile Augmented Reality Systems for Exploring the Urban Environment*, Department of Computer Science Columbia University New York, Anthony Webster Graduate School of Architecture, Planning and Preservation Columbia University, New York, 1997
[4] AZUMA, R., BAILLOT Y., BEHRINGER, R., BLAIR, M.: *Recent Advances in Augmented Reality*, HRL Laboratories, NRL Virtual Reality Lab/ITT Advanced Engineering, Rockwell Scientific, Georgia Institute of Technology, 2001
[5] KARLEKAR, J., ZHIYING Zhou, S., LU, W., NAKAYAMA Y., HII D.: *Mixed Reality on Mobile Devices*, Interactive Multimedia Lab., Dept. of ECE National University of Singapore, 2010
[6] ROBERTSON, C.: *Augmented Reality* , 2000
[7] *Augmented Reality – Linking Real and Virtual Worlds*, University of Southern California, 2000
[8] *The 6th Sight Report Magazine– The future of imaging*, September 2010
[9] MULLEN T.: *Prototyping Augmented Reality*, John Wiley & Sons, 2011
[10] MAAD, S.: *Augmented Reality*, Financial Services Innovation Centre, University College Cork UCC, Cork, Ireland, 2010
[11] CRAIG A.B., SHERMAN, W. R., Will, J. D.: *Developing Virtual Reality Applications*, Morgan Kaufman Publishers, 2009
[12] http://www.artlex.com
[13] http://www.guggenheimcollection.org/site/concept_Collage.html
[14] http://handheldar.icg.tugraz.at/argraphics.php

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# OCTAGONS AND SACRED NUMBERS IN THE FLOOR XI CENTURY MOSAICS OF THE CATHEDRAL OF AQUILEIA

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**Abstract.** In this paper we discuss some remarkable Geometrical and Arithmetical "coincidence" that can be found when looking at the structure of an octagonal mosaic in the floor of the Basilica of Aquileia (Northern Italy), that dates back to XI Century AD.

**Key words.** Regular Polygons, Octagon, Integer Numbers, Irrational Numbers, Theorem of Pythagoras

*Mathematics Subject Classification:* Primary 00A66; Secondary 01A35

## 1 The Cathedral of Aquileia

***Aquileia*** was an ancient Roman city in that part of Northern Italy that nowadays takes the name of *"Regione Friuli Venezia-Giulia"* (this last name comes from the name of the famous Roman Emperor, *Julius Caesar*). In native Friulian language its name is *Aculiee* (or also *Aquilee* or *Aquilea*), while during the Slovenian domination it took the name of *Oglej.* The Italian name comes from the Latin word *Aquila* (that means *Eagle* - and was in fact the symbol of the town). The old Aquileia was an important town in the Roman Empire, in fact the most important harbor of the Northern Provinces of the whole Empire. After the fall of the Roman Empire Aquileia maintained its importance as one of the most flourishing towns of the Eastern Roman Empire and it was in fact an important centre of Christianity. Situated near the head of the Adriatic Sea, at the edge of the "Lagoons" that include Venezia, it is located about 10 km from the waters, along the river *Natiso* (in modern language it is Natissa or Natisone), the course of which has changed somewhat since Roman times. Even if the actual town is rather small (about 3500 inhabitants) nevertheless it was a fundamental place in between the XI and the XIII Century, when it was the seat of a powerful Archbishop. Large and prominent in Antiquity, it is in fact one of the main archeological sites of North-Eastern Italy.

Fig. 1 – The Exterior of the Cathedral of Aquileia - © Photo by Giovanni dell'Orto

Particularly important is its Cathedral, a flat-roofed "Basilica" (see Fig. 1), erected by the Patriarch *Poppo* in 1031 on the site of an earlier church, and later rebuilt (about 1379) in the Gothic style by the Patriarch *Marquard of Randeck*. The Romanesque-Gothic style façade is connected by a "Portico" to the so-called *Church of the Pagans*, and to what it remains of the 5th century Baptistery, in the form of a circular ambient in which, as usual, the motives of Hexagons and Octagons dominate (see later). But what is most important here for us is its interior, formed by a central nave and two aisles, with a noteworthy mosaic pavement (see Fig. 2) dating back to a period ranging from the 4th to the 11th Century AD in which, among several scenes of the Old and New Testament, a number of fantastic and artistically valuable geometrically inspired shapes appear. Each one of them would be in fact worthy of being discussed separately, but we shall here limit ourselves to comment about one of these mosaic pieces, that is particularly significant because of its Geometrical, Arithmetical and Esoteric characters – see [1].

Fig. 2 – The Interior of the Cathedral of Aquileia - © Photo by Giovanni dell'Orto

## 2 Geometry and Numbers in some Mosaics in the Basilica of Aquileia

As we said, apart from a very rich mosaic art decoration concerning histories of Christianity, on the floors of the Basilica one can see a number of very regular patterns that have a pure Geometrical origin. This is true especially in the lower part of the left nave, to which we shall now devote our attention. For reasons related to their esoteric meaning, many of these mosaics explicitly or explicitly should refer to the themes of the ***(Regular) Octagon*** and of the ***Number Eight*** that is intimately related to the Octagon, it being the Number of its (identical) sides. The main reason of this choice – since it was a choice of the artists and not an accident - resides in the fact that, in early Christianity, the Number Eight was related to a number of hidden “sacred” meanings. In early Christianity, in fact, the Number Four is the “Number of the Body” and the “Double Quaternary” hidden in the Number ***Eight*** (the double of ***Four***) was considered as the “Union of the Human Body and the Spiritual Body”, i.e. a way to conjugate the Material and the Spiritual essence of Mankind. Four is in fact the Number of *Evangelists* and it appears frequently also in the *Book of Apocalypse* by *Giovanni Evangelista* (where, as in the *Bible* and in many other esoteric traditions of Antiquity) this Number refers also to the *Number of Elements* (Earth, Water, Air and Fire) and to the *Number of Directions* (North, West, South and East). Among the symbols of Christianity the ***Cross*** is also fundamental, and it is again related to the Number Four since it is characterized by exactly *Four Vertices* (while the vertical and the horizontal arms of the Cross evoke the Earth – horizontal – and the Sky – vertical – so that the Cross is also a “bridge” between Mankind and God). The Number Eight assumes, therefore, the valence of a *“Double Four”*; it is mostly related to the notion of *“rebirth”* and *”regeneration”* (since, according to the *Gospels*, a period of exactly Eight Days passed between the entrance of Jesus Christ in Jerusalem (before Crucifixion) and his Resurrection.

Because of this relation with “regeneration” the Octagon has increasingly assumed importance in Christian symbology, so that many “sacred” constructions are more or less explicitly based on the figure of a Regular Octagon, where to the power of the Number Eight one adds the strength of a very regular geometric shape. Most of the old *Baptismal Fonts* do in fact have an Octagonal shape, as well as several Churches and other buildings of early and also recent Christianity as well as in the Middle Age (among which, even if much less related to Religious aspects, we should also mention the famous *Castel del Monte* (Fig. 3), due to the Emperor Federicus II, with an Octagonal plant and eight octagonal towers at each vertex) – see [1].

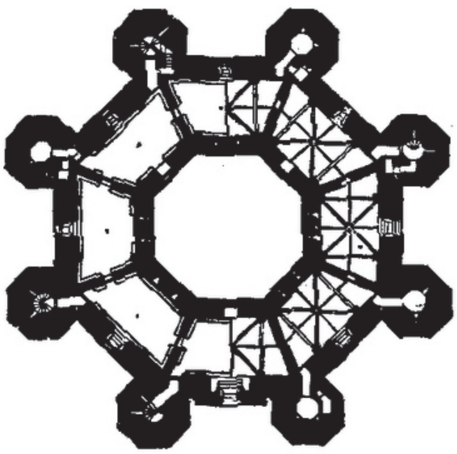

Fig. 3 – Castel del Monte – the Plant

Among many images in the pavement of Aquileia Cathedral (all of them would be worthy of being carefully analyzed and described, as we already mentioned – see Fig. 4 for an overall view of these mosaics) we have chosen a single one that seems to us to be particularly relevant (see Fig. 4). It has the shape of a Regular Octagon filled by a chessboard of small black and white squares (some of them being cut to Triangles along the oblique sides of the Octagon). The esoteric meaning of the black and white tessellation is well known: Black and White refer in fact to the eternal ***Duality*** between Darkness and Light, between Night and Day, between Evil and Good.

Fig. 4 – An overall view of the Geometric Shapes in Aquileia Mosaics - © Photo by Marcella Giulia Lorenzi

The Octagon can be embedded into a larger Square by just prolonging its parallel sides; when embedded into the Square it can be easily obtained by "cutting" four Triangular pieces at the Four vertices of the Square. These Triangles are (identical) Isosceles Right-Angled Triangles. If we assume that the side of the Octagon is Unity, thence the Triangles will have 1 as Hypotenuse, so that by the Theorem of Pythagoras each Cathetus will have length $l_{triangle} = 1/\sqrt{2} = \sqrt{2}/2$ ($\approx 0.7071$) and the side of the Square will thence be $l_{square} = 1 + \sqrt{2}$ ($\approx 2.4142$). Drawing the parallel lines that join horizontally and vertically the Eight vertices of the Octagon, one divides it into exactly ***Nine*** pieces: a central Square of side 1, Four Triangles identical to those that have been cut out from the larger Square and Four Rectangles with sides 1 and $1/\sqrt{2} = \sqrt{2}/2$. Cutting away the Four Triangles, the remaining ***Five*** pieces do form a Cross (see Fig. 5). Let us incidentally notice that also the Numbers Five and Nine bear particular esoteric meanings according to Plato's and Pythagoras' conception [2] (Five is, in fact, the Number of "Fundamental Components of Matter" – the Four Elements plus *"Quintessence"* - and the Number of *"Platonic Solids"* – see also [1]; while Nine is the *"Square of Three"*, i.e. a "Triple Triad", and Three is the fundamental Number of Christianity because of its relation with the *Mystery of Trinity*).

Look now at the Octagon in the mosaic of Aquileia (Fig. 6). One realizes that there are exactly Seven rows and columns of small black and white squares, Two in the left and right columns that correspond to the cut-out Triangles and Three into the central Cross. One can wonder: *Why…?* Notice also that the Octagon contains exactly ***Sixteen*** small black squares and ***Twenty-One*** white ones, for a total of ***Thirty-Seven*** small squares. However, there are black half-Squares along the oblique sides. Counting the halves of black squares there are Eight more triangular pieces, that sum up to Four more black squares (bringing the total to ***Twenty*** black and ***Twenty-One*** white "mosaic tesserae" and the overall total to ***Forty-One*** "tesserae"). Again one can wonder: *Why…?*

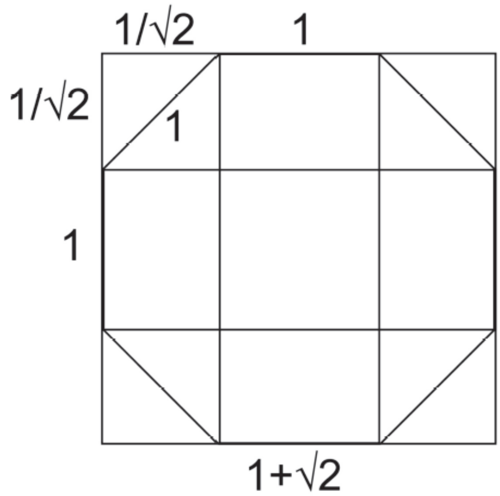


Fig. 5 – The Cross in the Octagon

The reason is very simple: it resides in the ***"Square Root of Two"***, i.e. in what was in fact the origin of Pythagorean Philosophy on *Irrational Numbers* (see [1]). And, of course, it resides in the practical necessities that were encountered by the antique mosaic-makers in constructing a black and white inner tessellation in this Geometrical Figure – practical reasons due to the fact that they had to cope with small mosaic "tesserae" having a finite size that could not be reduced at will.
Let us the analyze the linear measures of these figures and explain first why it was necessary (or at least "convenient") to divide the left and right parts into Two columns while the Central Cross was divided into Three columns. It is of course an approximation on exact Irrational Numbers, but in a suitable sense it is "the best approximation" under the aforementioned circumstances. Let us first notice that $1/\sqrt{2} \approx 0.7071 \approx 0.6666\ldots = 2/3$, so that with a reasonable approximation the number of parts into the side of the Octagon has always to be approximately 3/2 times larger than the number of parts of the side of the Triangle that is cut out; the choice ***Two*** and ***Three*** is of course the simplest, although one could use any other "proportional pair" (e.g., 4 and 6, or 6 and 9, and so on). As far as the ratios between linear sizes are concerned one should moreover notice that $l_{triangle} = 1/\sqrt{2} = \sqrt{2}/2$ ($\approx 0.7071$) $\approx 7/10 \approx 2/3$ (with an error of $7/10 - 2/3 = 1/30$, i.e. less than 4 % - in fact, $3.\underline{3}$ %). This explains further why one can divide the side of the Octagon into Three parts and the side of the Triangle into Two parts ($2 + 2 + 3 = 7$). Notice also that $1 + \sqrt{2}$ ($\approx 1.4142$) $\approx 1.5 = 3/2$, that is the inverse of 2/3 (these Ratios appear in the genesis of the Pythagorean Musical Scale).

Furthermore, let us notice that $l_{square} = 1 + \sqrt{2}$ ($\approx 2.4142$) $\approx 24/10 \approx 72/30 \approx 7/3$, which further explains why one can divide the side of the Square into ***Seven*** parts that approximately split into Two in each "lateral" side and Three in the center (corresponding to the side of the Octagon). Because of this, the "big" Square is divided into 7 x 7 = 49 (***Forty-Nine***) small squares, of which exactly 33 (***Thirty-Three***) are in the central Cross and 16 (***Sixteen***) in the Four squares at the four edges. Of these 16 remaining small squares, Eight are outside and Eight inside, divided in turn into Four + Four whole squares and Eight + Eight (triangular) halves (as we commented before). As a different calculation shows, one has also $l_{square} = 1 + \sqrt{2}$ ($\approx 2.4142$) $\approx 24/10 \approx 12/5 = 2 + 2/5$, while the remainder in the two small segments (left and right) is $\sqrt{2}$ ($\approx 1.4142$) $\approx 14/10 = 2 \times 7/10$. As a consequence, we might also remark that approximating instead $1 + \sqrt{2}$ as $1 + \sqrt{2} \approx 7/3 = 2 + 1/3$ shows that the error done is just $2/5 - 1/3 = 1/15$ (say less that 7 % of the exact measure).

Let us now come to discuss the superficial measures involved. The Four (isosceles and rectangular Triangles) that are cut at the Four edges have side $1/\sqrt{2} = \sqrt{2}/2$ ($\approx 0.7071$) and Hypotenuse equal to one, so that heir area is 1/4 (i.e., half of $1/\sqrt{2}$ times $1/\sqrt{2}$). The total area cut out from the Square is thence Unity; since the Square has total area $A_{square}$ equal to $(1 + \sqrt{2})^2 = 3 + 2\sqrt{2}$

($\approx 5.8284$), the area $A_{octagon}$ of the Octagon is given by $A_{octagon} = A_{square} - 1 = (3 + 2\sqrt{2}) - 1 = 2 + 2\sqrt{2} = 2(1 + \sqrt{2})$ ($\approx 4.8284$).

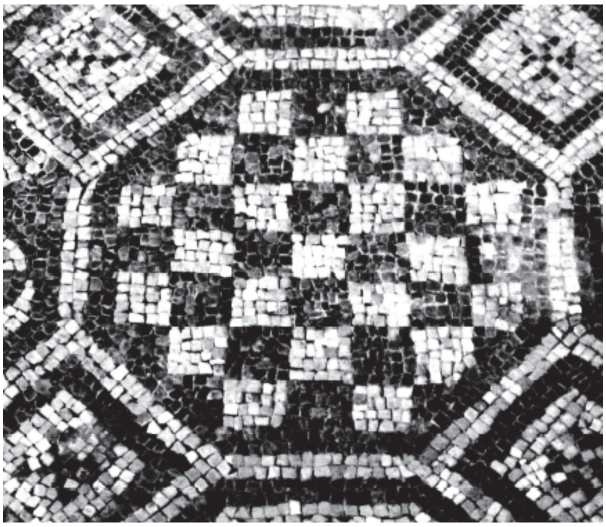

Fig. 6 – The Octagon in Aquileia Mosaics - © Photo by Marcella Giulia Lorenzi

On the other hand, taking verticals and horizontals at all the vertices of the Octagon we divide it into the aforementioned 9 parts (4 Triangles with area $A_{triangle}$ equal to 1/4, four Rectangles with area $A_{rectangle}$ equal to $1/\sqrt{2} = \sqrt{2}/2$ ($\approx 0.7071$) and the central "small" Square of side 1 and area 1 as well). The Rectangles and the small Square form the Cross, whose total area $A_{cross}$ is thence $A_{cross} = A_{octagon} - 1 = A_{square} - 2 = 1 + 2\sqrt{2}$ ($\approx 3.8284$).

We can now express the Ratios between these areas. First we have the important relation $A_{square} / A_{octagon} = A_{square} / (A_{square} - 1) = 1 + 1/(A_{square} - 1) = (3 + 2\sqrt{2}) / (2 + 2\sqrt{2}) = - (3 + 2\sqrt{2})(2 - 2\sqrt{2}) / 4 = 1/2\ (1 + \sqrt{2}) \approx 1.2071 \approx 6/5 = 1 + 1/5$, so that one has also $1/(A_{square} - 1) \approx 1/5$ (i.e., the Ratio $A_{square} / A_{octagon}$ exceeds Unity of about 20%). Analogously one has the important relation $A_{square} / A_{cross} = A_{square} / (A_{square} - 2) = 1 + 2/(A_{square} - 2) = (3 + 2\sqrt{2}) / (1 + 2\sqrt{2}) = - (3 + 2\sqrt{2})(1 - 2\sqrt{2}) / 7 = 1/7\ (5 + \sqrt{2}) \approx 1.5224 \approx 6/5 \approx 3/2 = 1 + 1/2$, so that one has also $2/(A_{square} - 2) \approx 0.5224 \approx 1/2$ (i.e., the Ratio $A_{square} / A_{cross}$ exceeds Unity of approximately 1/2, say the Square is approximately 50% larger than the Cross since their Ratio is approximately 3/2).

Now, dividing the "big" Square in small squares gives exactly 49 "tesserae". On the other hand one has $A_{square} / A_{octagon} \approx 6/5 = 48/40$, but 48 is not a Square Number. The nearest Square Number is of course 49, so that one can think to approximate 48/40 by 49/41, since the following holds: $49/41 \approx 1.1951 \approx 1.2 \approx 6/5 = 48/40$. This explains why 41 among the 49 small squares will fill up the Octagon with a rather good approximation. To say it better, this explains why among the 49 small squares into which one divides the "big" Square there will be 41 inside the Octagon (33 whole Squares + 16 halves) while 8 more (4 + 8 halves) will instead remain outside. The genesis of the Number 33 is now very simple: we said that the Square is approximately 50% larger than the Cross, so that 49 will be 33 + 16, being 16 more or less the half of 33; in other words, $A_{square} / A_{cross} \approx 3/2 \approx 49/33$ ($\approx 1.4848$) with an error of $3/2 - 49/33 = (99 - 98)/66 = 1/66$ (i.e., less than 1 %). Accordingly, 33 small Squares fill up the central Cross (according to the decomposition $6 + 6 + 6 + 6 + 9 = 4 \times 6 + 9$).

Coming to exact proportions, of course, the sizes of the columns (and rows) calculated in this way are a little bit different. In fact, dividing the central Cross into three columns and three rows requires to use 1/3 as measure, i.e. $0,\underline{3}$. On the other hand dividing the left and right portions of the Octagon into two columns requires to use $1/2 \times 1/\sqrt{2} = \sqrt{2}/4$ ($\approx 0.3535$) as measure. The relative difference between these two units is $\sqrt{2}/4 - 0,\underline{3} \approx 0.3535 - 0,3333 \approx 0.0202$, i.e. about 2 %

of difference (something that is of course difficult to see when looking at the mosaic from a distance).

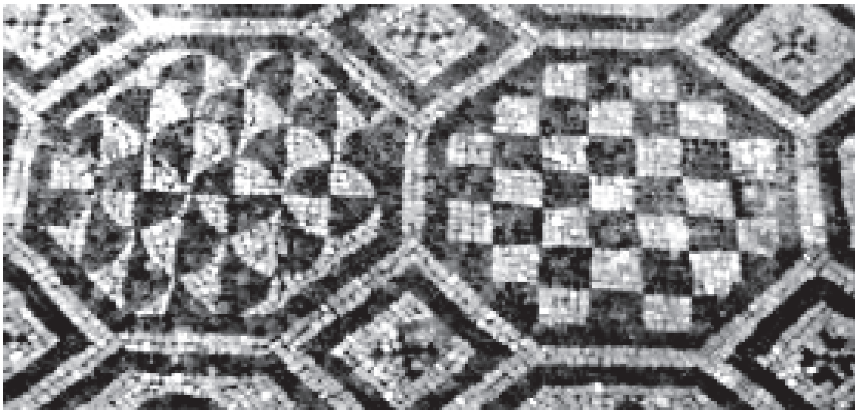

Fig. 7 – The two Octagons in Aquileia Mosaics - © Photo by Marcella Giulia Lorenzi

Up to here nothing is so strange. Nevertheless, with these simple and extremely "natural" approximations – that, we emphasize, are dictated by the "Square Root of Two" – the Octagon (which by itself has a profound esoteric meaning) generates a Cross with a tessellation of Thirty-Three small Squares – 17 white ones and 16 black ones, with a non casual although small prevalence of the white against the black ones). And Thirty-Three is the *Age of Jesus Christ* when, according to *Gospels*, he was put on the Cross…! The total of ***Forty-One*** is therefore seen as the sum 41 = 33 + 8, a decomposition in which the tessellation of the Octagon is formed by the sum of two rather important Numbers of Christianity: the Number ***Eight*** and the Number ***Thirty-Three***.

As a final remark about these numerical coincidences we would like to remark that 41 is the $13^{th}$ Prime Number (and Thirteen is again an important Number in Christian symbology. It being related with the theme of *Last Supper*); while 41 is also subtly related with Octagons, since it is also the $4^{th}$ *"Octahedral Gnomon"* (i.e., the growing figure of so-called *"Octaedral Numbers"* - [1].

We would finally like to remark that (see Fig. 5) a fully analogous decomposition that involves the same Numbers (8, 33, 41 and 49) is in fact visible in the mosaic of the Octagon that is constructed immediately at the left of this Octagon but using *Curvilinear Triangles* instead of small Squares (see Fig. 7).

Fig. 8 – Other Octagonal Geometrical Shapes in the Aquileia Mosaics - © Photo by Marcella Giulia Lorenzi

It is also worthwhile to notice that the motive of the Octagon appears implicitly into other Geometrical Shapes in the Aquileia Mosaics. For example, it is recognizable in all the motives of

Fig. 8, that among many other interesting shapes contain also the so-called ***"Knot of Salomon"*** (Fig. 9), which perfectly fits into an Octagon (as one can easily see by looking at the Figure).

Fig. 9 – The "Salomon Knot" in the Aquileia Mosaics - © Photo by Marcella Giulia Lorenzi

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**References**

[1.] FRANCAVIGLIA, M., LORENZI, M.G.: *Dal Cosmo al Numero e alla Geometria Euclidea,* Technai **(1)**, pp. 23-27, 2009 (in Italian) – Proceedings of the Conference „Scienza e Tecnica nell'Antichità Greca e Romana", CNR Roma, 3-4 Giugno 2008 – ISSN 2036-8097.

[2.] BOORSTIN, D.: *The Seekers. The Story of Man's Continuing Quest to Understand his World,* New York, Vintage Books, Random House Inc., 1999.

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# THE TWENTIETH CENTURY THE USE OF GEOMETRY IN SEARCH OF THE PURE FORM IN SCULPTURE

**OANCEA Ileana (RO), OANCEA Catalin, (RO)**

**Abstract**: In the twentieth century the search for the perfect form gets many artists closer to geometry; sometimes the artistic approach leads them towards mathematics, other times this is a direct source. Connections with mathematics and geometry go far back, but now the sculptural form gets closest to their rigorous structure. This play between art and geometry is, in fact, the subject of this paper.

**Key words**: sculptural forms, geometry, rhythm, volume, abstract.

*Mathematics Subject Classification:*

## 1. Introduction

Historically speaking, the twentieth century is divided into two great periods: the first from 1905 to 1945, at the end of the World War II, and the second from 1945 to the present. The beginning of the twentieth century is characterized by a spread of the African art, a complete rejection of the academic style, a disruption in the known aesthetic principles, a complete freedom of expression, the disappearance of the notion of school and an exaggerated individualism, a search for form—new, genuine, different, shocking solutions and tools—an interest in the abstract and sometimes simultaneous appearance of numerous groups, trends, movements which coexisted or quickly succeeded one after another. “Labels” appear such as: the Fauves, the Expressionists, the Cubists, the Abstractionists, the futurists, the Suprematists, the Dadaists, the Purists, the “de Styl”, the Realists, the Surrealists and others. These basically follow two main directions: some are anchored in romanticism and symbolism trying to expose the inner world, that of the personal “I” and that of the irrational (the Symbolists, the Expressionists, the Surrealists, and part of the Abstractionists), others have a rational view with influences from Mathematics, trying to organize and structure human experiences, to use pure geometrical forms, to dehumanize the image, to render movement and the fourth dimension (the Cubists, the Constructivists, the Raionists, the Futurists).

It is, however, important to mention that the connections between art and different sciences, Mathematics included, are quite old; and without writing about their history we should mention Pierro della Francesca, Italian painter of the second half of the fifteenth century, who believed just like his contemporaries that the bases of painting lie in the scientific observation. He is the author of the first mathematical treatise, in which he showed how the stereometric bodies and the architectural forms could be applied to the human body. This mathematical approach becomes apparent in his whole work. When he drew an arm or a head or a drapery, he regarded them as variations or combinations of geometric forms: spheres, cylinders, cones and pyramids, giving nature one of the most impersonal views. He may be a predecessor of some Abstractionist artists of the twentieth century, who also employ simplifications of the organic forms.

At the beginning of the twentieth century, Rodin declaimed the love for nature, for study; however, some of the painters who followed him rejected his ideas and the organic criteria in order to create other values—the ultimate values of the pure form, forms which are to be found in geometry. At the middle of the twentieth century, Herbert Read noticed that the three-dimensional art was no longer sculptural, moulded, but could be built in the same way as the architectural monuments or a machine (architecture being based on mathematical laws).

We will now show the most significant examples in the history of art, from the field of the three-dimensional, to illustrate this obsessive search of the perfect form using the geometric forms. Some artists came closer to the mathematical rigour, others less close, but this connection has always led to balanced works, beautifully and organically structured, their creation, their play with the strict forms representing the real amount of geometry we find in nature.

## 2.

The first and one of the most representative supporters of this idea is **Constantin Brancusi** (1876 – 1957), Romanian sculptor working and living in France. He pursued two main goals: the universal harmony and the truth of materials. The form, he said, is determined by physical laws in the process of growing. The ovoids and the birds prove a continuous and tireless attempt to find the perfect form, to reach an ideal or even more, to find an ideal which will contain data of natural volumes, trying as he himself confessed, "to solve the difficult and crazy idea of having all the forms in a single one. […] a real form should suggest the infinite. Surfaces should appear as if they continuously pointed to the future and left the mass for a more perfect, more complete existence". The search for this perfect volume, this unique form, drew the artist closer to the geometric forms. He simplified the living structures by means of an unusually rare synthesizing power.

We do not see any optic tricks or unusual materials in his work, he did not develop complicated themes, but used essential forms from nature and he succeeded in finding a common element between these and the geometric forms, the result equally depending on the former as well as on the latter—an equilibrium which led him to essence, to idea. Simplifying the form to get the essence, searching for an expression of volumes, the attempt to say the most using the least characterized and defined Brancusi, whose life served "the complete unity between material and form", as Barbara Hepworth regarded his work.

"We are within a sphere, we play with other spheres, we combine them, we make them blink". (C. Brancusi)

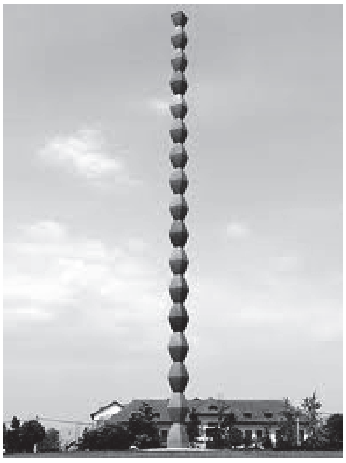
Brancusi did not set off looking for geometry, but gradually arrived there looking for the natural data of models, simplifying and giving up more and more details; the results were, however, synthetic forms which suggest the mathematical ones.

“Maiastra”

“Bird in Space”

“Boy Torso”

“Endless Column” (Tg Jiu)



“Newborn”

“Blind/The Beginning of the World”

“Torso”

In the above-mentioned works, we included two from the “Birds” series to illustrate the artist’s way. All his life he claimed that he could not find “The Gold Bird”; it seems that this search did not enjoy the final result, but even so, the way Brancusi worked became apparent: if in the from beginning sculpture got close to nature, in the end it expressed the essence, the flight itself. The switch from natural forms to geometrical ones represents the switch from the mimetic representation to the representation of the essence, of the idea. “Athena Tacha said: ‘By examining the evolution and the gradual simplification of “The Bird” one can realize that Brancusi reduced the form to the last limit of the possible. His art is a fragile equilibrium between the abstract and the figurative […]. As far as “The Bird” is concerned, he evolved from the motionless being to the act of flying itself…’”. (Petrin Genaru Adrian*, Image and Symbol in Brancusi,* Meridiane Publishing House, Bucharest, 1983).

The same simplification is found in “Bust of a Boy” which is reduced to a combination of three cylinders. In the same way the stone “Torso” or the ovoids with the origins in various portraits became forms related to the geometrical ones.

The sculptural ensemble from Tg. Jiu, “The Column of the Infinite”, “The Gate of the Kiss”” and “The Table of Silence” are perfectly geometrical works, elements from them being frequently found in many pedestals for the small sculptures, an integrated part of the sculptures themselves. In these works, Brancusi worked on one element which was then multiplied, vertically for “The Infinite Column” and horizontally for “The Silence Table”. Repetition created rhythm. Nevertheless, they represent philosophical ideas, whereas in other artists’ works the geometric forms are just instruments of arranging the artistic pleasure.

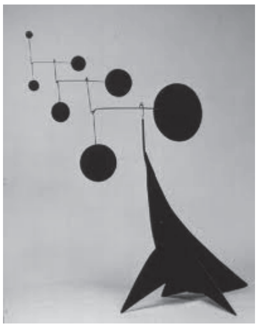
**Archipenko** (1887-1964) and **Lipchitz** (1891-1973) show special interest in the mass, in the sculptural object regarded firstly as volume. They systematically analyzed and constructed it from smaller volumes, each applying a different kind of synthesis in an attempt to see it geometrically. In painting, **Cezanne** (1839-1906) regarded nature from the sphere's, the cube's and the cylinder's views. They did not go too far away from the recognizable, but used the straight and curve lines as a system or arranging the real data, without getting further.

„Flat Torso"-A. Archipenko

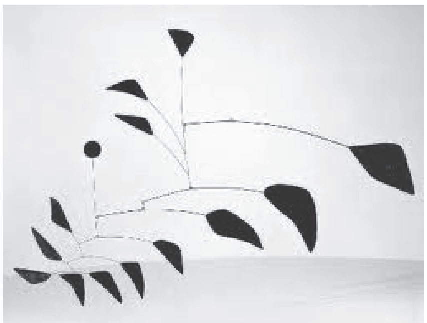
„The Guitar"-G. Lipchitz

There can appear similarities between cubism and futurism; whereas the former expresses the natural forms and places them in space, the latter tries to create linear elements, main lines, to suggest the effect of movement. **Boccioni**'s synthesis would be the systematization of the interpretation of planes, the foundation being architectural not only in the construction of masses, but in such a way that the sculptural block contains in itself architectural elements of the sculptural environment where the subject lives, as the author himself explained.

1930 is the year when **Alexander Calder** (1898-1976) created bi and three-dimensional geometrical forms which are dancing; he introduced the living movement into the geometrical abstraction. He later added colour. The relations between the volumes, both of size and of form, as well as their relation with space are in a constant movement. These relations come from mathematics.



Untitled- A. Calder



„Big red"- A. Calder

While Brancusi went from nature towards geometry, **David Smith** (1906-1965), in America played directly with basic geometrical structures such as the cylinder, the prism, the cube to eventually get to the "Cubi" series, constructions from geometrical forms which triumph over gravity. In Smith, we can say he structured space and played with geometric volumes entirely within the abstract. The lines are strict, straight, and too little in motion.

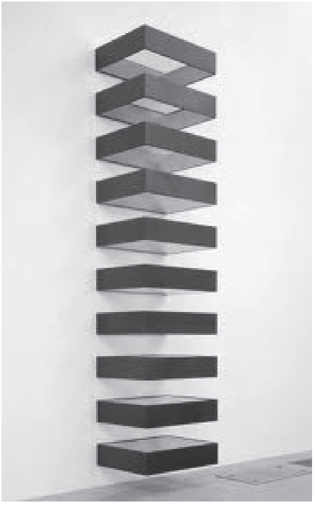
"Cubi XXVIII" - D. Smith

"Cubi XIX"- D. Smith

With the "Albany" series, begun in 1959, and the "Zig" series in the following year, Smith's work became more geometric and monumental. In "Zigs", his most successful Cubist works, he used paint to emphasize the relationships of planes, but in his "Cubi", begun in 1963, his last great series, Smith relied instead on the light of the sculptures' outdoor surroundings to bring their burnished stainless steel surfaces to life. These pieces abandon two-dimensional planes for cylinders and rectilinear solids that achieve a sense of massive volume. Smith joined these cubi-form elements at odd and seemingly haphazard angles, in dynamically unstable arrangements that communicate an effect of weightlessness and freedom. (Encyclopaedia Britannica Online, http://www.britannica.com/EBchecked/topic/549706/David-Smith)

**The Minimalism** ('60-'70) is mainly developed in America and represents the depersonalization of the artistic gesture, distance from the affective implication, the scarcity of means and expression. Sculpture goes back to basic, geometric forms, from already existing, series materials, and sometimes involving new technologies for example in Fleming's fluorescent tubes. There is one repetitive principle which later led to the appearance of Installationism. They do not reduce the world to simple forms, but are restricted to the interaction space-object-viewer and his/her experience. Perhaps here, more than in other trends one can talk about structures, rhythms, concave/convex spaces, hollow and full, since the minimalists tried them all, generally at a monumental scale, arranging bigger and bigger spaces. We witnessed a conceptualization of the artistic effort. The role of mathematics and particularly of geometry is not at all negligible. It helped to depersonalize the act and represented one of the new methods of approaching a work of art. The already existing rules within mathematical sciences started to be employed, firstly, relations and geometrical forms.

**Donald Judd** (1928-1994), American artist, one of the representative figures of minimalism, disciplined the aesthetic pleasure without denying it. He stimulated our perception through sizes, alternated materials, surfaces, colours, closed or open, concave or convex forms. The geometric volume was the ordering element of our senses, the basic structure. He gave up the warmth of natural forms in the favour of mathematical rigour.



Untitled (stack)- D. Judd

“Beams”-R. Morris

**Robert Morris** (born 1931) impresses the critics with his massive, gray, geometrical volumes, which make the visitor aware of the space around him. His passive-aggressive volumes arrange and rearrange the space; sometimes a simple L in different positions has different effects. Geometry is used to structure the displaying space itself when he exhibits in a gallery, or to organize the outdoor spaces.

**Sol Le Witt** (1928-2007) used boxes as basic forms which are arranged apparently at random, but whose pattern can be seen after careful examination. The cube or the parallel curve or straight lines are most common elements in the American artist’s work, whether in three-dimensional constructions or in structuring the displaying spaces.

Geometric Structure 2-2,1-1 – Sol LeWitt

Wall Drawing 1136– Sol LeWitt

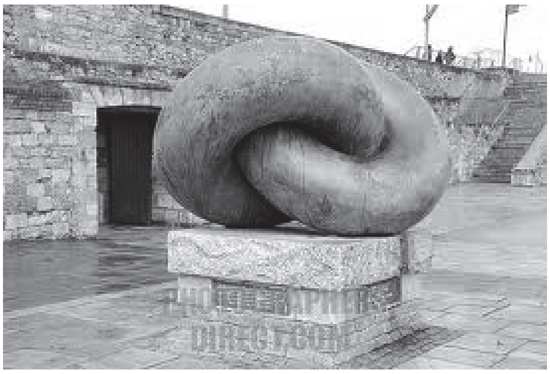
Somewhere between minimalism and land-art, the works of **Richard Serra** (born1939) and **Alexander Lieberman** (1912-1999) organize, structure and animate vast spaces either in parks or in towns, in connection to buildings. Art refuses to be restricted to conventional museums and sets off to conquer the town space, greets the passer-by, and participates in the day-to-day life. Generally strict, precise forms, the volumes succeed in setting the atmosphere into motion and they even bring life to places that previously had walls only.

“The matter of time”-R. Serra

“The way”-A. Lieberman

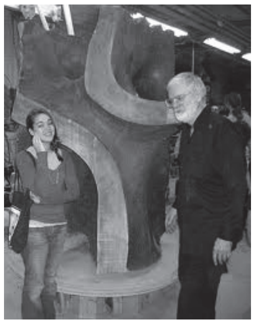
**Land-art** (60’-70’) developed on even more spectacular parallel coordinates. It represents the attempt to escape the workshop, a conventional space, to conquer the outdoor space, generally situated far from the inhabited areas. It is the return to nature, a combination between the megalithic primitive structures, archeology and geology. Land-art artists, just like minimalist artists looked for simple, geometrical forms, to which the use of natural materials confers movement, growth, change. (Michael Heizer, Richard Long, Dennis Oppenheim, Walter de Maria, Robert Smithson)

**John Robinson** (1935-2007) began his artistic career late, when he was 35. He was interested in astronomy and mathematical relations. Proportions, lines, rhythm, the agreement between titles and shapes are remarkable. Many mathematicians from topology and toruses see mathematical relations in his sculptures. The “Gordian Knot” and “Bands of Friendship” prove complex mathematical theories transferred to bronze. The “Rhythm of Life” comes from the experiments using a tape enfolded inside a tube whose ends meet. The “Genesis” was born from the attempt to represent the three Borromean rings, carved from one block, which cannot be separated without breaking the whole work to pieces.  Many of his works share the theme of humanity—in “Dependent Beings” there are two squares that curl on a circle, resulting in two contrasting textures. He worked in bronze, wood, steel, marble, trying to emphasize various mathematical models: the Moebius’ band, the Borromean rings, the Brehm model or knots. He started from the mathematical relation and headed for the geometric-artistic image. He did this backwards, but the result is nonetheless impressing, because mathematical relations have rhythm, alternation and structure. Volumes read easily, they are well individualized; the rigour of mathematical rules changing into a play of geometrical bodies.



“First fleet”-J. Robinson “Creation”- J. Robinson

**Helaman Rolfe Pratt Ferguson** (born 1940) is an American sculptor, a digital artist, especially an algorist, a scientist with a PhD in Mathematics. His research brought to light one of the best known algorithmic formula. He works in bronze and stone and represented mathematical forms. In spite of this, the resulted forms are organic, textural, mysterious, and not at all similar to those described in J. Robinson’s case, for example. Geometry is contained within the irregular volumes, while mathematical precision is eluded.



"Invisible Handshake II"-Helaman Rolfe Pratt Ferguson

**3**.

This relation with geometry is more frequently seen than we could initially guess and we cannot cover everything here, but we tried to include the most representative works to emphasize the various possible relations between the artistic three-dimensional form and mathematics, more precisely geometry.
Mathematics means pattern and rhythm so that one should not be surprised at how much mathematics there actually is in art. Whether it is about sculpture or another artistic field, mathematics can be the tool, the inspiration or simply the natural fact contained in structure.
The geometrical form still represents an artistic guideline, a starting point for some works (Boccioni), a final representation for others (John Robinson), a compositional structure (David Smith) or the essence of the living form (Brancusi). Artists have always been drawn, consciously or not, to various aspects of sciences, geometry being one of the most common. The innate aesthetic sense, on the one hand, and the study of nature, on the other hand has drawn these artists closer to the scientific rigour of this side of mathematics. We should say that even the running away from the strict rules of geometry represents the acknowledgment of its presence in the artistic discourse.

"Geometric sculpture can be a potent tool for communicating ideas in a visual and tactile manner. While all art should be enriching and thought provoking, mathematically based art displays additional internal resonance through underlying relationships, which appeal to one's sense of system and logic. It draws upon and celebrates the visual and structural modes of thinking which bind art with mathematics". (George W. Hart, mathematician and constructivist sculptor, "Mathematical Awareness via Geometric Sculpture", Math Awareness Month, April, 2003, http://mathaware.org/mam/03/essay2.html)

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**References**

[1] ARNHEIM, Rudolf, «The Power of the Visual Center», Ed. Meridiane, Bucharest 1995
[2] «Art Now», Ed. Taschen, Koln 2002
[3] BOORSTIN, Daniel J., «The Creators», vol. I, II, Ed. Meridiane, Bucharest 2001
[4] BRAUDEL, Fernard, «the Grammar of the Civilizations», vol. I, II, Ed. Meridiane, Bucharest 1994
[5] CONNOR, Steven, «Postmodern Culture», Ed. Meridiane, Bucharest 1999
[6] «Dictionary of the Arts», Ed. Helicon, Oxford 1995
[7] GEIST, Sidney, «Brâncuşi», Ed. Meridiane, Bucharest 1981
[8] JANSON, H.W., «History of art», Ed. Taschen, Koln 1999
[9] OANCEA, Ileana, «The Sculptural Volume, Physical and Spiritual Limits», PhD Thesis 2006
[10] PERRUCHOT, Henri, «L'Art moderne a travers le monde», Ed. Hachette, Paris 1968
[11] READ Herbert, «A Concise History of Modern Sculpture», Ed. Thames and Hudson, London 1979
[12] RIEGL, Alois, «History of Arts as History of Styles», Ed. Meridiane, Bucharest 1998
[13] Ruhrberg/Schneckenburger/Fricke/Honnef, «Art of the 20-th century», Ed. Taschen, Germany 1998
[14] «Sculpture», vol. I, P. Bruneau/ M. Torelli/ X. Barral i Altet, Ed. Taschen, Koln 1999
[15] «Sculpture», vol. II, B. Ceysson/ G. Bresc-Bautier/ M. F. dell'Arco/ F. Souchal, Ed. Taschen, Koln 1999

**web:**

http://mathaware.org/mam/03/essay2.html
http://www.britannica.com/EBchecked/topic/549706/David-Smith
http://www.popmath.org.uk/sculpture/sculpture.html
http://www.helasculpt.com/theartist/index.html

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# MATHEMATICAL SURFACES MODELS BETWEEN ART AND REALITY

**PALLADINO Nicla, (I)**

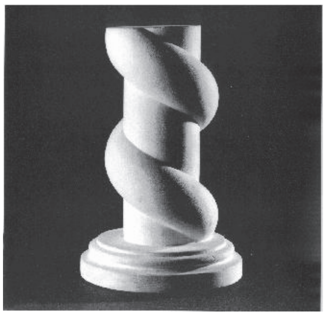
**Abstract.** In this paper, I want to document the history of the mathematical surfaces models of surfaces used for the didactics of pure and applied "High Mathematics" and as art pieces. These models were built between the second half of nineteenth century and the 1930s. I want here also to underline several important links that put in correspondence conception and construction of models with scholars, cultural institutes, specific views of research and didactical studies in mathematical sciences and with the world of the figurative arts furthermore. At the same time the singular beauty of form and colour which the models possessed, aroused the admiration of those entirely ignorant of their mathematical attractions.

**Key words.** Mathematical models, surfaces, history, NURBS.

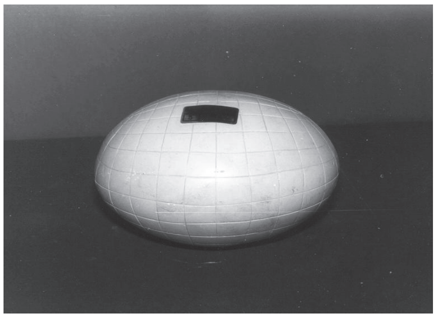
*Mathematics Subject Classification:* Primary 01A99, 01A60; Secondary 01A99.

## 1 The mathematical models

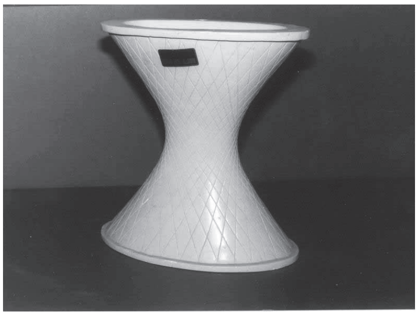
Looking back in the history of mathematics, it sometimes seems that it is possible to find ideas till usable or reusable in a modern context. Historical research on approaches to the teaching of Geometry is an activity that does not fail to clarify points, curious paradoxes and startling evidence, that run through a process of transformation from the works in which the Descriptive geometry was coded- Gaspard Monge, *Geometrie Descriptive*, 1794 – to the affirmation of the analytical methods - Gaston Darboux, *Geomètrie Analytique,* 1917. From descriptive representations by ruler and compass in the second half of the nineteenth century, mathematicians have turned to use mathematical models, initially produced by artisans and then by methods for mass production. Their story begins roughly in the second half of the nineteenth and ends in the thirties of the twentieth century. The models were made in different materials: brass, plaster, cardboard, wire or natural fiber, wood and strips of wood, celluloid, metal foils; they were used in numerous fields of pure and applied mathematics, in order to show results of researches or to teach "high mathematics" in Universities and Polytechnics. They were used in a lot of fields of mathematics: descriptive and analytic geometry, topology, optical geometry, theory of functions, ... Using the plastic models was useful for enhancing “intuitive” aspects of Mathematics and Geometry, as is well described in the book *Anschauliche Geometrie* by D. Hilbert and S. Cohn-Vossen.[1]



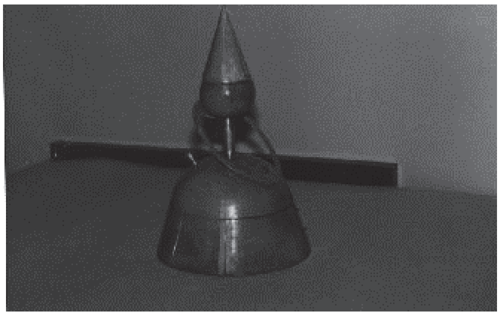
Helicoidal surface in chalk



Ellipsoid in chalk



Hyperboloid of one sheet in chalk



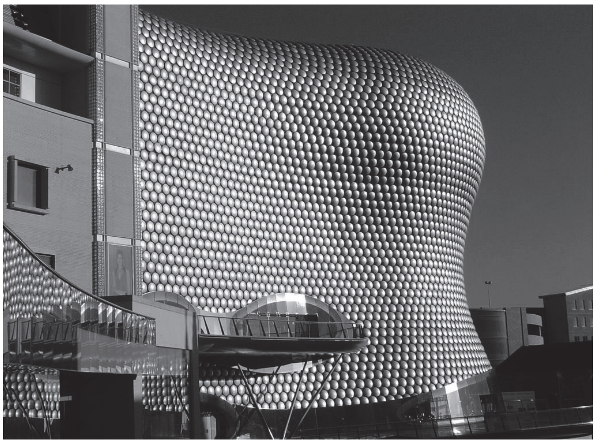
Surface for Dandelin's proof

## 2 Are mathematical models extinct?

The history of mathematics tells us that, after 1920, the mathematical models of surfaces were progressively abandoned. Asking why, we come in contact of the "Bourbaki" mathematical group – in particular Weyl, Brower and Beth- with the psychologist Jean Piaget and the philosopher Michel Fucault who even in mathematics require one of the cornerstones of Martin Heidegger’s philosophy[2] *-Language is the house of the truth of Being.* But Jacques Herbrand’s results are opposed: in 1830 he found the first-order theories in which the model is contained entirely in their language.[3]
These results complement the research undertaken in the previous century by Ludwig Wittgenstein and induce the Russian-French school mathematicians to consider unnecessary any interpretation that extends beyond the boundaries of linguistic structures up to the radical positions of the Russian Sylow, which in the his Course of Analysis says that mathematics is the study of the structures of the mathematical thinking.[4]
Consulting the texts preserved in the departments of mathematics and various branches of the Italian Mathesis (Italian Society of Mathematical and Physical Sciences, founded in 1895) leads to the conclusion that the mathematical models are extinct; but the evidence shows us that the most

[1] [8].
[2] See [6].
[3] See [7] and Herbrand‘s paper held in *Ecrits Logiques*, Paris, 1968.
[4] [14].

important buildings of modern architecture are inspired by mathematical models of surfaces as well as all the latest structures contain quadric surfaces made by *NURBS -Non Uniform Rational B-Splines*. School design followed by world famous architects impose themselves as the vanguard in contemporary architecture for the intensive use of mathematical models by *NURBS* in their projects. If the didactics of mathematical models was essentially their construction, it means that somewhere someone has continued to teach. Without missing much, judging from the results of their pupils.



Selfridges store, Birmingham

Swiss Re Building, London

Sydney Opera House, Sydney

Oceanographic aquarium, Valencia

Mathematics is not like in the Thirties, both in the accumulation of its results and -especially- in epistemic interpretations. For example, it is interesting to compare *l'art de bien raisonner sur des figures mal faites* by Henri Poincaré,[5] who shows the task of topology:

*On a dit souvent que la géométrie est l'art de bien raisonner sur des figures mal faites.* [...] *Mais qu'est-ce qu'une figure mal faite? C'est celle que peut exécuter le dessinateur maladroit dont nous parlions tout à l'heure; il altère les proportions plus ou moins grossièrement; ses lignes droites ont des zigzags inquiétants; ses cercles présentent des bosses disgracieuses; tout cela ne fait rien, cela ne troublera nullement le géomètre, cela ne l'empêchera pas de bien raisonner.*

[5] In [12].

*Mais il ne faut pas que l'artiste inexpérimenté représente une courbe fermée par une courbe ouverte, trois lignes qui se coupent en un même point par trois lignes qui n'auraient aucun point commun, une surface trouée par une surface sans trou. Alors on ne pourrait plus se servir de sa figure et le raisonnement deviendrait impossible.* [...]

*Cette observation très simple nous montre le véritable rôle de l'intuition géométrique; c'est pour favoriser cette intuition que le géomètre a besoin de dessiner des figures, ou tout au moins de se les représenter mentalement. Or, s'il fait bon marché des propriétés métriques ou projectives de ces figures, s'il s'attache seulement à leurs propriétés purement qualitatives, c'est que c'est là seulement que l'intuition géométrique intervient véritablement.*

Building a course in Geometry with mathematical models is to understand what has happened between the 30s - when mathematical models are gone into hiding- and 90s, when the mathematical models have reappeared –by *NURBS* surfaces, as a canonical expression in the avant-garde architecture. Among closed surfaces only *NURBS* can be dynamically deformed.

*Architectural topology means the dynamic variation of form facilitated by computer-based technologies, computer-assisted design and animation software. The topologising of architectural form according to dynamic and complex configurations leads architectural design to a renewed and often spectacular plasticity, in the wake of the baroque and of organic expressionism.*[6]

Among the works of Hilbert and Giuseppe Peano on the foundations of geometry in the late nineteenth century and the attempt to found the Tarskian geometry with meta-mathematical techniques of elementary language, fifty years have passed.
At the beginning of last century mathematical logic has focused on the problem of foundations and demonstrations of consistency theories: as the study of the consistency of geometry could be attributed to the consistency of arithmetic, meta-mathematical reflections on geometry and its axiomatization rather than on analysis of the consistency, they have developed in different directions. Pasch axiomatic approach was aimed to allow intrinsic characterization of the geometry in order to make it independent from empirical representation, i.e. from analysis and physical. In Peano and his school, the axiomatic formulation of geometry has led not to the study of relations between different theories, but to the study of definability of a term in relation to the primitive terms of the theory. Hilbert's axiomatization was not linked to the conception of geometry as a purely formal science: purpose of the *Grundlagen* was rather the characterization of that class of objects formed by the entities of elementary geometry. Tarski has taken Padoa's criterion of definability and with an appropriate limitation of language that would make it possible to define the involved concepts (definability and transformations on models), he established the axioms of elementary geometry, based on the concepts of "being between" and equidistance. The choice of the limitation of language made by Tarski has a reason: the meta-mathematical notion of definability covers both the figures synthetically defined and those of analytic geometry.

## 3 The artistic interest in the Twentieth Century

Always plastic mathematical models, especially those made of chalk, as well as being useful to the mathematical sciences, provide inspiration to painters, sculptors, architects and designers. Alfredo Franchetta, professor of geometry at the University of Naples, graduated in Rome where his teacher was Federico Enriques; Franchetta was asked at the beginning of the Nineties for information about

[6] [4].

Mathematical models of surfaces, and he said that Enriques had repeatedly stressed that designers of *Cinecittà* (born in Rome in 1937) came often to the Institute of Mathematics (today at the University "La Sapienza") to observe, in order to to draw inspiration, the models kept there. Some elements of documentation on the artistic side can be found in *Fantasia e logica della matematica*[7] by Luigi Campedelli. This popular volume contains illustrative tables of sculptures in chelk by Alberto Viani (1906-1989) and a picture of Atanasio Soldati (1896-1953), inspired inspired by the geometric patterns of mathematical models.
In the review of the text by Gerd Fischer, written by Jeremy Gray,[8] there are photos of sculptures by Umberto Boccioni (1882-1916) –in his *Visioni simultanee*, of 1911, it seems that some traits, the purple color faded, were inspired by the *Surface of Riemann*[9]–, by Naum Gabo (1890-1977), and by Costantin Brancusi (1876-1957).
In a paper on Man Ray (1890-1976), Isabelle Fortuné[10] speaks of his discovery, with Max Ernst (1891-1976), in 1934, of mathematical models exposed at the *Institut Poincaré*, in Paris.[11] In her paper, the author highlights the artistics photos by Man Ray and a *collage* by Max Ernst (with title *Man Ray, peintures et objets*, of 1935) that reproduced the *Enneper minimal surface*, *Riemann surface*, *Clebsch diagonal surface*, extracted from the *Katalog* of Walther Dyck; she remember that Ray painted, between 1948 and 1954, from his photographs, a series of tables grouped under the title *Équations shakespeariane*.
Interesting and significant is also the work of Angela Vierling-Claassen on *Models of Mathematical Surfaces*[12] where he also talks about *Mathematical Models and Art in the Early 20th Century*.
In 2005 the sculptor Cayetano Ramírez López restored the mathematical models of the University of Groningen, in Holland.
Isabelle Fortuné explains that the attraction for mathematical models was generated because they appeared as points of intersection of the two major trends that marked the surrealism in the Thirties: reflections on the notion of object and on the discoveries of modern science.

## References


[1.] E.W. BETH and J. PIAGET, *Epistémologie mathématique et Psycologie*, PUF, Paris, 1961.
[2.] L. CAMPEDELLI, *Fantasia e logica nella matematica*, Scienza Nuova, Feltrinelli, 1973.
[3.] E. CASTELNUOVO and M. BARRA, *La matematica nella realtà*, Boringhieri, 1975.
[4.] G. di CRISTINA, *Architecture and Science,* London, Wiley-Academy, 2001.
[5.] G. FISHER*, Mathematische Modelle/Mathematical Models (vol. 1-2) edited by Gerd Fischer, reviewed by Jeremy Gray*, «The mathematical Intelligencer», vol. 10, n. 3, 1988, pp. 64-69.
[6.] M. HEIDEGGER, *Brief über den Humanismus*, in Platons Lehre von der Wahrheit, Bern, 1947.
[7.] J. HERBRAND, *Recherches sur la théorie de la démonstration*, Prace Towarzystwa Naukowego Warszawskiego, Warsaw, 1930.
[8.] D. HILBERT and S. COHN-VOSSEN, *Anschauliche Geometrie*, Berlin, Springer, 1932.
[9.] N. NEGROPONTE, *Being multimedial,* MIT Press.
[10.] M. PASCH, Vorlesungen über neuere Geometrie, I. Aufl., Leipzig 1882.
[11.] H. POINCARE, *Analysis Situs*, 1895, in Journal de l'Ecole Polytechnique.


[7] [2].
[8] [5]
[9] *Zweiblättrige einfach zusammenhägende Riemann'sche Fläche*, in [13] Serie XVII, n. 10.
[10] *Man Ray et les objets mathématiques*, «Études photographiques», 1999, *etudesphotographiques.revues.org*
[11] Université Paris VI "Pierre et Marie Curie".
[12] *http://math.harvard.edu*.

[12.] H. POINCARE, *Oeuvres de Henri Poncaré publiées sous les auspices de l'Académie des Sciences*, Paris, Gauthier Villars, 1853.
[13.] M. SCHILLING, *Catalog mathematischer Modelle*, Leipzig, Verlag von M. Schilling, 1911.
[14.] G.E. SYLOW, *Matematičeskij Analiz,* MIR, Mosca, 1978.

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# SILENT RHYTM OF SPACE AND COLORS IN MONDRIAN PAINTING

**PĂUN Marius, (RO)**

**Abstract.** Mondrian was a theosophist. ”There is no religion higher than truth”, the motto of his philosophy, gives him the frame in which he delivers his art. We try to follow the evolution of the conception of reflecting reality from image to nonillusionism and, using some mathematics, to reshape the nonillusionistic painting back to reality.

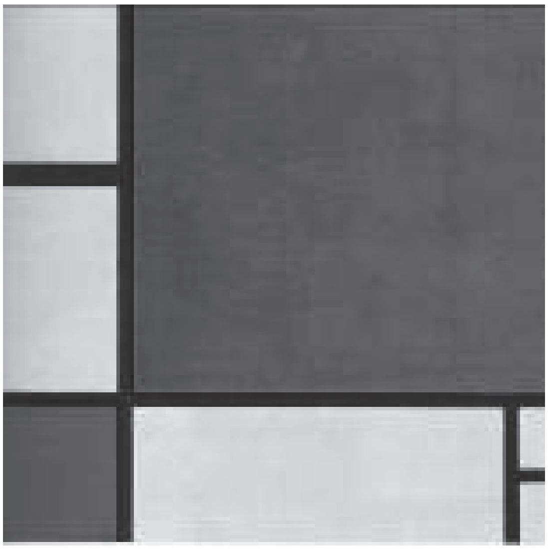
**Key words.** Shape’s space, colour’s space, graph, topological filter

*Mathematics Subject Classification:* 91F99

## 1 From representation to nonillusionism

Some hundred paintings separate the „Landscape with mill” (1904) from „Composition in (with) red blue and yellow” (1936) and almost all are a step ahead in the evolution of the perception that Mondrian had about representing reality in art. As young artist, at the turnaround of the 20-th century, he was connected to the contemplation of the Dutch stable landscape. The image of a motionless, windless mill near a lake underlines the late idea. Mondrian nails the landscape in a pure Cartesian frame with the axes suggested by the bridge and the mill itself parallel with the frame of the painting. The reflection of the mill in a mirror like water surface reinforces the stability of the composition. Small segments and grid pattern repeated in different parts of the painting induce a strong filling of rhythm, a rhythm without sounds, a unifying tool between space and shapes between ”universal and individual”. The strong lines of the bridge are like an empty stave waiting for notes to be placed. The composition is asymmetric and intense colours are replacing unimportant details. Linear brushstrokes delimit the aesthetical collectivities: mill, water, landscape. And there is more: the projection of the mill in the water induces the Mondrian’s incipient idea of flattering out the space.

The second painting was chosen because it is the same mill in another technique of painting. We can ignore the technique and imagine that the artist is preparing us, let’s say, for the moment of truth. Shapes are no longer the images of objects. “Mill in sunlight” (1908) look like a sand castle. The composition is very fragile, straight red lines keep the mill in a piece but the stability of the landscape is weak. The rhythm is present in this painting in the dancing spots of light. There is a measure of representation in this painting but space, light and shapes are preparing to be decomposed in essential sensations: the fundamental colours yellow, red and blue. Aesthetical collectivises are not anymore represented by objects, water and landscape but by colours. The blue of the water merge in the body of the mill, the bridge and the frame of the mill merge in the straight red lines, land and light merge in yellow spots. The sky, symbol of the surrounding space became more and more grey.



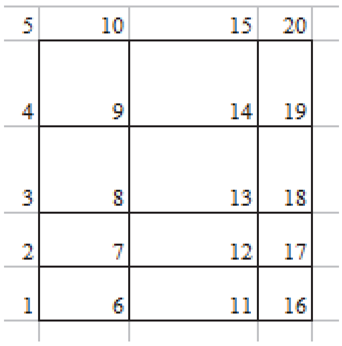
”Composition in red blue and yellow” is the ideal representation of the mill. The constructive components of the ”Landscape with mill” are all present here. The Cartesian frame given by the bridge and the mill is not suggested but physically present. The asymmetries of the first painting are maintained and shapes float and eventually melt in colours. It is a vision of reality that Mondrian considers his task: expressing real perception in a purified artistic absolution, replacing shapes by colours.

In this nonillusionistic type of painting, varying dimensions of areas and with the balance of colours and the rhythm imposed by the spiral of colours he believes such liberation can be accomplished. Aesthetic collectivities are presented in a matrix. The title of this painting indicates exactly what is on the matrix of space. Piet Mondrian had a dream and a strong confidence that he can realize a unified vision of space and forms in this space. The artist purified the expression of forms, imperfectly experienced in nature, in perfect tones of red, blue and yellow. He introduces a strong regularity of subspaces in rectangular shapes included in an irregular quadrature cotrolled not by the direct perception but by his particular interpretation of balance and equilibrium. In the varying dimensions of the rectangular areas with no irregularities traceable to the hand and with a perfect balance of colors he fulfilled his dream. He characterizes his painting "Space becomes black or gray; form becomes red, blue or yellow".

## 2 From non-illusionism to representation

Let us apply some very simple mathematics and trace back from „Composition in red, blue and yellow" to the „Landscape with mill".

First we shall superimpose a grid over „Composition in red, black and yellow" with the property that every line in the grid contains a black segment of the painting. This grid has 20 nodes denoted from bottom to up and from left to right.




From those nodes we discard the set {2, 14, 15, 19} which are not present in the painting.

We denote by A={1,3,4,5,6,7,8,9,10,11,12,13,16,17,18,20} and we consider a partition of the set A as follows:

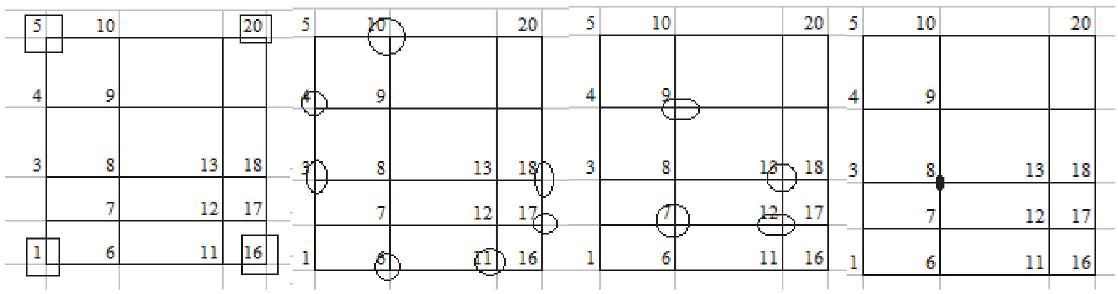
$A_1 = \{1,16,20,5\}$ - the set of nodes delimiting the space of the painting

$A_2 = \{6,11,17,18,10,4,3\}$ - the set of nodes on the frontier delimiting forms on the space of the painting

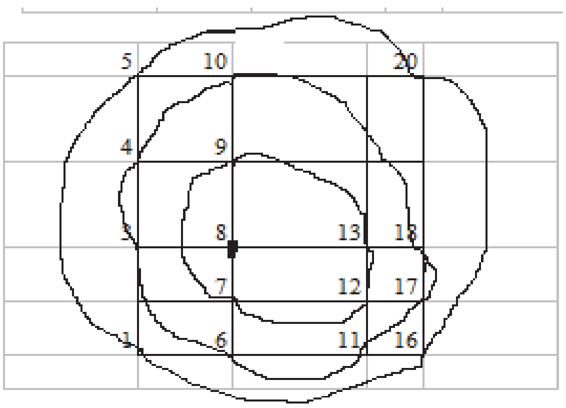
$A_3 = \{7,12,13,19\}$ - the set of interior points delimiting forms

$A_4 = \{8\}$ - the origin of the Cartesian frame associated with the painting.

To every set of this partition we attach a level of "importance" given by the numbers of connections each point in the set realizes.



For the set A we can define a graph whose edges are given by the cycles defined using the partition $(A_i)_{i=1,2,3,4}$. Drawing the mentioned cycles we obtain:




If we attach to every cycle the level of importance than this can represent a map of the painting "Composition in red, blue and yellow". We succeed so to add volume to this painting which is essentially plane.

Let us consider now the set of colours in Mondrian painting:

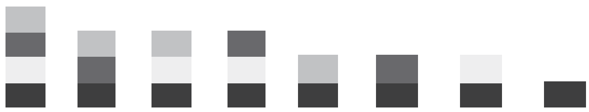
*C= {red, blue, yellow, grey, black}.*

We can consider on C the discrete topology.

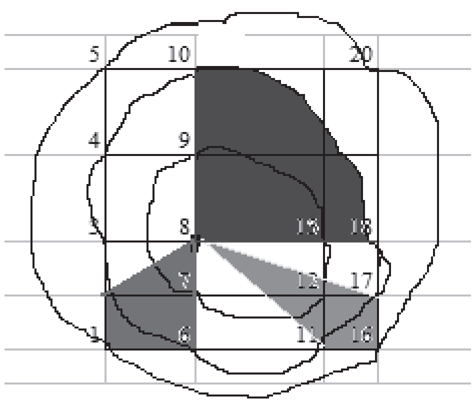
Remember that if we have $(X,\tau)$ a topological space then a family $\varphi \subseteq \tau$ is a filter if:

1. $\varphi$ is closed to finite intersections
2. if $U \in \varphi$, $V \subseteq X$ and $U \subseteq V$ then $V \in \varphi$
3. the empty set does not belong to $\varphi$.

The filter $\varphi$ is said to be convergent to $x \in X$ if the filter of the vicinities of $x$ is contained in $\varphi$.
With respect to the discrete topology defined above the filters of vicinities of “red” is convergent to “red” and similar for “blue” and “yellow”.
We can imagine these filters like a funnel which permits only to one colour to pass by.
An image of such a filter is, for example:

 (the red filter).

If we place on map the three convergent filters (red, blue, yellow) so they converge in the point 8 we obtain:




To the map of the environmental space we added the volume of forms and colours. The colours used to represent forms are, as there were for Mondrian, a limit. A limit of the power of expression of an artefact, a limit of aesthetical representation and in the same time a new frontier of imagination. The unified vision of space and forms was achieved.

**References**

[1] L. WITTGENSTEIN, *Tractatus Logico-Philosophicus*, Editura Humanitas (Bucureşti, 1991)
[2] M. PAUN, *Folding and Unfolding Symmetry*, APLIMAT - Journal of Applied Mathematics, **4** (2011) –ISSN 1337-6365; in: “Proceedings 10th Int. Conference APLIMAT 2011” (Bratislava, February 1-4,
[3] H.P. ALESSO & C.F. SMITH, *Connections. Patterns of Discovery*, John Wiley & Sons, Inc. (2008)
[4] C. LUPU & G. SAMOILĂ, *Interconnection – a Scientific Scenario to Reception/Generate Aesthetical Structures,* APLIMAT - Journal of Applied Mathematics, **4** (2011) – ISSN 1337-6365; in: “Proceedings 10th Int. Conference APLIMAT 2011” (Bratislava, February 1-4, 2011); M. Kovacova Ed.; Slovak University of Technology (Bratislava, 2011), pp. 1019-1028

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# PATTERNS THAT LAST

**RICHTÁRIKOVÁ Daniela, (SK)**

**Abstract.** The paper focuses on patterns found in art, their origin, philosophical meaning, mathematical description, and one of the methods generating fractal outputs.

**Key words.** mathematics and art, patterns in art, neolithic symbols, *Mathematica*, Gustav Klimt.

*Mathematics Subject Classification:* 00A66

## 1 Introduction


Fig. 1 Gustav Klimt

In 2012, 150th anniversary of Gustav Klimt's birthday is celebrated and in artistic world whole the year is dedicated to this exceptional Austrian artist. Vienna, the Klimt's hometown, provides a series of special exhibitions for visitors besides the permanent ones [17.]. Gustav Klimt (1862-1918), fin de siècle genius and pioneer of Viennese Modernism was maturing in Vienna, the 4th largest European city that time, cultural centre of the Austro-Hungarian Empire, concentrating artists and intellectuals of enormous creativity such as Sigmund Freud, Otto Wagner, Gustav Mahler and Arnold Schoenberg. Influenced by new liberalizing ideas as well as obstacles posed by traditional conservatives, he gradually formed his own style as one of the most important representatives of Art Nouveau, founding member and the president of the Vienna Secession. Instead of earlier naturalistic styles, he enriched the traditional with modernity, made use of allegory of symbols and symbolic elements to psychological

ideas emphasizing the "freedom" of art condensed in the main motto standing above the main entrance of the Vienna Secession building: “Der Zeit ihre Kunst, der Kunst ihre Freiheit“ (To every age its art, to art its freedom). The main themes of his work included people's mental and physical conflicts and suffering, life, reproduction, growth and death, i.e. the fundamental issues of the abstract and mysterious essence of people [5.], embodied mostly by the model of a woman, pictured in her beauty and eroticism.
Gustav Klimt was one of the most talented and the most successful artists already during his life and after. Even as a student he worked on decorations of the Kunsthistorisches Museum, the Burgtheater, the ceiling of the spa house in Karlovy Vary, or the first government commission, decoration of Villa Hermes, the Empress Elizabeth's country residence. Eventhough his appointment to be a professor at the academy had never been accepted and his three paintings for university were not understood and had to be for their controversy removed, a lot of his works were awarded by various cultural bodies of that time. In Vienna he won the Gold Medal for his artistic creations, and the Imperial Prize for his work at the old Burgtheather. The rejected painting for university “Philosophy” was awarded the Grand Prize at the World Fair in Paris. He became a winner of the Great Prize in Belgium, for his work at the Esterhazy theater and also got the First Prize at Rome's Universal Exhibition, with his painting Death and Life [7.]. Moreover, a number of his paintings (the most decorations of buildings and walls and portraits) were commissioned and purchased by representatives of high society, government and flourishing bourgeoisie. His works were and are desired by exhibitions and museums. The Kiss, 1907/1908 (Fig. 4), possibly the best known and the most important work of his “golden phase” has been compared to Mona Lisa. Nowadays, the Klimt’s paintings belong to the most expensive artworks in the world. In 2006, the 1907 portrait, Adele Bloch I (Fig. 5). was purchased for the Neue Galerie in New York for a reported US $135 million [16.].

After the general criticism launched by scandal about university paintings, Klimt not allowed to paint the nude so he started to cloth the figures into fanciful gowns covered by floral motifs and rich ornamentation consisting of various arrangements of rectangles, triangles, circles, arcs, undulating curves and the most noticeable spirals, mystical whirlpools and bright assorted shapes, later enriched with gold elements drawing attention to sexuality more than before.

*“I have the gift of neither the spoken nor the written word, especially if I have to say something about myself or my work. Whoever wants to know something about me -as an artist, the only notable thing - ought to look carefully at my pictures and try and see in them what I am and what my intentions are."*

**Gustav Klimt**

The controversy that bore his work, the choice of topics, all in the vein of this statement offer the material for never ending discussions and study.

Fig. 2 Tree of life, 1905/1906

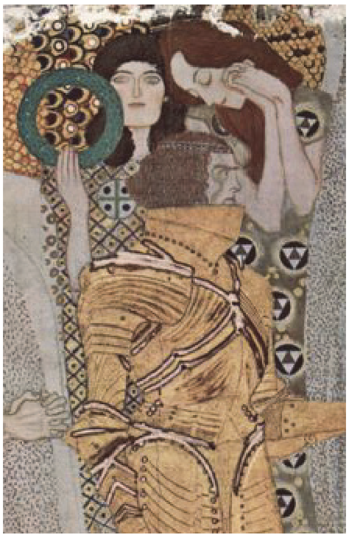

Fig. 3 The Beethoven Frieze (detail), 1902

Fig. 4 The Kiss (detail), 1907/08

Fig. 5 Portrait of Adele Bloch-Bauer I, 1907

Fig. 6 The Bride (detail), 1917-18

## 2 The Origin

Elementary geometric patterns appeared for the first time in art during the period of Neolithic revolution. The era of approximately 7 millennia (10 000 – 3 000 BC) brought cardinal changes in evolution of mankind; it was when material and intellectual nuclei of advanced cultures of Mesopotamia, Egypt, China, Japan, and Ancient America were formed. Adoption of agriculture and settling down to permanent location had a significant influence also on spiritual life. Particular appearances of male divinities symbolizing mostly the power of animal male (a bull) and female divinities as the holder of life and fertility (Venus and great mother figurines) were gradually changing into more abstracted forms. In the most advanced agricultural civilizations, depictions of nature were fully replaced by geometric abstractions: circle, oval and curves. Undulating or zig-zag lines symbolized water; universe was angular with four angles representing four compass points: east, west, north and east; and triangle symbols stood for fertility. Dwelling places, burial grounds, pottery and other objects of daily use – tools, arms, etc. – were the place for expression of art at that time. Ornaments were not only simple decorations; they had their own meaningful value [3.].

The shape development of this epoch is very well documented by the patterns of ceramic vessels excavated on the Greek territory. The first geometric motifs observed on pots were triangles which can be seen already on pottery coming from the Early Neolithic period (6 500-5 800 BC.). Later, during the Middle Neolithic period (5 800-5 300 BC.), the stepped and checkerboard motifs appeared as well as squares and zigzag lines, which resulted in zigzag or parallel linear layout (Fig. 7, Fig. 8) On seals (Fig. 11), the meander-maze motif was the most characteristic for this period. The Late Neolithic (5 300-4 500 BC.) brought radial and spiral patterns, concentric cycles and cruciform patterns with additional stacked chevrons in the quadrants. Among painted pot decorative motifs, the spiral and the checkboard pattern predominated, while incised pottery used decorative motifs both from weaving and basketry [11.]. The spirals appeared here for the first time (Fig. 12, Fig. 13) and could become the everlasting pattern occurring evidently in the most frequent and the most mystic manner.

Later, during Geometric style period (about 1 000 - 700 BC), the pottery art was strongly influenced by the development of a faster potter's wheel and a compass as a drawing tool. Vases of this style (Fig. 14). are characterized by several horizontal bands, between which linear designs of angular patterns, hook maeander systems, rows of dots, zigzags, crosses, svastikas, strokes, stars, triangles, rhombi, etc. are placed. Circles and rosettes are neatly made with the compass [18.]. Human figures also gain abstract geometric shapes. Males are depicted with a triangular torso and long cylindrical thighs and calves. Curves of female bodies are drawn as circle arcs.

Fig. 7 Early and Middle Neolithic pottery with painted decoration [14.]

Fig. 8 Middle Neolithic pottery with painted decoration [14.]

Fig. 9 Late Neolithic pottery with painted and incised decoration [14.]

Fig. 10 Late Neolithic pottery with painted decoration [14.]

Fig. 11 Neolithic seals [14.]

Fig. 12 Triple spiral and rhombuses on entrance stone, Newgrange, Ireland (3 100-2 900 BC.)

Fig. 13 jade weapon ornament, China, British museum 700 -500 BC [12.]

Fig. 14 Geometric Greek Dipylon Vase [18.]

## 3 The explanation

According to Andries Van Onck, Dutch designer living and working in Italy, some simple Neolithic motives can be semiotically explained with respect to symmetry operations: translation (T), rotation (R), and dilatation (D) [8.]. He joined Marija Gimbutas's classification in four categories following the main functional attributes of the Deity (dispenser of Life, the eternal renovating Earth, death and regeneration, energy and development) with kinematics approach.
The primordial point – a dot representing "the mark of a presence; a beginning or an end; a centre, a source of power, the silent counterpart of a single sound (perhaps the beat of a drum)" – moves and creates the simple patterns of dots and lines standing single or multiplied, evidencing repetition, alternation, rhythm, or number. In his theory, the straight line results from a T operation. It represents an object, a presence in the sense of the Peircian qualisign or sinsign. It is the documentation of the very concept of being, having a meaning, revealing a manifold.

A curved line resulting from an R operation represents the division ‘me/other’, me being at the inside of the curve. It indicates that communication between this world and the other world is possible. The signs are associated with the Peircian category of Legisign; i.e. signs that are put in relation to other signs for some purpose. An expanding form as a spiral exemplifies the D operation - dilatation that represents a ‘dominion’, or influence and extension of a power. This is the superior level of signs, as symbols or arguments in the Peircian classification.
Looking at transformations, the translation is considered to represent multiplicity or manifold being. As examples Van Onck gives parallel lines (translation of the straight line motive) to be a symbol of rain or a river meaning fertility. A four legged cross (rotation of the straight line motive) is meant to be a symbol of the ever rising sun meaning regeneration and the V sign (rotation of the straight line motive) or its reinforced edition (obtained by translation as a chevron) as an emblem of the Bird Goddess.
Derived from a triangle (pubic triangle, vulva), symbol of the White Goddess, that generates life, death and rebirth of all beings The Bird Goddess symbol has been in use since the Superior Palaeolithic. Rotation means connection between the earthly world and the Goddess: curved line indicates giving (of offers) or receiving; the serpentine line signifies the serpent or the eternal return, and multiple arcs are the passages to the other world, labyrinths (in combination with the operation of dilatation). Dilatation operation of a central point and concentric circles indicates a dominion and its growth, symbol of all encompassing divine power. Spiral, labyrinth, and maze are symbols of the reign of the triple Goddess, its growth, life, and energy; and the difficulties and dangers of its access. Multiple arcs (dilatations of the simple arc) represent transmutation, trance or delirium. The zigzag line can be considered as a translation of the V sign. It signifies water and in ancient Egypt it was known as a Hieroglyph with the meaning dispenser of life. An undulating line consisting of translation and rotation is a symbol of the serpent, meaning life and death, guardian of the source of life.

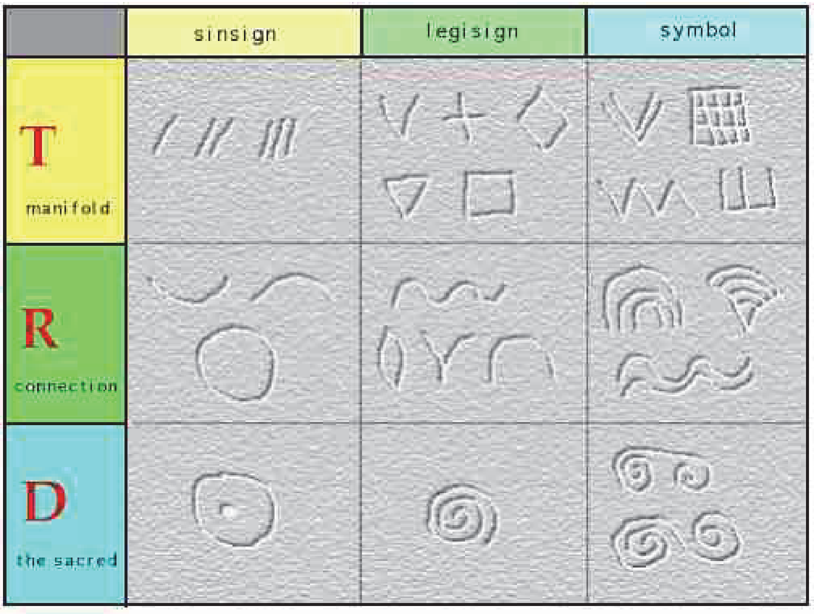


Fig. 15 Classification of Neolithic signs with respect to symmetry operations [8.]

The symbols of Neolithic period persist in further civilizations and some of them can be found also today in the form of e.g. science, engineering or religion signs. Their meaning slightly differed with

respect to topical culture, religion or other field of application. Description of wide range of symbols can be found in Encyclopedia of Western signs and Ideograms [6.].
Looking at patterns, Gustav Klimt used in dressing his figures and objects, the most repeated and the most striking one was a spiral. He used it in various forms: single, in pairs, triple as well as in multiple layouts often as one line drawing.
Spirals have fascinated people since the very early times. A first double spiral with clockwise rotation (Fig. 19), made up by rows of dots, has been found engraved on an amulet of mammoth tooth 24,000 years old. This fact points out that it must have been engraved already by Cro-Magnon mammoth hunters [6.]. Basic element, the spiral (of constant angular velocity) inscribed in a circle (Fig. 18), also lying at the centre of amulet, is being associated with eclipsed phase of the sun. One of the explanations says that the Neolithic rock carvings document configurations of the planets and stars at times of total solar eclipses.

Fig. 16 Ra The Sun God of Egypt

Fig. 17 Roman Mosaics - Infinity Symbol, Regional Museum of History, St. Zagora [13.]

A circle belongs to the most ancient symbols, representing wholeness. Having no end no beginning, it depicts an eternal cycle symbolizing female power, the Goddess, Mother Earth, or human spirit. With dot in the centre, or without, it occurs probably in every cultural sphere, and represents the sun (Fig. 16).
Spiral, or the dot inside, stands for its energy and power. Joining two spirals together makes connection between spiritual and material worlds, and allows mixing their energies. The triple spirals drawn in one continuous line, often found at graving places (Fig. 12), suggest reincarnation, afterlife, and continuous movement of the universe within eternity. The Klimt's Tree of Life (Fig. 2) with branches in the shape of spirals is usually considered to express the idea that everything in life is mutually connected, science, religion, philosophy, and mythology. In Western ideography, the clockwise spiral (Fig. 20) is strongly associated with water, power, independent movement, and migrations of tribes. Like a double spiral it belongs to the oldest engravings. Its mirror image (Fig. 21) is an Egyptian hieroglyph for, among other meanings, the number one hundred and in the earliest Chinese ideography it was used with the probable meaning return or homecoming. Two arcs standing face to face put things or ideas together, it is the universal symbol for unity, and a square is an ancient symbol for land, field, ground, or the element earth. The square within a square means keep, retain, keep inside, or close in. Lines crossing at the edges of rectangles express the union of horizontal female and vertical male principles. In Egypt, the "compass" represented Osiris, the male god of the Egyptians, while the "square" was the symbol representing Isis, the female goddess of Egypt — both sexual symbols. To the contrary, in many

earth-centered cultures, the square represents male qualities, while the circle and spiral symbolize female spirit.


Fig. 18


Fig. 19


Fig. 20


Fig. 21

## 4 Art patterns and science

Historically, the patterns of symbols originated on observations of various events in nature. They depict the balanced phases of event states. Manifesting the energetically balanced component arrangements, the patterns show generally more interesting ways how the equilibrium has been reached. The nature patterns caught in art gradually evolved into abstracted simple forms. Since causes of nature phenomena were not understood, the abstracted forms of patterns got to represent them and started to represent their properties and the power. The elementary geometrical patterns became the main tool of everlasting magic. On the other hand, drawing tools, as liner or compass, allowed lines drawn in hand to be changed into their idealized forms and created the space for people to study them and establish the first forms of research. Revealing scientific background, many of the nature dependences have been managed to be described by science language – mathematics. Nowadays, it is no wonder to see traditional patterns with origins in ancient ages as graphs of mathematical formulas. (Fig. 22 - Fig. 24).

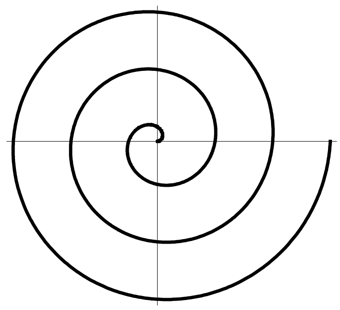
Fig. 22 Archimedean spiral

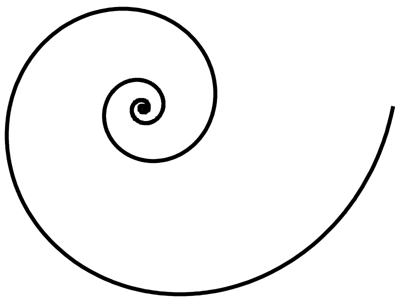
Fig. 23 Logarithmic spiral


Fig. 24 Tripple spiral [15.]

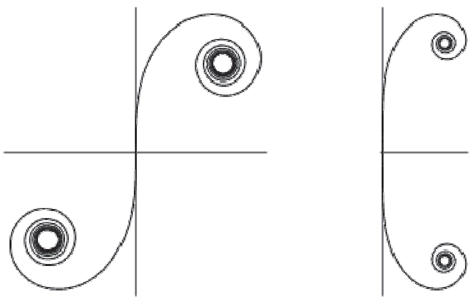
Fig. 25 Cornu spirals

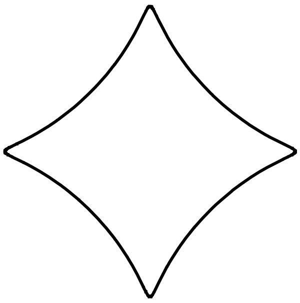
Fig. 26 Astroid

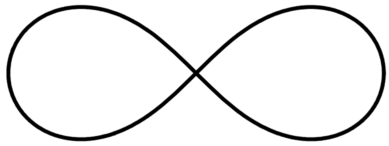
Fig. 27 Lemniscate of Bernoulli

Capturing more than nature's relationships, or depicting the graphical shapes, mathematical formulas represent a powerful tool that could be nowadays widely utilized by computer processing, development of which significantly influenced the progress in science and also influenced art. One of the scientific branches, which was established and bloomed upon entrance of computer technologies, was fractal theory.

Patterns based on the iteration of simple rules, trajectories of chaotic behaviours (Fig. 28) uncover further principles of laws in nature and societies and help to understand them. They demonstrate the behaviour of mathematical codes. It turned out that traditional patterns can be also seen in graphs of functions in complex plane (Fig. 29).

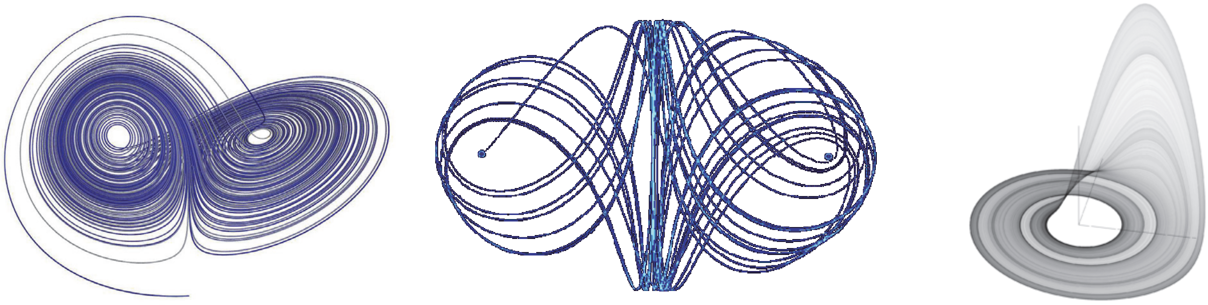

Fig. 28 Trajectories of chaotic behaviours

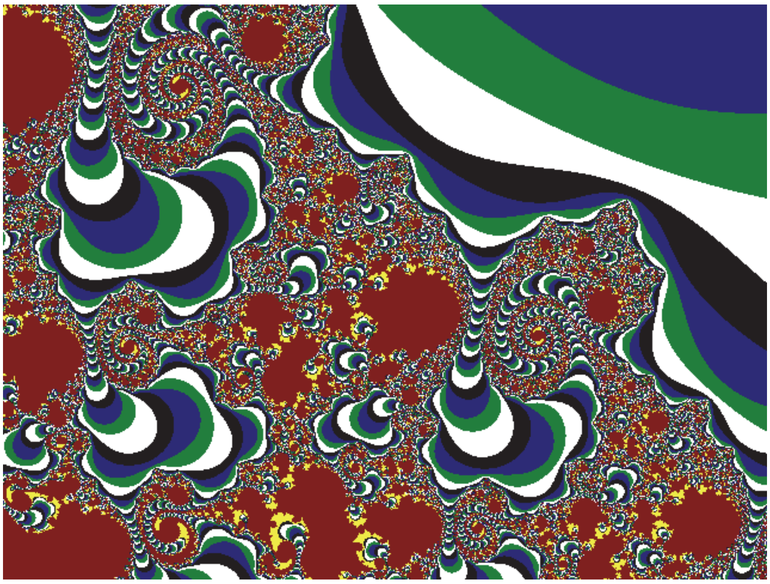

Fig. 29 Spirals in Mandelbrot set, graph of the function $z_{n+1} = z_n^2 + c$ in complex plane

Compositions of self similarities into units could be also observed as early as in Neolithic. Theory elaboration and accessible ability of computer visualization provide possibilities of the pictures generation that have attracted general public as well as interested parties of art.

Fig. 30 Sierpinski triangle construction: the first iteration in Klimt's Beethoven Frieze; the second iteration on Neolithic vessel, and the final attractor

The onset of computer technology and its rapid and boundless expansion into the professional, academic and even home conditions brought an increased interest in mathematics and mathematical formulas even for non expert interested party. Working with mathematical mappings beautiful images were produced that stimulated and attracted the interest.

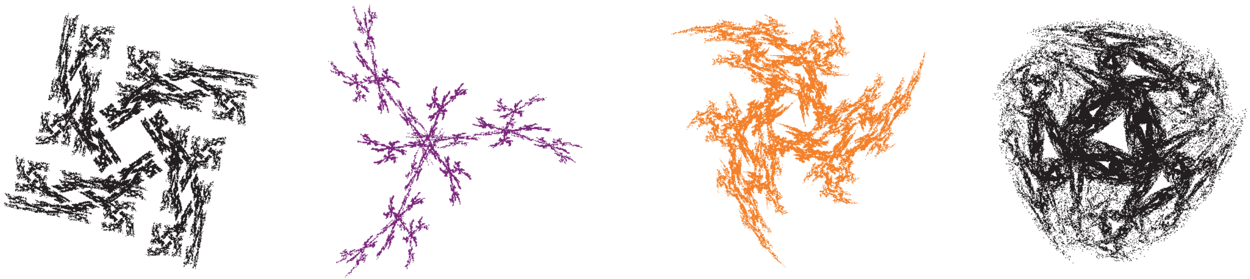

Fig. 31 Fractal Rosettes – IFS composition of basic affinity and two sets of iteration functions: set of rotations, set of homotheties

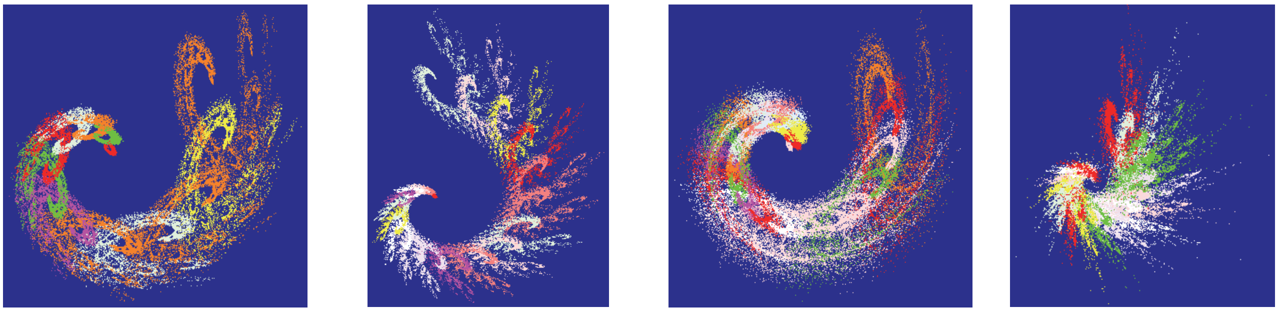

Fig. 32 Fractal Rosettes – IFS composition of basic affinity and two sets of iteration functions: class of rotations, class of homotheties

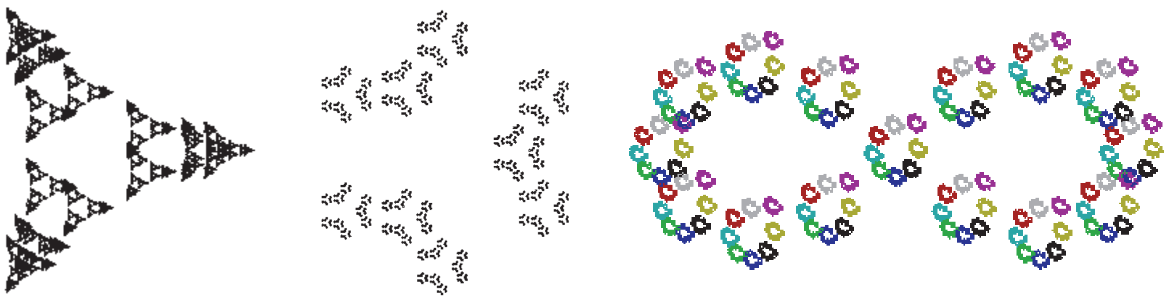

Fig. 33 Fractal Translations – IFS composition of basic affinity and class of translation along a curve: hypocycloids and Lemniscate of Bernoulli

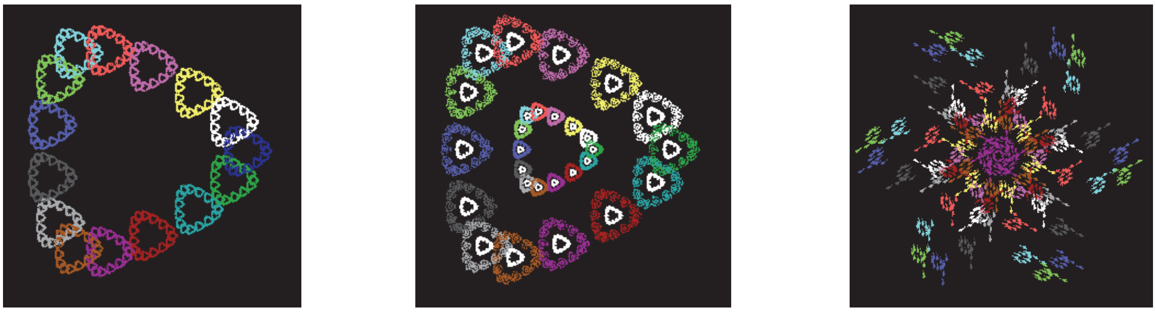

Fig. 34 Fractal Translations – IFS composition of basic affinity and class of translation along a curve: basic; enriched with class of homotheties; enriched with class of rotations

Visualizing formulas, the ideal forms of patterns are created. They are able to record the natural or technical dependencies with machine accuracy. Comparing with outer observation usually they are not seen in such ideal shape by common glance in nature. The reason is prosaic. The ideal forms strictly respect mathematics rules and are rid of side effects. Maybe this clarifies why looking at art people like symmetries but not perfect symmetries – perfect symmetries reflect ideal condition – artificial state, and in many cases the handmade shapes representing ideal forms (which are to be reached) full of small defects (sometimes intentional) are preferred.

## 6 Conclusion and Patterns in Slovak contemporary fine art works

Industrial revolution in the late $18^{th}$ and $19^{th}$ centuries continuing by the period of technical and scientific revolutions started the great changes in the life style of common people. Apparently, as a reaction to the strain on rationality, new art movements arouse, focusing more on spirituality and sensual perception. Many artists concentrated more or less on occultism and mysticism, returning back to symbols of magic – simple geometric shapes. Ideas of Picasso, Braque, Brancusi, Matisse, Mondrian, Kupka, Kandinskij, Klimt, and others; later Vasarely, etc., significantly influenced the look of art. Besides spirituality and purity of soul there were also movements working on rational basement like the Bauhaus, combining technicality with art, demonstrating the unity of all the visual and plastic arts from architecture and painting; and also the movements from opposite side like constructivism, building art mainly on the basis of technical and scientific background. New abstract movements of $20^{th}$ and $21^{st}$ century (kinetic art, op art, virtual art, etc.) usually stand on both poles mixed them together, stressing more one or another side.
In Slovak contemporary fine arts we can recognize both influences. Mathematical and mainly geometrical and contstructivistic approaches are characteristic for a strong group of artists that has been formulating since 60-ies of $20^{th}$ century. Their activity is mapped by the exhibition of works of 41 artists as well by the publication with the same title “Boarders of Geometry” (see more in [1.]). Besides this, the basic everlasting patterns of simple geometric shapes like circle, triangle, rectangles, ovals, simple straight or undulating curves, not forgetting on spirals do not lose nothing from their power to express the artist’s feelings, desire for purity, or the perfection of shape reflecting beauty and universe of laws and rules. They still have their place in artistic pieces of work, education as well as in design art.

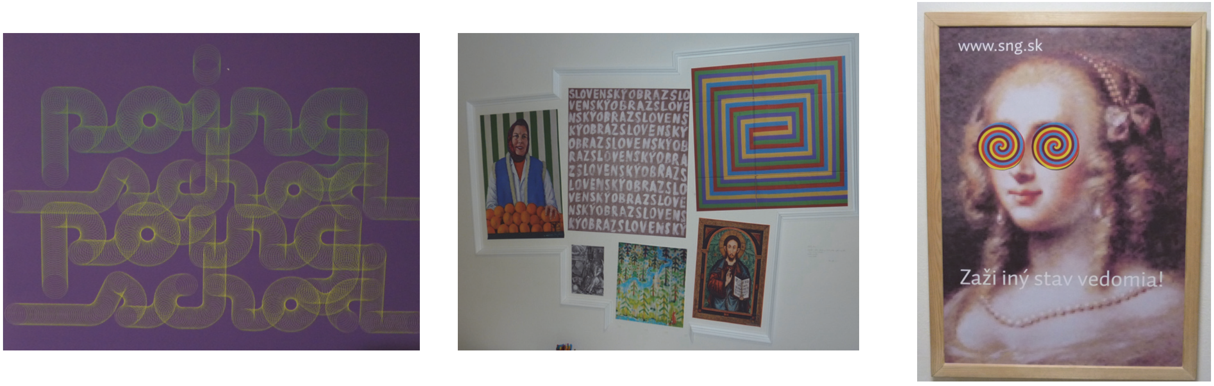


Fig. 35 Works of students, Academy of fine arts and design, Exhibition in Slovak national gallery, 2011

Fig. 36 Mgr. Jana Šulcová, The city, 2007; The study of shell 2007

Fig. 37 The tie, contemporary clothes design

## References


[1.] BELOHRADSKÁ Ľ., Trojanová E., *Hranice geometrie/ Borders of Geometry*, Petum, 2009, 432 pp., ISBN: 978-80-970160-1-2

[2.] BUSCHOR, E. *Greek Vase-Painting*. New York: Hacker Art Books, 1978. In Witmore, Ch., *The Geometric Style* . Available on internet: http://humanitieslab.stanford.edu/113/655

[3.] *Chronik Verlag in der Harenberg,Kommunikation Verlags und Mediengesselschaft* GmbH&Co.KG, First Slovak edition, *Kronika ľudstva*, Fortuna Print, spol.s.r.o., Bratislava, 1992, pp 19-23
[4.] FLIEDL, G., *Gustav Klimt*, , Taschen GmbH, Köln, 2006
[5.] *Klimt's Death and Life: Mystery and Sadness*, Available on internet: Fine Oil Paintings For You, http://www.wholesaleartmall.com/blog/?p=25
[6.] LIUNGMAN, C.G., *Online Encyclopedia of Western signs and Ideograms*. Available on internet: www.symbols.com, HME Publishing, Stockholm, Sweden.
[7.] MOFFAT, Ch., *Gustav Klimt*, The Art History Archive. Available on internet: http://www.arthistoryarchive.com/arthistory/symbolism/Gustav-Klimt.html
[8.] ONCK, A., *Analysis of Neolithic signs*. Available on internet: http://www.andriesvanonck.com/ neolithic.html#note%201
[9.] RICHTÁRIKOVÁ, D., *IFS Fractals by Translation along a Curve*. In Proceedings of 7th International Scientific Conference Mechanical Engineering 2003, Slovak University of Technology in Bratislava, CD-Rom, s8_04a, ISBN 80-227-1960-9
[10.] RICHTÁRIKOVÁ, D., *webMathematica Visualisation of Fractal Rosettes within Scienar*,. In Proc. of XXVIII International Colloquium on the management of Educational Process, Brno, 2010, , ISBN 978-80-7231-722-6
[11.] *Stone Age*, web pages of Foundation of Helenic World. Available on internet http://www.fhw.gr/chronos/01/en/index.html
[12.] *The British museum*. Available on internet: http://www.britishmuseum.org/explore/highlights/highlight_objects/asia/j/jade_ornament_for _a_weapon_hil.aspx
[13.] The travelin frog. Available on: http://www.flickr.com/photos/rossitza/with/2911391629/
[14.] THEOCHARIS, D.R., *Neolithiki Hellas,* National Bank of Greece, Athens 1973, pl. V.- XX. In: web pages of Foundation of Helenic World. http://www.fhw.gr/chronos/01/en/ index.html
[15.] *Tripple Spiral*. Available on internet: http://en.wikipedia.org/wiki/File:Triple-Spiral-Symbol.svg
[16.] VOGEL, C., *Lauder Pays $135 Million, a Record, for a Klimt Portrait Portrait of Adele Bloch-Bauer I*, NY Times report from June 19, 2006, Available on internet: http://www.nytimes.com /2006/06/19/arts/design/19klim.html?ex=1308369600&en=37eb32381038a749&ei=5088&partner=rssnyt&emc=rss. In: *Portrait of Adele Bloch-Bauer I*, wikipedia, Available on internet: http://en.wikipedia.org/wiki/Portrait_of_Adele_Bloch-Bauer_I#cite_note-0
[17.] *Welcome to Vienna*. Available on: http://www.wien.info/en/sightseeing/museums-exhibitions/ klimt2012/klimt-sonderausstellungen
[18.] WITMORE, Ch., *The Geometric Style* . Available on internet: http://humanitieslab.stanford.edu/113/655

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# THE FIGURATIVE ARTS STRIVING FOR MUSIC AND MATHEMATICS

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**Abstract**. The author starts with Hartmann's point of distinction concerning the fact that any work of art has a front plan (the modeled matter) and a back plan (the spiritual content). He notices that the front plan of the figurative arts (architecture, sculpture, painting, poetry) allows a representative shape that allude to ideas and finite feelings while the front plan of music (the sounds) is unrepresentative and alludes to general ideas and metaphysics.

The author assimilates the distinction between the limited figurate and the unlimited musical non figurate to the distinction Nietzsche made between Apollonian arts (fine arts) and Dionysian ones (music and dance). He also accepts Nietzsche's idea according to which the Dionysian style may be vaguely suggested by fine arts, as well.

Consequently, the author argues that, as long as the fine arts remain figurative, the way poetry and fine arts strive after music or mathematics as well, is possible through the combination of the figurative images according to the main elements of speech music – melody, rhythm and harmony, so as to suggest somehow the overtaking of the limited figurate towards the unlimited non figurate of music. The author underlines the fact that the figurative arts' striving for music, concerns also mathematics because the sounds of music can be expressed through mathematical shapes and quantitative rapports.

The author promotes and illustrates the idea that the figurative arts can strive for the highest level towards music only if they tend to pass radically from Apollonian to Dionysian and from figurative to nonfigurative.

**Key Words:** front plan, back plan, real, unreal/imaginary, Apollonian, Dionysian, figurative, non figurative, sublime

## 1. The encounter of the figurative arts with music and mathematics in the front plan, of the modeled material, through melody, rhythm, harmony

Beyond the differences, the main projects about the creation and the work of art signed by Hegel, Hartmann, Heidegger, Lukacs, or by Romanian estheticians George Călinescu and Tudor Vianu converge through the understanding of the work as a transfiguration of a sensitive matter through a spiritual content or, in other words, as an expression of a spiritual content through a sensitive matter that is, thus, transfigured. In this respect, it is very important the way Nicolai

Hartmann characterizes the work of art as compared with three or four terms: the modeled matter (*front plane*), its spiritual content (back plane/background), the living spirit of the author which correlates the first two terms as well as the receiver's spirit. The distinction between the front plan (Vordergrund) and the back plan (Hintergrund) is identified by Hartman as the distinction between *real* and *unreal*.

As for the modeled matter and the sensitive matter, the figurative arts use a representative shape, which being limited (in architecture, painting, sculpture, by drawing line, shape, color, volume, and in poetry through images suggested by words) suggests as a rule feelings that reveal the finite and the bounded. In return, the music, having an inner structure and a symbolic language of sounds but lacking a figurative shape is capable to suggest the transgression of the finite, towards a general and total perspective.

The idea of focusing the artistic figurative on the individual and the musical nonfigurative on the general, refers also the distinction Nietzsche made in his work *The Birth of Tragedy* between the *Apollonian style* redeemed expressively by *fine arts* and centered cognitively on individual and affectionately on measured feelings and the *Dionysian style* expressed most properly through *music* and *dance*, which by rolling sounds and, respectively, rolling movements redeem cognitively the integration of the individual in the general and affectionately offer the most powerful and tremendous feelings.

It was also Nietzsche who stated that the Apollonian can be redeemed by a certain music genre, which is a calm dripped music, while the Dionysian can be suggested by some works of art in which the figurative shapes contort, as in the Laocoön statue Group, in order to express movement, as an overcoming of the static finitude.

Indeed, according to Nietzsche's interpretation over a series of figurative works of art, such as *the* Laocoön Group, the paintings *Saint Cecilia*, *Madonna Sistine*, *Apotheosis* made by Rafael, or the engraving *Knight, Death and the Devil* of Dürer, these may suggest the Dionysian spirit insofar as they can combine images so as to undermine their original desire for customization and express, on a contrary, the individual's overcoming towards united and universal concepts.

Thus, a modality through which figurative arts strive for music and also for mathematics refers to the figurate images combination in accordance with the main elements of the musical language - the melody, rhythm and harmony – in order to suggest in a way how they overcome the limited figurate towards the unlimited non figurate of music. Indeed, the architectural elements, images and paint colors, the volume and sculpture shapes, the sounds of poetic words can be displayed in a melodious, rhythmic and harmonious way. Approaching musically a nonmusical material is, implicitly, a propensity for mathematics and its power of transcending the qualitative aspects through the quantitative ones because the sounds of music can be redeemed through mathematical shapes and rapports. Let's think of the harmony of an ancient temple which follows the so-called golden measure.

However, it must be specified the fact that the striving of figurative arts for music through melody, rhythm, harmony cannot be fully satisfied because the nonmusical works are relatively autonomous and irreducible, and no matter how the expressive is, the limited figurate cannot replace the unlimited nonfigurative of music.

Mozart sustained that, in a musical work, the poetry should be "the obedient daughter" of music. But it is difficult to find a strict correspondence between the musical motif and the poetic theme. Poetry cannot be limited to melodiousness and rhythm without spoiling its content.

There are Romanian poets who adapted to a great extend the musicality of the poems to their content (Eminescu*, Bacovia**, Arghezi***), whereas to other poets the melody breaks or modifies in favor of expressing the message (Minulescu****). The attempt of changing the poetry into music, in a pure sonority, leads to the diminishing of its ideatic meanings.

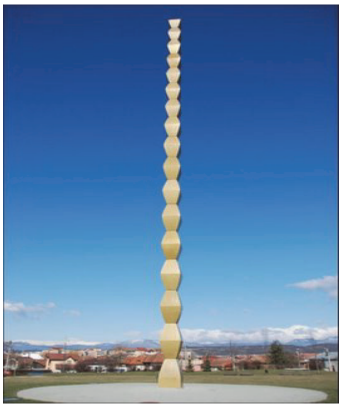
**2. The encounter of the figurative arts with music and mathematics in the spiritual back plan, by passing from figurative to nonfigurative and from Apollonian to Dionysian**

In what follows, we consider the nonfigurative arts longing to symphonic music, through Dionysian excellence, through radical shift from figurative to nonfigurative and from the Apollonian to the Dionysian.

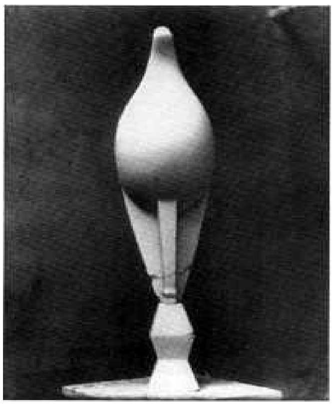
Because the symphonic music expresses, through melody torrents, a total metaphysical perspective, we sustain and argue that nonfigurative arts are captivated in the highest degree of music, of its ability to suggest everything, the whole, The Being, in so far tend in turn, to the deepest sense, to ideas and feelings that go beyond the finite. In this endeavor, they tend to move dramatically from Apollonian to Dionysian and from figurative to nonfigurative.

In architecture, the buildings cannot loose totally the figurative structure. But, through monumentality, through the rhythm of the elements and their soaring character, they may suggest the tendency for seamless. Gaudi's *Sagrada Familia* Cathedral can be compared to a symphony in the same way a symphony can have something from the monumentality of a work of architecture.

In sculpture, talking about Constantin Brâncuşi, the shape is no more such representative as in figurative art, which has only an allusive connection with, but it has its proper value and distinguishes itself through essentiality, rather by elimination or decrease than through increase as in his famous *Sleeping Muse*, *The Kiss*, *Mademoiselle Pogany*, *The Magic Bird*, *The Infinite Column* works.



The Column of the Infinite



The Magic Bird

The Magic Bird

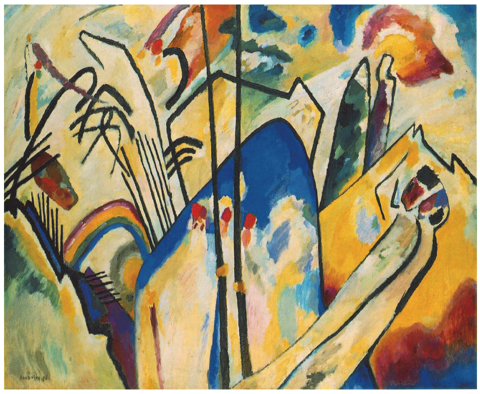
In painting, the temptation of getting distanced from the limitative character of the exterior shapes happened in different manners, among which, the Cubism, Abstractionism, Expressionism may by mentioned.

Having antecedents in Cézanne's painting *Mountain Sainte-Victoire*, where the nature is reduced to essential units, the Cubism started its extenssion after 1907 due to the paintings of Braque and Picasso. After having passed through a blue and a purple period, Picasso moves to Cubism by painting *Les Demoiselles d'Avignon*, where five naked women are presented under many angles, in a radical geometrical style. By geometrizing hidden aspects or displaying from the back the unknown aspects of the objects, his Cubism, on the whole, expresses the artistic need to capture everything and to make the unrepresentative to be representative or the unseen to become visible.

In another nonfigurative manner, the abstractionism deepened the passing from the qualitative shapes to the inner geometric and symphonic structures without abandoning the colors.

In his ten works entitled *Composition*, Wassili Kandinsky, one of the founders of the abstract painting, reduced the real objects to lines and geometrical elements with different shapes and colors. In his book, *Concerning the Spiritual in Art* (1912), he explicitly put together the color rapports and melodic sonority. It was also Kandinsky the one who, focusing on passing painting from the qualitative aspects to the geometrical quantitative ones, talked about *the shape*'*s sound* or *the triangle*'*s spiritual perfume*. [Wassily Kandinski, Spiritualul în artă. Meridiane, 1994. Translation and Preface by Amelia Pavel.]

Les Demoiselles d`Avignon



Composition IV, 1911

The painting also evolved towards nonfigurative through emphasizing the accent on a drawing instead on color, until the coloristic painting appeared, lacking any representative substrate, but generating inner soul images through the symphony of colors.

The striving of arts for music and the shift from the figurative towards nonfigurative meant the transition from *the sensitive phenomenal* to *spiritual essential* and further from *the limited Apollonian* to *unlimited Dionysian* or in kantian terms from beauty to sublime. According to Kant, *the beauty* is a feeling caused by intelectual ideas, a finite shape and the quality of representation, whereas *the sublime* is caused by reason, lack of shape and the quantity of representation. And "the sublime does not exist in any object from nature but only in our soul ..." [I. Kant, Critica facultăţii de judecare, Ştiinţifică şi Enciclopedică, Bucureşti, 1981, p. 158.]
Striving of arts for music and sublime is more obvious in poetry. One of the most fascinated by music poets was Paul Verlaine: „*De la musique avant toute chose,/ Et pour cela préfère l'Impair,/ Plus vague et plus soluble dans l'air,/ Sans rien en lui qui pèse ou qui pose.../De la musique encore et toujours!/...Vers d'autres cieux à d'autres amours.*" [Paul Verlaine, "Art poétique," in Georges Pompidou, Anthologie de la Poésie française, Librairie Hachette, 1961, p. 428-429.]

But again: as far as the arts, including also the poetry, have musical sublime nostalgia, in the same way the integrations of melody, rhythm and musical harmony in the means of expression are not adequate for specific Apollonian figurative finite images, but they serve to essentialize the image and to set it free in order to redeem "other heavens for other loves."

* Mihai Eminescu: 1850-1889, Romanian poet
** George Bacovia: 1881-1957, Romanian poet
*** Tudor Arghezi: 1880-1967, Romanian poet
**** Ion Minulescu: 1881-1944, Romanian poet

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# WEAVE AN AUGMENTED REALITY. ALGORITHMS AND FIBER ART

RUSU Alexandra-Andreea, (RO)

**Abstract.** The advanced AR (augmented reality) and VR (virtual reality) technologies have enhanced our current perception of reality testing senses and skills. This augmentation of environments has launched an interactive world that is changing to meet new technological requirements. Fibers and fabrics are now part of smart textiles technology. Defining wearable AR (cameras, sensors, displays) is no longer a finished process but rather one in constant development. Using organic electronics materials and fiber transistors has become compatible with textile manufacturing and opened a path for innovation.
The next step in layering virtual reality on real-time events is a fiber language using algorithms based software as a tool for AR technology. Evolution algorithm software has been used in textile design for some time but the language has been a method for "silent" structures. E-textiles should be more than a structure that enables digital components to be embedded in them. New fibers and new fiber properties could allow fabrics and fiber art to become more interactive, not just a medium but a language and a distinct technology.

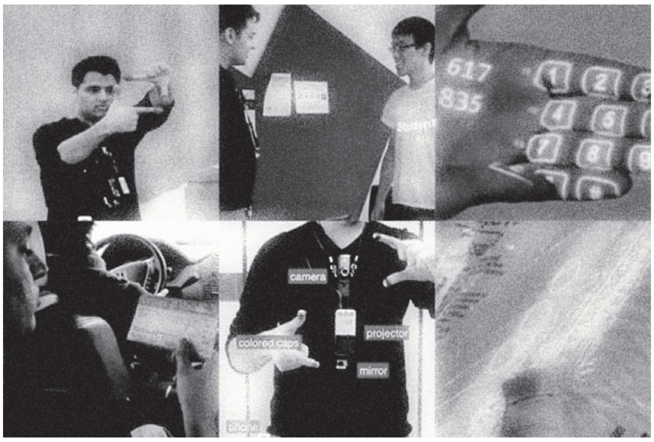
**Key words:** Augmented reality, QR Code, smart textiles, fiber art.

## 1 Augmented reality. An introduction

Augmented reality combines real-world environment and computer generated image or information so that users can feel more intuitive as they explore a continuum of reality-virtuality that spans from the real environment to the virtual environment.

Augmented reality technology augments the sense of reality by superimposing virtual objects and cues upon the real world in real time. The immersion is not limited to the sense of sight. AR technology can also augment or substitute user's missing senses by sensory substitution.
One of the most important aspects of augmented reality is to create appropriate technologies for intuitive interaction between the user and virtual content of AR applications. There are different ways of interaction in AR application: tangible AR interface, collaborative AR interface, hybrid AR interface and the emerging multimodal interface. Augmented reality applications were developed using projective (epipolar) geometry, rotation representation with exponential maps, nonlinear optimization and robust statistics. Software use programming constructs like variables, assignment statements and control loop functions.

Some of the most advanced AR applications developed in the recent years are tangible apps( VOMAR, TaPuMa, gloves, wristbands) and multimodal interfaces that combine real object input with naturally occurring forms of language and behaviors such as speech, touch, natural hand gestures or gaze (WUW gestural interface).




*TaPuMa or Sixth Sense technology*
http://asiknews.wordpress.com/2010/09/30/sixth-sense-the-latest-trend-in-technology/

The last decade showed important developments in camera systems and tracking systems that can analyze physical environments in the real time and relate positions between objects and environments.

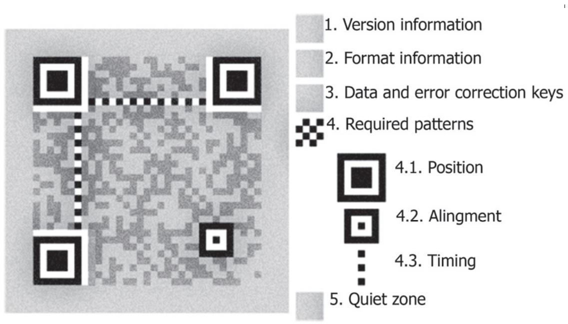
Tracking methods in AR depends mostly on the type of environment the AR device will be introduced to as well as the type of AR system (indoor/outdoor, mobile/static). Although visual tracking now has the ability to recognize and track a lot of things it mostly relies on other techniques such as GPS and accelerometers. A recent new approach for advances in visual tracking has been to study how the human brain recognizes objects, also called the Human Vision System (HVS) as it is possible for humans to recognize an infinite number of objects and persons in fractions of seconds.

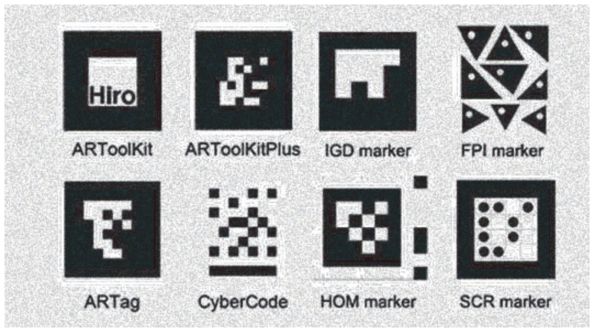
The difficulties faced by computer vision with most objects are related to their irregular shape, to lightning conditions and damaged codes.

## 2 QR Code technology

To display contents in a real world environment seamlessly a marker is needed to identify and position the designated graphics. Linking from physical world is termed hard linking or objects hyper linking.

QR Code (Quick Response Code) is a type of 2D barcode that has been widely used in many fields in the world. The amount of data (binary, alphanumeric, kanji symbols) that can be stored in QR Code depends on the character set, version and error correction level (up to 30%). There are over 40.000 versions of codes and each version has a different module configuration and number of modules.



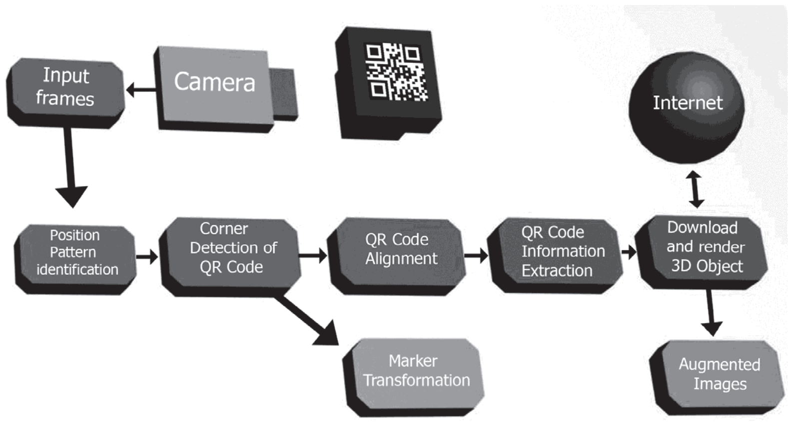

*QR Code structure* *Types of marker systems*

http://en.wikipedia.org/wiki/File:QR_Code_Structure_Example_2.svg, Furth, Borko(ed.), *Handbook of Augmented Reality*, Springer, New York, 2011

Unlike other barcodes which have to be read with physical scanners QR Code can be easily detected by a mobile device, provided the gadget is installed with photo-taking function and decoding software. Thus, QR Code fits perfectly well with mobile devices.

An image frame is captured through a video device, the captured image is then binarized and a marker pattern is found and matched with the registered markers in the database. Once registered and identified the system will locate the 3D coordinates of the marker pattern and place the corresponding virtual object on it. In the most commonly used marker systems (AKT, HOM, GD and SCR) new markers and its corresponding augmented content have to be registered before so that the system can identify the markers and display the augmented content.

In a QR Code based system, as soon as QR Code is scanned the system can decode the information and download the augmented content from the internet. Users can directly download information without having to go through a third channel. Service providers can also modify AR content in QR Code. In the process of tracking and identification the application provides detailed information on each marker such as the coordinates of the four corners of a certain square marker, the line equation of the four lines of it, and its 3D coordinates. Through the geometry of the three positioning patterns, plus information, it is possible to identify the correct positioning patterns in QR Code. Once the system successfully finds out the three positioning patterns, the coordinates of the fourth corner of the QR Code will be identified.




*From marker recognition to layering augmented data*

Current applications depend on fixed-threshold image-binarization in order to detect candidate fiducials for further processing. In an effort to minimize tracking failure due to uniform shadows and reflection on a marker surface there have been developed fast algorithm systems for selecting adaptive threshold values, based on the arithmetic mean of pixel intensities over a region-of-interest around candidate fiducials.

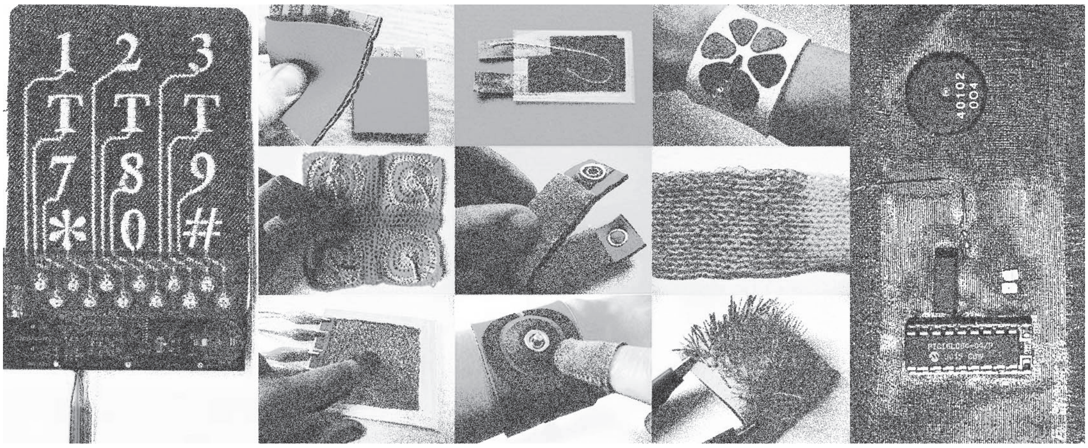
## 3 E-textiles and wearable technology

There is not an area of our world unaffected by the advances in technical textiles. Principles of textile science and technology merge with other research fields such as engineering, chemistry, biotechnology, material and information science.

Textiles are the natural choice for seamlessly integrating computing and telecommunication technologies to create a more personal and intimate environment. Although clothing has historically been passive, garments of the XXI century will become more active participants in our lives, automatically responding to our surroundings or quickly reacting to information that the body is transmitting. Clothing will automatically react and adapt to the surrounding environment.
Since clothing is pervasive and presents a „universal interface“ it has the potential to meet the emerging needs of today’s dynamic individual, interactivity, connectivity, ease of use and „natural“ interface for information processing. It can be the intelligent information infrastructure for the demanding end-user. Smart textiles can be passive (sensors) or active (sensing the stimuli from the environment and also reacting to them). New generation smart textiles have the ability to adapt their behavior or characteristics to the circumstances: adapt their insulation function according to temperature changes, detect vital signals, change color or emit light upon defined stimuli, generate or accumulate electric energy to power medical and other electronic devices.

The development of such novel textile materials is currently the subject of intensive research effort directed toward producing electrically conducting fibers, shape-memory fiber polymers, chemically responsive fibers, photosensitive fibers, color change materials and mechanical responsive textile materials.

Smart textiles technologies enable computing , digital components and electronics to be embed (wires, conducting textile fibers, transistors, diodes and solar cells, buttons, LED mounted on woven conducing fiber networks, organic fiber transistors).




*Various developments in e-textile technology: keypad, sensors, breadboard*
http://web.media.mit.edu/~rehmi/fabric/, http://www.instructables.com/id/Fabric-bend-sensor-1/, http://web.media.mit.edu/~rehmi/fabric/

E-Textiles offer inherent electrical connections for power and data in the fabric, but the issue of how to integrate conductive fibers must be addressed. Embroidery and weaving are the two main developments. Rehmi Post (Visiting Scientist at the MIT Center for Bits and Atoms) has developed the e-broidery technology that mixes electronic circuitry and wash-and-wear textile substrates.
Not as flexible as the embroidery systems Bekintex researchers used stainless steel wire or tinsel wire to replace thread in the loom.

The perfect situation for e-textiles would be to have all the components in fiber form: fiber microphone, batteries, etc.

The research field extends from pressure sensing fabrics to conductive fibers and follows the process from input to output and power devices. Elekson has developed a soft sensing fabric capable of interfacing to a multitude of devices. The fabric can be designed to interact with different components to form a system with a soft interface. Other examples of sensing fabrics can be found in recent developments: Mapper Garment ( a vest that maps the location of the user in relation to the room), Acoustic Beamformer Technology( designed to find the location and direction of motion and to report information), Shape Sensing( an e-textile that can sense its own form using accelerometers and piezo-electric sensors).

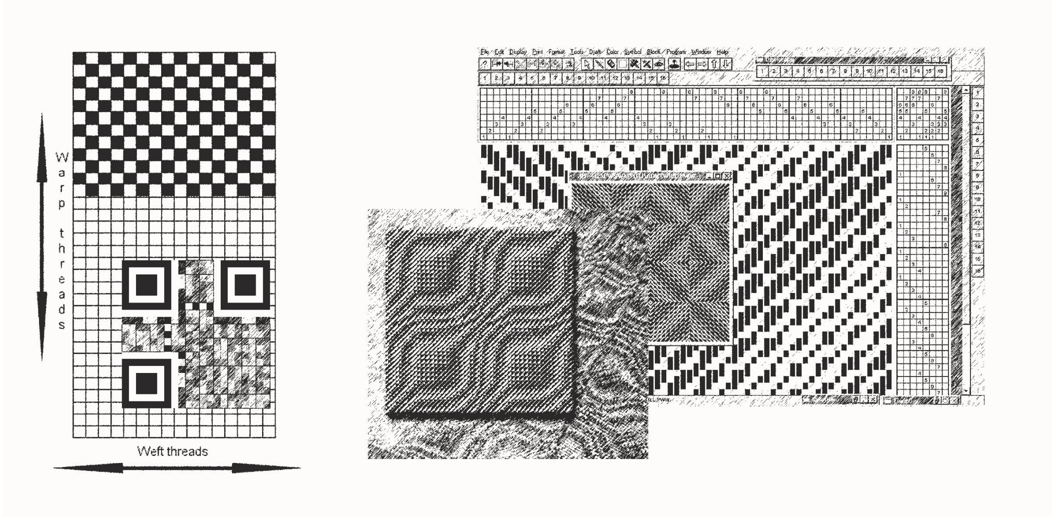
Some research projects have been successful with the case of conductive fibers. Metallic organza has been used in both the row and the column (weft and wrap) fabric.
Lightglove, ElekTex keyboard, Gesture Pad are only few of the input devices build to sense the movement of the fingers. Lite*Star is developing extremely thin batteries to power future wearable motherboard and integrated circuits made by thin film technique.

## 4 Fabric is the computer. Fiber art and technological innovation

Wearable technology and the telecommunication market will merge. Range of our wearable are compatible via Bluetooth mobile phones but AR systems that wish to step up from the labs to the industry will also be facing fashion issues as the users will not want to wear HHD or other visible devices. Fiber integration issues however are very challenging. Patterning in particular is a significant concern.

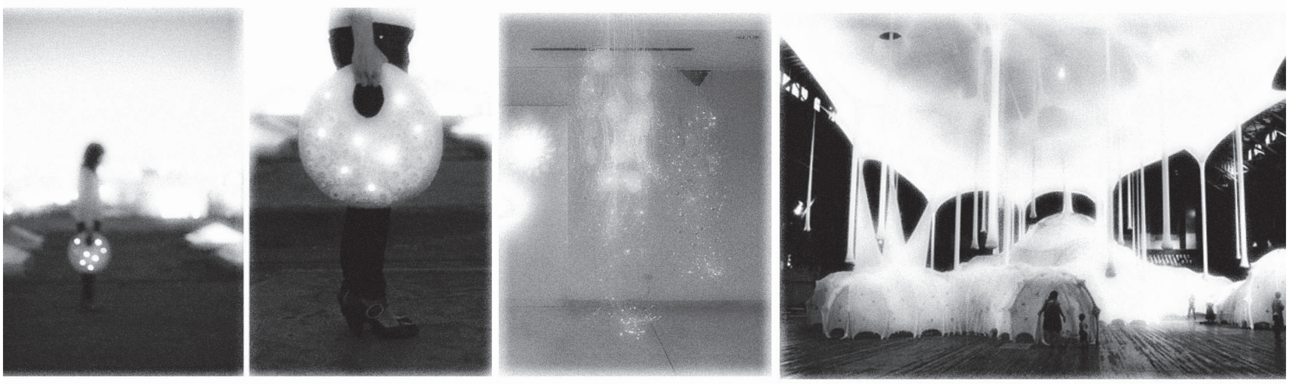
Fashion design, ambient and textile design are taking the experiments from the labs into the real world. From the first voice-controlled MP3 player, that can be sewn directly into shirts or jackets announced by Infineon in 2002, to actual displays from LED mounted on woven conducting fiber research attempts to transform the fabric into information surface.
Smart textiles have been used by designers for their interactivity and for synesthesia effects that build a continuum reality-virtuality.

The film industry develops new ways to immerse the viewers in the virtual world. 3 D effects are augmented using new cinema seat properties, engaging all senses. The convergence of smart textiles innovations will forever change the way in which we interact with the real and virtual world. The purpose is to gain maximum interactivity from common clothes/fabrics bought only by esthetical criteria. Handle and flexibility alteration is an impediment. The possibilities of embedding technology in fabrics at fiber scale decrease when using complex patterning for esthetic purpose.
Evolutionary algorithm aided textile design software is used to create a wide range of fabric patterns. Designers use programs to develop ideas, sketches and projects for textile production. But until now software has been used to create “silent fabrics”.




*Using textile design software to create "invisible" markers for AR technology*

Conductive fibers and augmented reality technologies revolution is in need for a new language, a language of interactive fibers. Weaved markers could support AR technology. Weaved codes using negative/positive type of weaving or combined patterns (plain weave/ twill weave) could be an alternative to common codes. Refining identification and tracking systems opens new possibilities for detecting weaved codes with layers of dyes or supplementary weaving systems. Clothes can communicate not only product information but could actually become a visit card, connecting to personal sites and profiles. The "custom made" definition would be changed radically. Advertising panels could be only white coated fabrics. The virtual content of codes could become the 3 D equivalent of a 2D work of art, or the kinetic instance of a static sculpture.



*Sessile bag-Grace Kim/Antonella Cimatti- optical fiber and porcelain/ Ernesto Neto installation:*
http://www.pleatfarm.com/2010/04/28/sparkly-felt-bag-by-grace-kim/,
http://antonellacimatti.blogspot.com/2010/06/21-maggio-29-agosto-ceramica-storia-di.html,
http://www.qartlog.com/?p=2653

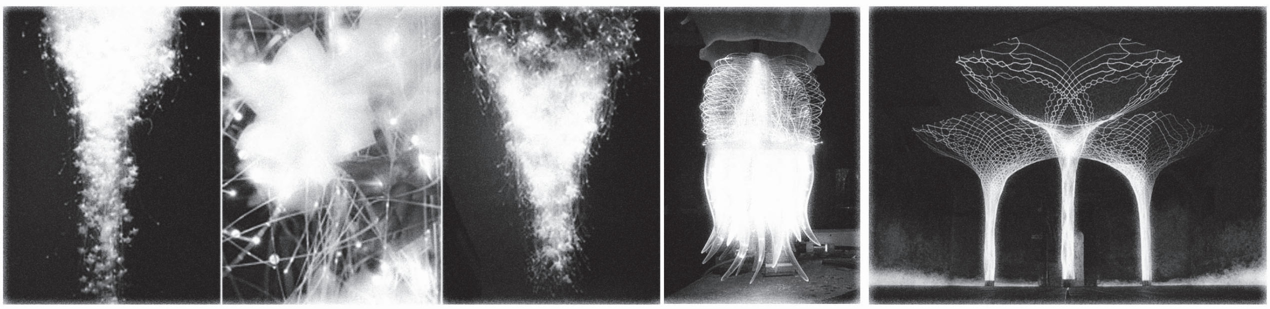
Art has always accompanied innovation and designers have foreseen the future in their ideas and prototypes. New fibers and new fiber technologies are already being used by fashion designers and fiber artist. Clothes, accessories, ambient devices and works of art are gaining new properties. They are no longer just one-purpose objects but aim to connect all our senses to the environment. It's all about feature fusion: function/esthetics, real/virtual content, sense fusion. Examples show as a developing smart textiles art.

The Kinetic Dress is part of the Transfor-Me collection developed for the NEMO Science Museum in Amsterdam. The Kinetic Dress changes color and pattern depending on the activity of the wearer. Other systems use Bluetooth technology to light up fabrics when calls are received.

Kinetic Dress Kate's Dress Galaxy Dress

http://www.oobject.com/best-interactive-clothes/kinetic-dress/657/,
http://popwatch.ew.com/2010/05/04/katy-perrys-met-gala-dress/
http://www.wired.com/gadgetlab/2009/11/led-dress/

Fiber artists have always used a multitude of materials and media to support their statement but now they are testing the limits of textile technology and fibers properties. Innovation, unexpected combinations, unusual materials and color are the hallmarks of XXI fiber art. Fabrics are now designed to respond to the interplay and activity of the people passing by or to cast light, play music. The atmosphere of musical rhythms, harmonies and luminous patterns can be composed by the visitors' movement - either active or passive. SonUmbra is one project that benefits from new technological developments being both a work of art and a solar-powered tree composed of strands of light-emitting fabric woven into a lucent web of branches. The installation's canopy of photovoltaic panels captures light during the day, and once the sun sets the tree blooms in an interactive flourish of light and sound. The light emitting fabric of the umbrella is crafted into a lacework of many electroluminescent fibres. This latticed pattern is animated in concert with the generated surround sound and visually illustrates the visitors' position within the constellation. Fiber art now attracts the wandering and passive viewers and engages them in a sense symphony. Everyone becomes a unique note in the composition of light, sound and space(either real or virtual).



Sharon Marston Dadona Studio loop-pH- Sonumbra

http://www.busyboo.com/2007/11/13/unique-lamp-shades-marston/,
http://www.daddonastudios.com/galleries/moldmaking_casting/pages/02.htm
http://www.pargy.co.uk/2011/08/

Developments to be foreseen in smart textiles follow two parallel paths: increasing fiber/fabric properties by integrating organic conductors, sensors, devices at small scale and creating “invisible” markers to redefine augmented reality. Either way fibers won’t be the silent witnesses we’ve known so far and will transform our environment more then ever.

**Acknowledgement**

The author wants to thank Professor Mauro Francaviglia for the advices during the elaboration of this paper. The paper was supported by Grant PN II IDEI “Maps of Time”.

**References**

[1] FURTH, Borko(ed.), *Handbook of Augmented Reality*, Springer, New York, 2011.
[2] GRUNDLER, DARKO, ROLICH, TOMISLAV, *Evolutionary algorithms aided textile design*, University of Zagreb,2003.
[3] JAYARAMAN, Sundaresan, KIEKENS, Paul(ed.), GRANCARIC, Ana Marija(ed.), *Intelligent Textiles for Personal Protection and Safety*, IOS Press, Amsterdam, 2006.
[4] Kan, TAI-WEI,Tong, CHIN-HUNG, CHEN, Y., Mike, *QR Code Based AR Applications, Springer*, New York, 2011.
[5] LEE, B., Josephine, SUBRAMANIAN, Vivek, *Organic Transistors on Fiber: A first step towards electronic textiles*, University of Berkley, 2003.
[6] LINGLEY, A.R., Ali, M., Liao1, Y., Mirjalili1, R., Klonner, M., Sopanen, M.,
[7] SUIHKONEN, S., SHEN, T., OTIS, B.P., LIPSANEN,H., PARVIZ, B. A., A *single-pixel wireless contact lens display*, IOP Publishing, Washington, 2011.
[8] NAKAD, S., Zahi, *Architectures for e-Textiles*, Virginia Polytechnic Institute, Blacksburg, 2003.
[9] PINTARIC, Thomas, *An Adaptative Thresholding Algorithm for the Augmented Reality Toolkit*, Vienna University of Technology, 2002.

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# GESTURE, PERFORMANCE AND GEOMETRY

SERBĂNESCU Adrian, (RO)

**Abstract:** The Action Painting style has brought freedom to the geometric structure of traditional painting composition. Although the compositions may seem to be chaotic, they are formed on geometric structures that generate rhythms and are meant to bring order into the apparent chaos. The current paper enforces this hypothesis by analyzing an action painting art work.

**Key words:** art, geometry, action painting, gesture

In the year 1963 Charle Bouleau published one of the most important books in the fields of art and modern science „The Secret Geometry of Painters". Unfortunately this book is nowadays generally forgotten. The book underlines the importance of the scientific and geometric structures of creating art, from the beginnings until the 20th Century.

Together with the beginning of the non-figurative style in the second half of the 20th Century, this materializing perspective of art has lost its appeal and has disapeared from the artistic research.

To demonstrate the importance of geometry in Modern painting I analyze the abstract expressionism and the gestualism of the 4th and 5th decades of the 20th Century, [1], [2], [3], [4]. In this case the geometrical character of the artistic gesture is seen as an ensemble of rhythms: "The gesture of human body, its capacity for lateral expans, dictates the character of the line. […] the marks in Pollock's paintings are intimately tied to the proportion of the a human body – his own".

With the rhythms of his own body Pollock suggests the rhythms of nature, as if he was attempting the break down the distinctions that segregate the self from the environment, the internal from the external. Again his retort to Hans Hofmann comes to mind: "I am nature." Such an equation of the self with the natural dimension may draw further allusions to or parallels with Lucien Lévy-Bruhl's concept of primal mind." [5]

Perhaps these action painting experiments which, apparently „propagate the unilateral unconscious self-representation of the existing impulse in the composition of painting, thus hiding anti-formalistic and anti-aesthetic tendencies" [6], and that simulate the atraction of chaos, the pure feeling of infinite, of movement (mechanical and biological in the same time [6], contain a „secret geometry" [7], that I will try to decode through a case study of my own abstract expressionist paintings.

My works are based on deformed space and surface, on gestualism, on matter. I use the practice of assembling, of supra-impressions and perforation, which I relate through well-determined recipes of geometric concepts.

I enrich the technique with subtle procedures, capturing the shades that develop in the perimeter, bringing forward the process of creating and not the result. I am looking to invent a new and original way of drawing and building new spaces.

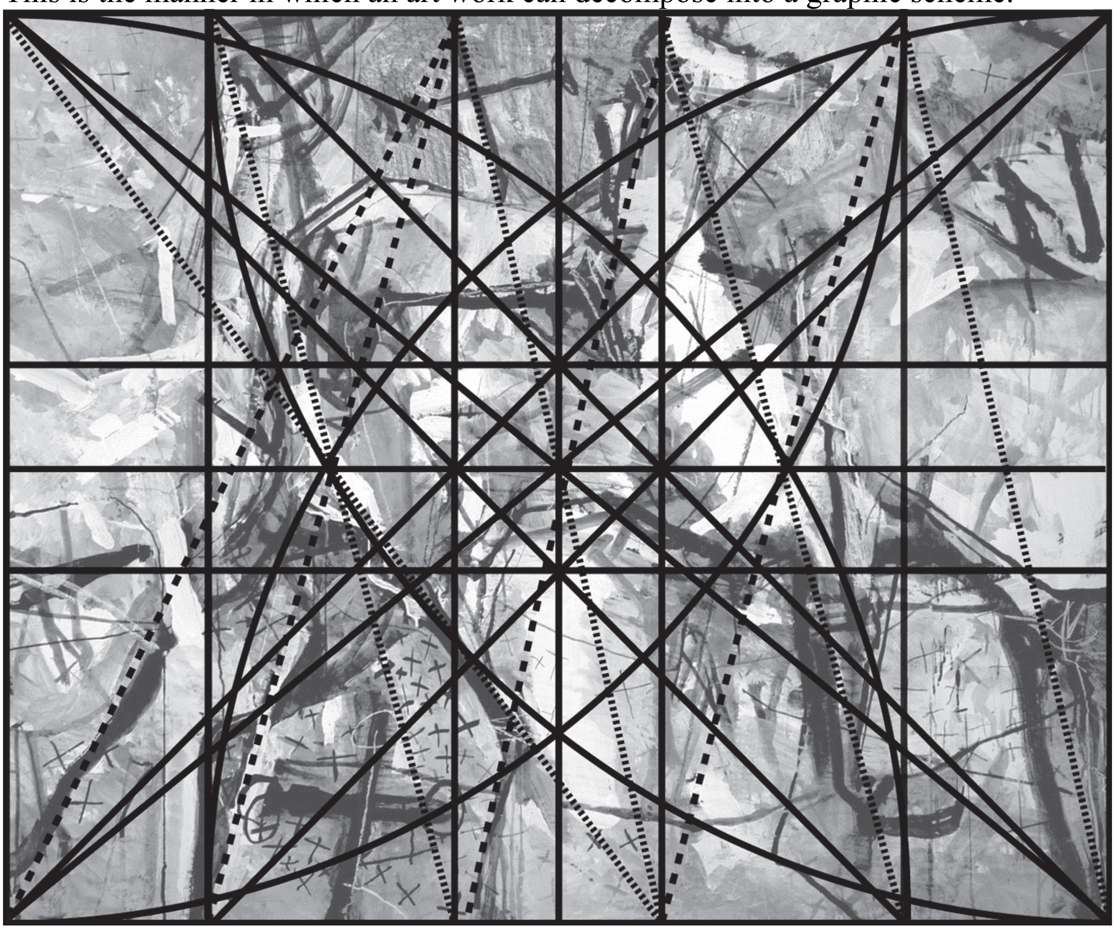
The color is not designed to describe or create the illusion of things; on the contrary, it has to live its own independent and excited experience, explicitly following the laws of art. Through superposing, using contrasts as well as correlations, I produce up-down and right-left movements, I create a game of spatial colouris relations, in a light flow or dynamic movement.

I continue this chromatic composition, this tensioning and de-tensioning of the properties of colours by superposing colour on color, allowing the color behind to transpare to the surface.

All these artistic gesture actions are based on a personal geometry: I wish for the factor generator line of movement to demonstrate it's capacity of transforming into zig zags, of receiving diverse angles, tensions and undulations, in a continuous flux -reflux flow.

The line has to describe paths and to execute aerial gestures, just as a metal spiral. This time the spatial process has to be perceived as the movement and rhythm of a free play that produces a dynamic impression to the viewer. The line runs, jumps, rolls and leaves, in its whimsical path rigorous geometric shapes or free signs. In an uninterrupted run, the line seems to adapt each time to the environment, to do as it pleases, in a changing tempo, transforming everything from a static existence into a dynamic becoming.

This is the manner in which an art work can decompose into a graphic scheme.



In order to show the (pointed) ascending and descending force lines of the work, the image's decomposition was presented with its compositional grid. The horizontal, vertical and oblique lines are projections of the golden ratio which prove the rhythm of the duct, of the contrasts that organize the composition.

Original work: „Untitled: nr.1" mixed technique
dimensions:149 x 179 cm, 2005, author: Adrian Serbănescu

This "secret geometry" [7] which is not explicitly presented in the artistic gesture sustains the dynamic composition of the art work. The curves and straight lines are structuring the composition, which apparently seems to result from a chaotic form. The geometry that hides behind the shades of color confers that intellectual (mathematical) structure that the viewer perceives at an unconscious level.

In the given case, the cognition of the artistic gesture is based, on the mathematical principles of the shapes of curves and straight lines that offer order in chaos [8], this being an unconscious process that takes place in the mind of the observer.

**Acknowledgement**

The author wants to thank Professor Mauro Francaviglia for the advices during the elaboration of this paper. The paper was supported by Grant PN II IDEI "Maps of Time".

## References


[1] KOZLOFF, Max, "6 American Painting during the Cold War". In *Pollock and after: The Critical Debate*, Harper & Row, New York, 1985.

[2] LEVIN, Gail, *Action Painting: Perspectives from Two Sides of the Atlantic*, Art Journal 67, no. 4: 119, 2008

[3] BERGER, Maurice, *Action/Abstraction: Pollock, de Kooning, and American Art, 1940-1976*, Yale University Press, 2009

[4] SEGEL, Katy, *Abstract Expressionism*, Phaidon Press, 2011

[5] CERNUSCHI, Claude, *Jackson Pollock: Meaning and Significance*, New York: Icon Editions, 1985

[6] CIRLOT, Eduardo, *Pictura Contemporana*, Meridiane, Bucharest, 1969

[7] BOULEAU, Charles, *The Painter's Secret Geometry: A Study of Composition in Art,* Harcourt, Brace & World, New York, 1963

[8] MĂNESCU, Mihai, Ordine si dezordine în limbajul graficii, Editura Tehnică, Bucharest, 2001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# FROM THE PION DECAY TO KANDINSKY'S "CIRCLES"

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**Abstract.** This paper shows a teaching project for High School students based on the study of antimatter through the qualitative and quantitative analysis of several images taken in a bubble chamber and propose a possible parallel with the painting *"Circles in a circle"* of Vasiliy Kandinsky (1926).

**Key words.** Math, physic, art, teaching, Kandinsky, antimatter.

## Introduction

This paper has the goal of presenting a teaching project for High School students based on the study of antimatter through the qualitative and quantitative analysis of several images taken in a bubble chamber. Moreover, the aesthetic beauty of geometric forms found in these images is exploited, to propose a possible parallel with the painting *"Circles in a circle"* of Vasiliy Kandinsky (1926). In this way one can make it clear to students that special geometric properties are often hidden both in Nature and in artistic phenomena.

Before describing in detail the testing performed by one of us (MT), we shall briefly report the ordered sequence of the topics discussed in the proposal:

1. A brief history of Particle Physics until 1932;
2. Description of the cloud chamber for the study of cosmic rays;
3. The discovery of the positron and the discovery of antimatter;
4. Qualitative and qualitative analysis of some images of trajectories of particles as seen in a bubble chamber;
5. Digression on *"Circles in a Circle"* of Vasily Kandinsky.

In particular, the goals one wants to pursue when teaching through this proposal are:

a) Knowing and being able to apply the physical laws concerning the Lorentz force;
b) Recognizing and being able to calculate the radius of the trajectory of a charged particle in a magnetic field;
c) Understanding the nature of cosmic rays;
d) Reasoning about antimatter and its features.

This educational proposal has been tested by one of us (MT) in the school year 2010-2011 for the Master Plan Level II IDIFO3 PLS (Scientific Degree Plan) sponsored by MIUR, with the fifth grade of students attending High School, in particular 5°E Institute "A. Volta" of Castel San Giovanni (PC), the 5°A of Respighi School of Piacenza, the Science and Technology 5°B ITI Cardano of Pavia, for a total of 86 students.
The experiment lasted for a total of 6 hours (divided into three lessons of two hours each) and was managed in conjunction with the Chair of the classes of teachers who had previously introduced in their classes preliminary aspects for trial, such as the definition of magnetic force, the "law of the right hand" and the Lorentz force.
Moreover, as far as the argument 5 proposed in the above list is concerned, we remark that it represents an innovative character that has not yet been directly tested in the context just described; in view of the facts developed, it nevertheless offers interesting insights for further interdisciplinary developments.

**The different phases of experimentation**

**PHASE I.** In the first lesson of the experiment it has been proposed that pre-test students had to be completed in 20 minutes (highlighted in bold are the correct answers):

***Question 1:***
*In nature do exist two types of magnetic charge, positive and negative (conventionally North and South). Charges of equal sign repel, opposite charges attract. True or **False**?*

| N° Students | Correct answers | Wrong answers | % wrong answers |
|---|---|---|---|
| 86 | 15 | 71 | 82,6% |

***Question 2:***
*North Pole and South Pole magnetic exist independently of each other (they can be isolated from each other). True or **False**?*

| N° Students | Correct answers | Wrong answers | % wrong answers |
|---|---|---|---|
| 86 | 64 | 22 | 25,6% |

***Question 3:***
*If we put a positively charged body near a current-carrying wire an interaction between string and body is observed! True or **False**?*

| N° Students | Correct answers | Wrong answers | % wrong answers |
|---|---|---|---|
| 86 | 31 | 55 | 63,9% |

***Question 4:***
*If we place a magnet (magnetic needles) near a Current-Carrying Wire we observe an interaction. **True** or False?*

| N° Students | Correct answers | Wrong answers | % wrong answers |
|---|---|---|---|
| 86 | 69 | 17 | 19,8% |

It is interesting to note that the results are obtained, albeit in part, in a similar way to those observed in other experiments on electromagnetism [7]; they show that:

1. Students do not have a specific vocabulary (at least for Magnetism): in fact, about 80% of students think that positive and negative charges are synonymous with the magnetic North pole and South pole and confuse the two terms, probably since the behavior of the magnetic poles (Law of the poles) is similar to that of the charges.
2. About 65% of students think that there is an interaction between a current-carrying wire and a stationary charged body placed in its vicinity: the students probably do not have a very clear distinction between the interaction of charged objects and the interaction between the wire (current flowing) and body load.

To summarize the impression on the pre-test, one can say that students who participated in the experiment do not know specifically (and with appropriate scientific vocabulary) the specific features related to the basic magnetic phenomena, as well as the effects of the interaction between magnet and current-carrying wire.

**PHASE II.** The remaining time of the first lesson was devoted to present the results of Particle Physics prior to 1932, the year of the discovery of the positron by Carl David Anderson. They form the historical basis for the analysis of Particle Physics that is done in this trial. In particular, we have briefly described the findings of Thomson and the work of Dirac in a qualitative way, without addressing the quantum nature of their discussions. As a part of the historical path it the existing link between cosmic rays and antimatter was also briefly mentioned (which will be detailed later), since this is the pretext for introducing the experiment of Anderson (1932). At that point was then possible to focus on the operation of cloud chambers (or a single cloud chamber), their history and technological evolution they had, stressing their importance played in studying particles in cosmic rays and, in particular, in the experiment of Anderson.

**PHASE III.** The second lesson was devoted to the reconstruction of the historical and experimental experience of Anderson in 1932, when for the first time the positron was seen. The existence of the electron's antiparticle was theoretically predicted in 1928 by Dirac, who deduced its existence (since the positron is an outcome of the solutions of Schroedinger equation). The track followed has revived some of the teaching contents in the original paper of Anderson [5], although presented in a simplified and more suitable way for a Secondary School. Initially the children were proposed a generic image analysis test conducted in a modern cloud chamber (Fig. 1). The students' attention was focused on three tracks left by two electrons and one proton in the cloud chamber. Students have said that within the cloud chamber there is a magnetic field - that can be estimated as being perpendicular - and inducing the formation of the figure. Students have noticed, not without some difficulties, that the particle trajectory appears in the corner, because the cloud chamber is the arena of a magnetic field and that the curvature depends on the charge. Everyone has also noticed that the

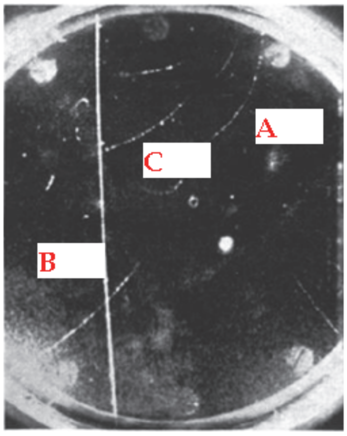


**Fig. 1: tracks left by a proton (B) and two electrons (A and C) in a cloud chamber**

trace left by the proton (Figure 1, label B) is much more marked: this is due to the direct proportionality between the charge of a particle and its “ability to ionization”.

After having studied the link between the Lorentz force and the curvature of the particle traces left in the cloud chamber, invoking the law of the right hand, the analysis of the figure (Fig. 2) was proposed for the experiment history of C. Anderson, 1932 (as reported in [7]). The students were given a larger A3 format of Figure 2, to work better in a quantitative way, and were also given information about the magnetic field in the cloud chamber used by Anderson [5]. The intention was to make students go through the doubts and difficulties encountered at the theoretical level (even initially) in 1932 by Anderson himself, whence he stood astonished in front of a particle leaving the trace of an electron, but having the curvature proper of the proton…! To understand this phenomenon, Anderson inserted a lead plate 6 mm thick into the particle cloud chamber, to realize the way it comes from the particle, and used this information along with the direction of the magnetic field to conclude that it was a particle similar to the electron, but of positive charge, namely a “positron”.

The students were apparently carried out at exactly this situation of stalemate through a series of questions:

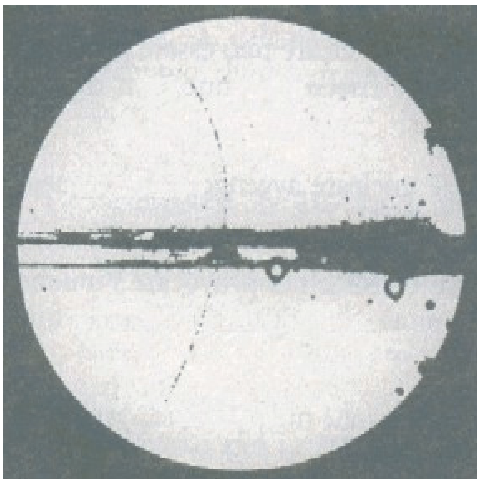

**Fig. 2: the positron by Anderson (picture taken from the original article [5] of 1933)**

a) Where did you get the particle? Measure the radii of curvature of the particle above and below the lead plate;
b) What happens to the particle when it crosses the lead plate? Remember and use the relation between the momentum of the particle and its radius of curvature.
c) Find reasonings that help in deciding whether the energy of the particle comes from the bottom or top of the image, and infer from this his behaviour.

As for a), some students came to the conclusion that it was possible to estimate the radius of curvature of the particle using the perpendicularity between the tangent to a circle and the radius: in this way, after locating the center, this could be fairly estimated. The exact radius of the particle is above or below the lead plate.

Many students have however misinterpreted the figure going in search of a common center for two different radii of curvature, showing not to have sensed that the trajectory of the particle was changed when passing through the lead plate. It was precisely the question b) to address the fact that the particle came from the bottom of the image: about 80% of the students immediately came to this conclusion by reasoning about the size of the curvature radius of the particle: if above the lead plate, it was less than that below the slab. Using energy conservation (question c) of the above list) allowed students to deduce that the particle, crossing the plate, cannot buy energy, but needs instead to lose some, and therefore it cannot come from above but from below the image. After these conclusions, the students have been told that what they had analyzed was in fact a positron, the “antiparticle of the electron” – that was hence described in purely qualitative characteristics and applications of “antimatter”.

The fact that the majority of students come eventually to the correct conclusion is - according to the sequence of cognitive proposal - a confirmation of the goodness of the proposed teaching path, that has therefore shown that it is possible to tackle with confidence also with the last part of the experimentation.

**PHASE IV.** In the last lesson we addressed a "pion decay" ($\pi^+ \to \mu^+ + \nu_+ + e^+ + \nu_e$) through the analysis of an image that depicts the collision between a positron and an atom of Neon (Ne), taken in a streamer chamber (containing precisely Neon). The analysis of this image (Fig. 3) allowed students to work again over antimatter, and the conservation of momentum in collisions between subatomic particles. It has also given rise to interesting further discussions about the nature of cosmic rays and technological applications of antimatter (TAC and PET).
Similarly to what was done with the image of Anderson, we provided students with a A3 magnification of Figure 3. They were asked to think about the particles in it. In particular, they were asked to reason about paths, recalling what was done in previous lessons. A qualitative impact on it for further analysis skills would be needed through the use of Relativity Theory, a subject that was not yet covered in the classes in which the testing was conducted .
To overcome this problem the boys were asked to achieve approximate results concerning relations between the momenta of the particles (in this way the relativistic Lorentz factor simplifies) and not between the velocities (which are, of course, relativistic). Stepwise teaching has therefore followed this sequence of questions:

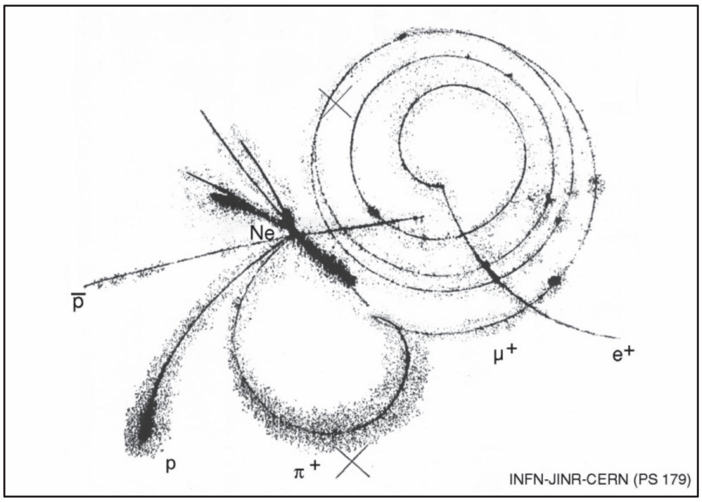


**Fig. 3: Pion decay**

a) Indicate what is the direction of the particle and the direction of the magnetic field in the streamer chamber;
b) Put the particles in an ascending order based on their momentum;
c) Look to how this zooms in a significant picture (Fig. 4A and 4B – that is a particular collision). Draw the momentum vectors before and after the collision and calculate their intensities for each particle (the students were given all the necessary numerical information: magnetic field strength, charge and mass of particles).

Concerning point a) about 90% of the boys realized the work done in parallel with the second lesson (phase III) and, with the help of the trace left by the positron, they immediately replied that the streamer chamber contains a magnetic field perpendicular to the sheet and departing from it. The question b) did not originate big problems among students, and about 75% of them managed to sort in ascending order the momenta of the particles (using their radius of curvature in the reasoning), as it was already done in the second lesson.
It was instead the question c) to provoke a broader discussion, since only a few students were effectively able to draw the vector momentum of the particles highlighted before and after the collision in a correct way. To convince the boys that the momenta of the particles were smaller after the collision, all the information about particles have been provided first and students were then asked to calculate the radii of curvature of the trajectories as already done for the positron. In this way we tried to overcome the problem of speed for these particles, that in fact follow the laws of Relativity. After these operations, only 20% of the students guessed that, since the momentum is conserved during the

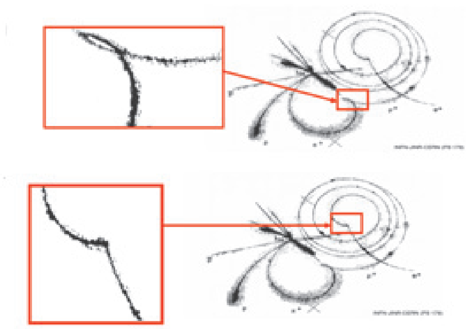

**Fig. 4 (A, B): details of pion decay**

collision, there must be something that acquires the momentum of $\pi^+$ and $\mu^+$. Of course, the boys have not been able to give a name to this new "partner" and it was our pleasure to tell them that it was a "neutrino".
Some students, having read informative articles, concluded that they left no trace in the streamer chamber because they have no charge, confirming that the work of Phase III was appropriate and educationally effective.

**PHASE V**. The final lesson (of the last 20 minutes) was devoted to a final test, containing the following questions (the correct answer is highlighted in bold next to each answer; the table shows the number of students who have chosen it):

*Question 1: The figure shows a positively charged particle that moves to the right at velocity v in a magnetic field B.*

*Which answer best describes the force F acting on the particle?*

| | | |
|---|---|---|
| *Answer a* | *0* | *points up* |
| *Answer b* | *0* | *points right* |
| *Answer c* | *6* | *F=0* |
| ***Answer d*** | *71* | ***perpendicularly out from the paper*** |
| *Answer e* | *9* | *enters perpendicular to the paper* |

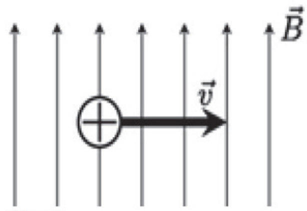


Fig. 5

*Question 2: A positively charged particle that moves to the right at velocity v between the North and South poles of two magnets.*

*Which answer best describes the force F acting on the particle?*

| | | |
|---|---|---|
| *Answer a* | *7* | *points up* |
| *Answer b* | *1* | *points right* |
| *Answer c* | *2* | *F=0* |
| *Answer d* | *8* | *perpendicularly out from the paper* |
| ***Answer e*** | *68* | ***enters perpendicular to the paper*** |

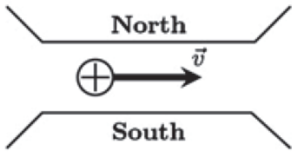


Fig. 6

*Question 3: The figure represented the north pole and south poles of two magnets. Which best describes the responses of the magnetic field B at P?*

| | | |
|---|---|---|
| *Answer a* | *5* | *B point up* |
| ***Answer b*** | *70* | ***B point down*** |
| *Answer c* | *8* | *B=0* |
| *Answer d* | *0* | *B perpendicularly out* |

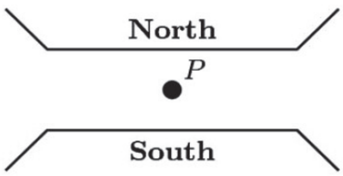


Fig. 7

| | | |
|---|---|---|
| *Answer e* | *3* | *from the paper B enter perpendicular to the paper* |

*Question 4: If we place a positive charge stop between the North and South poles of two magnets is observed that the charge ...*

| | | |
|---|---|---|
| *Answer a* | *1* | *approaches to the magnet* |
| *Answer b* | *4* | *moves away from the magnet* |
| ***Answer c*** | ***69*** | ***stays*** |
| *Answer d* | *2* | *moves up* |
| *Answer e* | *10* | *moves down* |

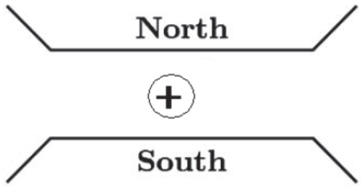

**Fig. 8**

*Question 5: If we put a current-carrying wire near a magnet it is observed that the wire ...*

| | | |
|---|---|---|
| ***Answer a*** | ***75*** | ***interacts whit the magnet*** |
| *Answer b* | *0* | *is not subject to forces* |
| *Answer c* | *8* | *interacts only if it moves with respect to the magnetic field* |
| *Answer d* | *3* | *I cannot decide because I do not know the intensity of current flowing in the wire* |
| *Answer e* | *0* | *I cannot decide because I do not know the intensity of the magnetic field* |

*Question 6: Consider an electron moving with some velocity v in a uniform magnetic field B perpendicular to the sheet.*

*The electron travels a circular orbit. If we increase the magnetic field B it is observed that ...*

| | | |
|---|---|---|
| *Answer a* | *4* | *the trajectory does not change* |
| *Answer b* | *1* | *the trajectory becomes an ellipse* |
| *Answer c* | *6* | *the trajectory is circular, but the radius increases* |
| ***Answer d*** | ***74*** | ***the trajectory is circular, but the radius decreases*** |
| *Answer e* | *1* | *the trajectory becomes an hyperbole* |

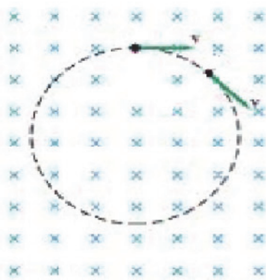
**Fig. 9**

For a quick read of the results one can immediately see that, in the post-test, the number of correct responses increases, indicating that the trial had a "positive teaching" result. The first two questions of the post-test on the skills, required students to recognize the direction of the magnetic force acting on a particle moving in a magnetic field. They have been taken from a research performed at the University of Ohio among 110 High School students [7]. The histogram in Fig.10 refers to the answers given by Americans students to whom the same questions have been proposed. As one can see, less than half of them answered correctly to both the first and the second question. It should however be pointed out that these questions were, for American students, nothing but a test of Physics concepts inherent with the program they were being taught, and not a questionnaire proposed after an experimental teaching (as we did instead). As reported by the author [7] after two weeks of lessons on this subject their correct answers increased by about 20%.

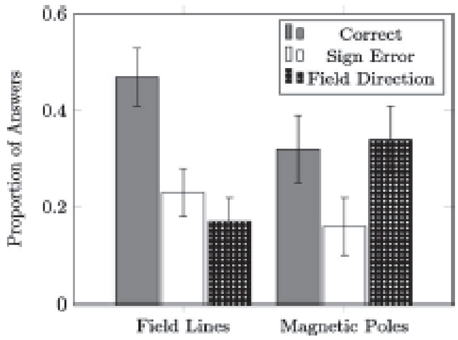


**Fig. 5: answers given by students at the University of Ohio to a questionnaire similar to the final test of our experimentation**

Also the third question of our post-test was taken from [7]; it represents a good way to tell whether students have understood the direction of the magnetic field lines in unconventional situations, or not. In this case the percentage of incorrect answers is almost coincident: in the case of our post-test it is 12%, while in the case of [7] it is equal to 15% and, as stated in [7], errors come out either as a misrepresentation of the field lines or by confusion about their direction. Regarding our post-test we should also add that a good percentage of those who gave an incorrect answer would probably think that since the charge is in the exact center of the magnet the North pole and South pole effects are equal and therefore cancel out.
The other three questions of the post-tests have been useful to see whether the students had modified some of their conceptual errors after the trial and whether they had undertaken the conceptual argument that in fact is a "bridge" between the two parts of the experiment, i.e. the magnetic force and the magnetic field effect on a particle moving with a certain speed. About 80% of students showed that the trial was successful and had a good education and disciplinary fallout, as it was evident from the "uniformity in content" and the "linearity" of their presentation.

**Possible insights and future prospects**

**Fig. 6: Vasily Kandinsky "Circles in a circle" (1926)**

Given the success of the trial, the intention of one of us (MT) is to present it again in the school year 2011-2012. To this purpose, some schools in the province of Alessandria have already offered their availability and it has been proposed that they start testing in the spring of 2012. The intention is to expand the test both in its content and in the number of experimentation hours required for its implementation (as in the past we already assumed this requirement). Probably the number of hours will be increased to 8-10 . As a new and extremely original content we shall add an intriguing parallel with the painting by Vasily Kandinsky's "Circles in a Circle" in 1926. It is in fact evident, also on the basis of a strict temporal coincidence, that the newly found light traces left in bubble chambers do, in a sense, follow the same mathematical rules that Kandinsky used to develop his

revolutionary "geometrical painting" (see, e.g., [10],[11]).
In addition in order to appreciate the charm of that painting, one can in fact bring the attention of students towards the "geometric analogy" that links the work of Kandinsky (presented in Fig. 3) to the images of a pionic decay – see also [12]. In fact all the trajectories of the particles in the streamer chamber are circular, but the snapshot shown offers the opportunity to appreciate only one. In addition, it has been reported that Kandinsky had created his painting by just thinking to planets orbiting the Universe in accordance with Newton's Universal Law of Gravitation.
This link between pions and Kandisky's work can also be a good further way to talk about the geometric essence that often hides into physical phenomena and also to more easily and visually introduce the concept of "symmetry" in experimental sciences. In addition, such an approach will be able to enforce the interdisciplinary nature of the project, moving in the direction of a wide-ranging education sector and not, as too often happens in Italian schools (at all levels), in restricted contexts.
Additional sources for the educational development of this proposal, in addition to the research direction already mentioned above, can also be found in local health authorities, who wish to stimulate visits by the future participants of the trial to their facilities where equipment for TAC and PET are located (another interesting connection, where one can again develop an interdisciplinary analysis of recent researches on the nature of cosmic rays and on astroparticles physics).
We think that the various possibilities to connect the main subject of the experiment described above with other topics within the same subject of Physics, Mathematics and other scientific disciplines might be a good idea both for expanding the activities already begun and also in view of broader scopes. The experiment performed in Alessandria has in fact revealed itself to be "educationally functional", but it will be necessary to continually update it and further implement the experimentation in order to consolidate and confirm the good results already obtained. We believe that the link with Art discussed above will help a lot in this direction.

## Acknowledges

The authors are indebted to Prof. Marisia Michelini (Udine) and to her Master Project IDIFO3, that has allowed this fruitful collaboration. One of the authors is partially supported by INdAM-G.N.S.A.G.A. (MGL) while the other author (MF) is partially supported by INdAM-G.N.F.M.

## References

[1] P. A. TIPLER, *Invito alla fisica*, Zanichelli.
[2] HALLIDAY D., RESNICK R, WALKER J., *Fondamenti di Fisica*, Casa Editrice Ambrosiana.
[3] F. S. DELLI SANTI, *Introduzione all'astronomia*, Zanichelli.
[4] TAYLOR E. F., WHEELER J. A., *Fisica dello spazio-tempo*, Zanichelli.
[5] ANDERSON C. D., *The positive electron*, Phys. Rew., 13, 1933.
[6] BERTUCCI B., *I raggi cosmici*, Le Scienze, 494, 2009.
[7] SCAIFE T. M., HECKLER A., F., *Student understanding of the direction of the magnetic force on a charged particle*, Am. J. Phys., 78 (8), August 2010.
[8] www.astroparticle.org.
[9] www.telescopearray.org/olutreach.html
[10] M.G. LORENZI, M. FRANCAVIGLIA, *The Role of Mathematics in Contemporary Art at the Turn of the Millennium,* APLIMAT - Journal of Applied Mathematics, 4 (2011) – ISSN 1337-

6365 In: "Proceedings 10th International Conference APLIMAT 2011" (Bratislava, February 1-4, 2011); M. Kovácová Ed.; Slovak University of Technology (Bratislava, 2011), pp. 997-1018 ISBN 978-80-89313-51-8 (book and CD-Rom) - abstract in: "Book of Abstracts", ibidem, p. 135 – ISBN 978-80 89313-52-5

[11] M.G. LORENZI, M. FRANCAVIGLIA, *Geometry as a Source of Inspiration in Contemporary Art,* in: Bridges Coimbra (Mathematics, Music, Art, Architecture, Culture), Conference Proceedings; R. Sarhangi & C. Sequin Ed.s, with P. Machado as co-Editor; Tessellation Publishing (Phoenix, AZ, USA - 2011), pp. 365-372, ISBN 978-0-9846042-6-5; ISSN 1099-6702

[12] L. SHLAIN, *Art & Physics (Parallel Visions in Space, Time, and Light)*, Harper Perennial (New York, USA, 1991) – ISBN 978-0-06-122797-4

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# ABSTRACT ALGEBRA ART VISUALIZATIONS

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**Abstract**. Paper presents few ideas on visualization of abstract algebraic operations in more dimensional spaces in the form of geometric structures with certain artistic value. Abstract geometric forms and figures created as compositions of views of more dimensional Riemannian manifolds that are generated by abstract set algebraic operations in more dimensional spaces are presented, regarded as certain artistic visualizations of abstract algebraic relations, concepts and patterns.

**Key words.** abstract algebraic set operations, modelling of more dimensional Riemannian manifolds, visualisation of algebraic operations

*Mathematics Subject Classification:* Primary 00A66, Secondary 00A71.

## 1 Introduction

Minkowski sum of two point sets is an abstract binary algebraic-geometric operation defined on point sets in the $n$-dimensional Euclidean space $\mathbf{E}^n$, and it can be determined and interpreted in various ways. Most commonly appearing definition is based on the concept of well defined vector sum in the associated vector space over $\mathbf{E}^n$. Respective points in the operand sets can be attached their positioning vectors related to a reference point in a unique way, regardless the chosen coordinate system with the origin assigned for the fixed reference point. No restrictions are imposed on the two operand sets, which can be e.g. finite sets of discrete points, or any sets of smooth sub manifolds in $\mathbf{E}^n$ in general.

## 2 Minkowski sum of point sets

Let $A$ and $B$ be two point sets in the $n$-dimensional Euclidean space $\mathbf{E}^n$ considered with the usual orthogonal Cartesian coordinate system $\langle O, x_1, x_2, \ldots, x_n\rangle$ and associated vector space $V^n$ with the orthonormal basis $\langle O, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\rangle$. Relation between the two algebraic structures can be realised as described in the following.

Any point $m \in \mathbf{E}^n$ determined by Cartesian coordinates $m = \left[x_1^m, x_2^m, \ldots, x_n^m\right]$ can be associated with its positioning vector **m** in a unique way by the following mapping

$$\begin{aligned} &\pi : \mathbf{E}^n \to V^n, m \in \mathbf{E}^n \mapsto \mathbf{m} \in V^n \\ &m = \left[x_1^m, x_2^m, \ldots, x_n^m\right] \mapsto \pi(m) = \mathbf{m} = x_1^m \mathbf{e}_1 + x_2^m \mathbf{e}_2 + \ldots + x_n^m \mathbf{e}_n = \left(x_1^m, x_2^m, \ldots, x_n^m\right) \end{aligned} \tag{2.2}$$

Denoting by $V(A)$, $V(B)$ sets of all positioning vectors of points in $A$, $B$ respectively, the following definition of Minkowski sum of point sets $A$ and $B$ can be precisely formulated.

**Definition 1.** Minkowski sum of two point sets $A$ and $B$ in $\mathbf{E}^n$ is a point set, which is the sum of all points from the set $A$ with all points from the set $B$, i.e. set of points

$$A \oplus B = \{a + b; a \in A, b \in B\}. \tag{2.1}$$

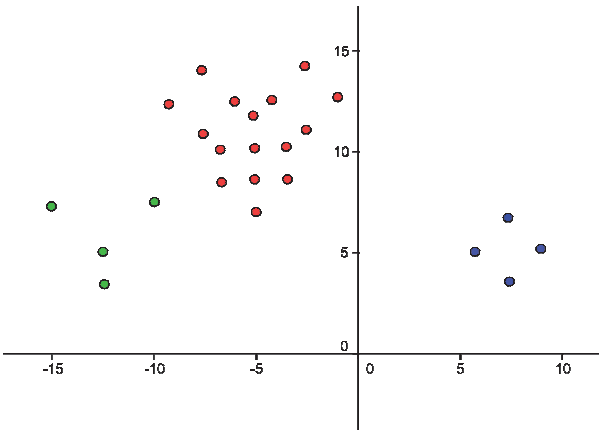
**Definition 2.** Minkowski sum of two point sets $A$ and $B$ in $E^n$ is the set

$$A \oplus B = \bigcup_{b \in B} A^b, \tag{2.3}$$

where $A^b$ is the set $A$ translated by the vector $\mathbf{b} \in V^n$ associated as positioning vector to one fixed point $b \in B$

$$A^b = \{m \in \mathbf{E}^n : \mathbf{m} = \mathbf{a} + \mathbf{b}, \mathbf{a} \in V(A)\} = \{a + b; a \in A\}. \tag{2.4}$$

Two examples of Minkowski sum of two finite point sets in $\mathbf{E}^2$, each consisting of 4 discrete points determined by Cartesian coordinates, is presented in the fig. 1. Symmetric structure of points in the operand point sets distribution is a geometric invariant of the Minkowski sum and leads to the symmetric complex structure of point distribution in the resulting point sets. In this way, invariance of symmetric distribution can be considered as one of the important modelling parameters of the Minkowski set operation shaping the resulting point set distribution and can be used on purpose to obtain certain artistic aesthetic values.



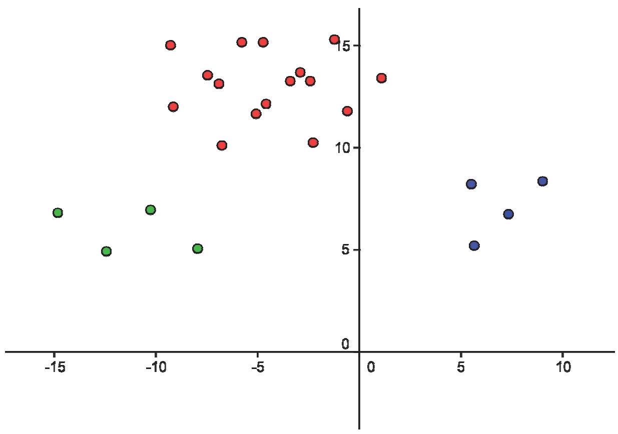




Fig. 1 Minkowski sum of two finite point sets

Considering two point sets composed as finite sets of smooth curve segments in $\mathbf{E}^2$ one can design unusual planar patterns, as is illustrated in fig. 2 on two sets of four different circles.

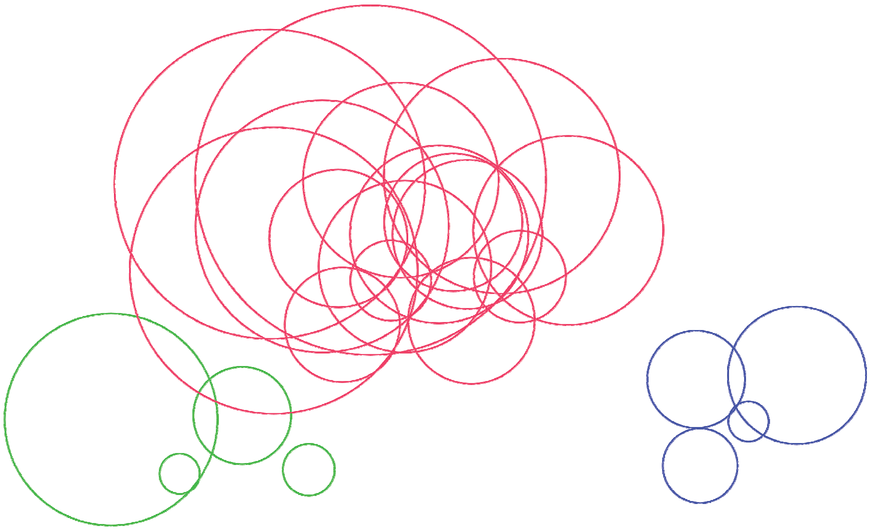


Fig. 2 Minkowski sum of two sets of circles

## 3 Minkowski linear operator on point sets

Let us now focus our considerations to the linear combination of vectors that is a well defined abstract concept of vector space algebra and can be extended to a Minkowski linear combination of point sets. Based on the operation of multiplication of a vector by any scalar, the concept of scalar multiplication of a point set can be determined.

**Definition 3.** Multiple of a point set $A$ by a scalar $k \in R$ is the set $A_k$ in $\mathbf{E}^n$ of points defined as

$$A_k = k.A = \{k.m : \mathbf{m} = \pi(m) \Rightarrow k.m = \pi(k.\mathbf{m}), m \in A, k \in R\}. \tag{3.1}$$

Based on this definition a new concept of Minkowski linear combination of 2 point sets can be defined and used as more complex operation on point sets.

**Definition 4.** Minkowski linear combination of two point sets $A$ and $B$ in $\mathbf{E}^n$ is the point set $C$ in $\mathbf{E}^n$ defined as follows

$$C = k.A \oplus l.B = A_k \bigcup B_l = \{k.a + l.b, a \in A, b \in B\}. \tag{3.2}$$

Illustration of an arbitrary Minkowski linear combination of the same two 4-point sets as in fig. 1, is presented in the fig. 3, while $k$, $l \neq 1$ whereas the equality holds for the usual Minkowski sum of the two point sets. Notice that the invariance of symmetric distribution remains as a strong tool in modeling point sets by means of Minkowski linear combination of sets. For negative values of coefficients $k$ and $l$ one can consider the operation of Minkowski difference of point sets $A$ and $B$, which can be defined as Minkowski sum of point sets $A$ and -$B$, where -$B$ is the multiple of the point set $B$. This must be strictly distinguished form other possible definition of Minkowski difference defined as operation inverse to operation of Minkowski sum of two point sets.
Concept of Minkowski linear combination of two point sets leads to a generalisation in the form of the structure of Minkowski linear operator defined on the set of point sets in the Euclidean space.

There exists more than one possible applicable forms of such definitions, while the most natural way will be used to define a simple Minkowski linear operator.

**Definition 5.** Minkowski linear operator $L_{k,l}$ is a mapping defined on $\mathbf{E}^n \times \mathbf{E}^n$, in which any pair of point sets $A$, $B$ in $\mathbf{E}^n$ is attached a point set $C$ in $\mathbf{E}^n$ in the following way

$$\begin{aligned} &L_{k,l} : \mathbf{E}^n \to \mathbf{E}^n \\ &(A,B) \mapsto C = L(A,B) = k.A \oplus l.B,\ A,B,C \subset \mathbf{E}^n \end{aligned} . \tag{3.3}$$

Visualisation of the Minkowski linear combination of the same point sets as in the fig. 1 is available in the fig. 3, while in the fig. 4 we can see visualisation of the Minkowski linear combination of the same two sets of circles as in the fig. 2.

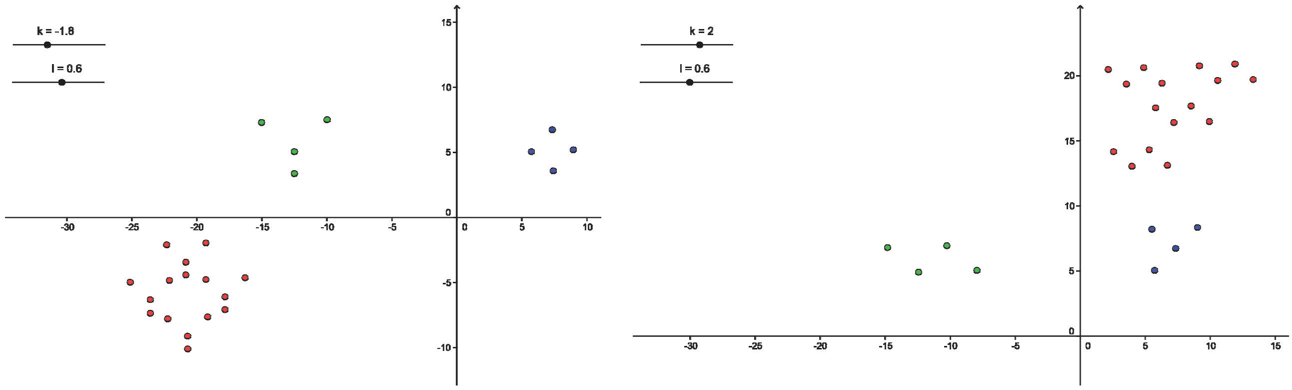



Fig. 3 Minkowski linear combination of two finite point sets

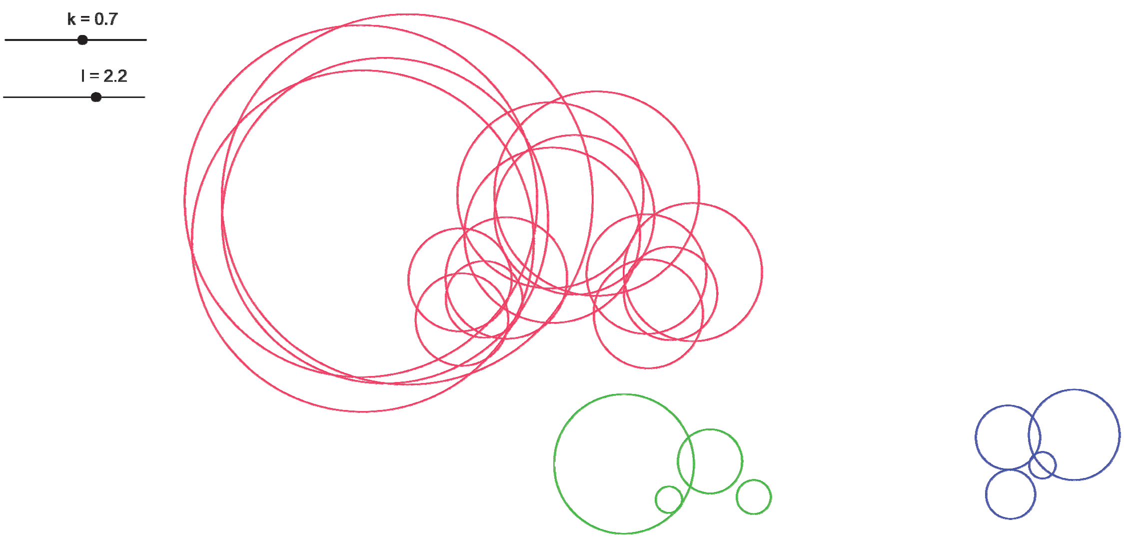



Fig. 4 Minkowski linear combination of two sets of circles

**Definition 6.** Matrix multiple of a point set $A$ by a regular square matrix $\mathbf{M} = \left(a_{ij}\right)_{i,j=1,\dots,n}$ of rank $n$ is the set $A_{\mathbf{M}}$ of points in $\mathbf{E}^n$ defined as

$$A_{\mathbf{M}} = \mathbf{M}.A = \left\{\mathbf{M}.m : \forall m \in A, \mathbf{m} = \pi(m) \Rightarrow \mathbf{M}.m = \pi(\mathbf{M}.\mathbf{m}^T)\right\}. \tag{3.4}$$

Another general form of Minkowski operator, Minkowski matrix operator $L_{\mathbf{M,N}}$ can be defined on the base of matrix multiple of a set of points, as given in the next definition.

**Definition 7.** Minkowski matrix operator $L_{\mathbf{M,N}}$ is a mapping defined on $\mathbf{E}^n \times \mathbf{E}^n$, in which any pair of point sets $A$, $B$ in $\mathbf{E}^n$ is attached a point set $C$ in $\mathbf{E}^n$ in the following way

$$\begin{aligned} &L_{\mathbf{M,N}} : \mathbf{E}^n \to \mathbf{E}^n \\ &(A,B) \mapsto C = L_{\mathbf{M,N}}(A,B) = \mathbf{M}.A \oplus \mathbf{N}.B,\, A,B,C \subset \mathbf{E}^n \end{aligned} \tag{3.5}$$

Minkowski matrix combinations of two discrete point sets and two sets of circles are illustrated in the fig. 5.

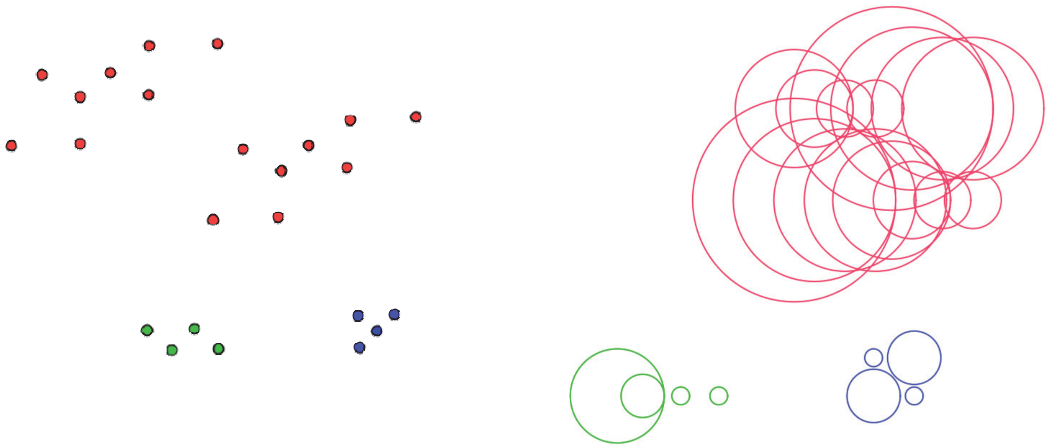


Fig. 5 Minkowski matrix combination of two sets of points (left), circles (right)

## 4 Examples from higher dimensions

Let us consider now two infinite point sets $A$, $B$ defined as compact Riemannian manifolds in the space $\mathbf{E}^n$. Minkowski linear combination of sets $A$, $B$ can be achieved in two different ways.
Let $A$ and $B$ be two curve segments in $\mathbf{E}^n$ defined by their vector representations on unit interval

$${}^1\mathbf{r}(t) = \left({}^1xr_1(t), {}^1xr_2(t), \ldots, {}^1xr_n(t)\right), t \in \langle 0,1 \rangle \tag{4.1}$$

$${}^2\mathbf{r}(t) = \left({}^2xr_1(t), {}^2xr_2(t), \ldots, {}^2xr_n(t)\right), t \in \langle 0,1 \rangle . \tag{4.2}$$

Minkowski sum of curve segments $A$ and $B$ according to def. 1, def.2 can be derived as a curve segment $A \oplus B$ defined on the unit interval by vector representation

$$\begin{aligned} &C = A \oplus B = \mathbf{r}(t) = {}^1\mathbf{r}(t) + {}^2\mathbf{r}(t) = \\ &= \left({}^1xr_1(t) + {}^2xr_1(t), {}^1xr_2(t) + {}^2xr_2(t), \ldots, {}^1xr_n(t) + {}^2xr_n(t)\right), t \in \langle 0,1 \rangle \end{aligned} \tag{4.3}$$

Simple example can be given by Minkowski sum of set $A$ consisting of one circle and set $B$ consisting of two parabolic arcs in $\mathbf{E}^2$ with vector representations

$${}^1\mathbf{r}(u) = \left(a_1 + r\cos 2\pi t, a_2 + r\sin 2\pi t\right), t \in \langle 0,1 \rangle \tag{4.4}$$

$${}^2\mathbf{r}_1(v) = \left(b_{11}t + c_{11}, b_{12}t^2 + c_{12}t + d_{12}\right), t \in \langle 0,1 \rangle \tag{4.5}$$

$$ {}^{2}\mathbf{r}_2(v) = \left(b_{21}t + c_{21}, b_{22}t^2 + c_{22}t + d_{22}\right), t \in \langle 0,1 \rangle . \tag{4.6}$$

Vector representation of the resulting Minkowski sum $C$ in $\mathbf{E}^2$ is set of 2 curve segments with vector functions

$$\mathbf{r}_1(t) = \left(a_1 + r\cos 2\pi t + b_{11}t + c_{11}, a_2 + r\sin 2\pi t + b_{12}t^2 + c_{12}t + d_{12}\right), t \in \langle 0,1 \rangle \tag{4.7}$$

$$\mathbf{r}_{21}(t) = \left(a_1 + r\cos 2\pi t + b_{21}t + c_{21}, a_2 + r\sin 2\pi t + b_{22}t^2 + c_{22}t + d_{22}\right), t \in \langle 0,1 \rangle . \tag{4.8}$$

Visualisation of the above example can be seen in the fig. 6, together with analogous illustrations of different results of Minkowski linear operator $L_{k,l}(A, B) = k.\ A \oplus l.B$ for arbitrary values of real constants $k$ and $l$ applied to sets $A$ and $B$.

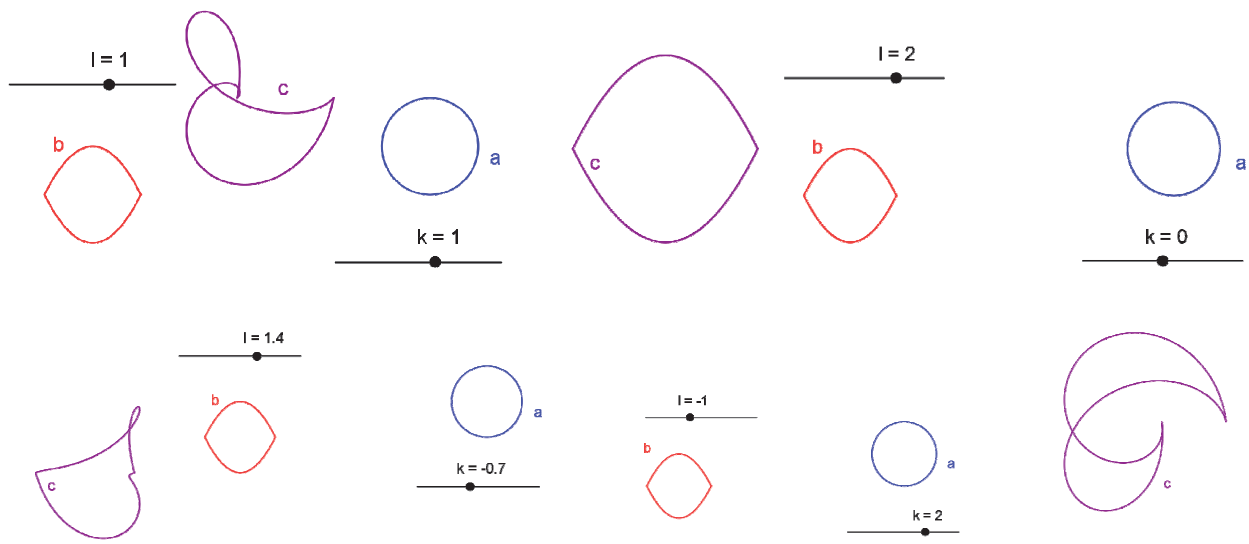



Fig. 6 Minkowski linear combinations of two compact manifolds

More complex geometric forms can be achieved for point sets consisting from several different, though elementary curve segments. Examples can be seen in the fig. 7, in which two different examples are visualized as designed on the base of the Minkowski linear operator $L_{k,l}(A, B)$ of sets: $A$ – set of 4 circles, $B$ – set of 4 parabolic arcs.

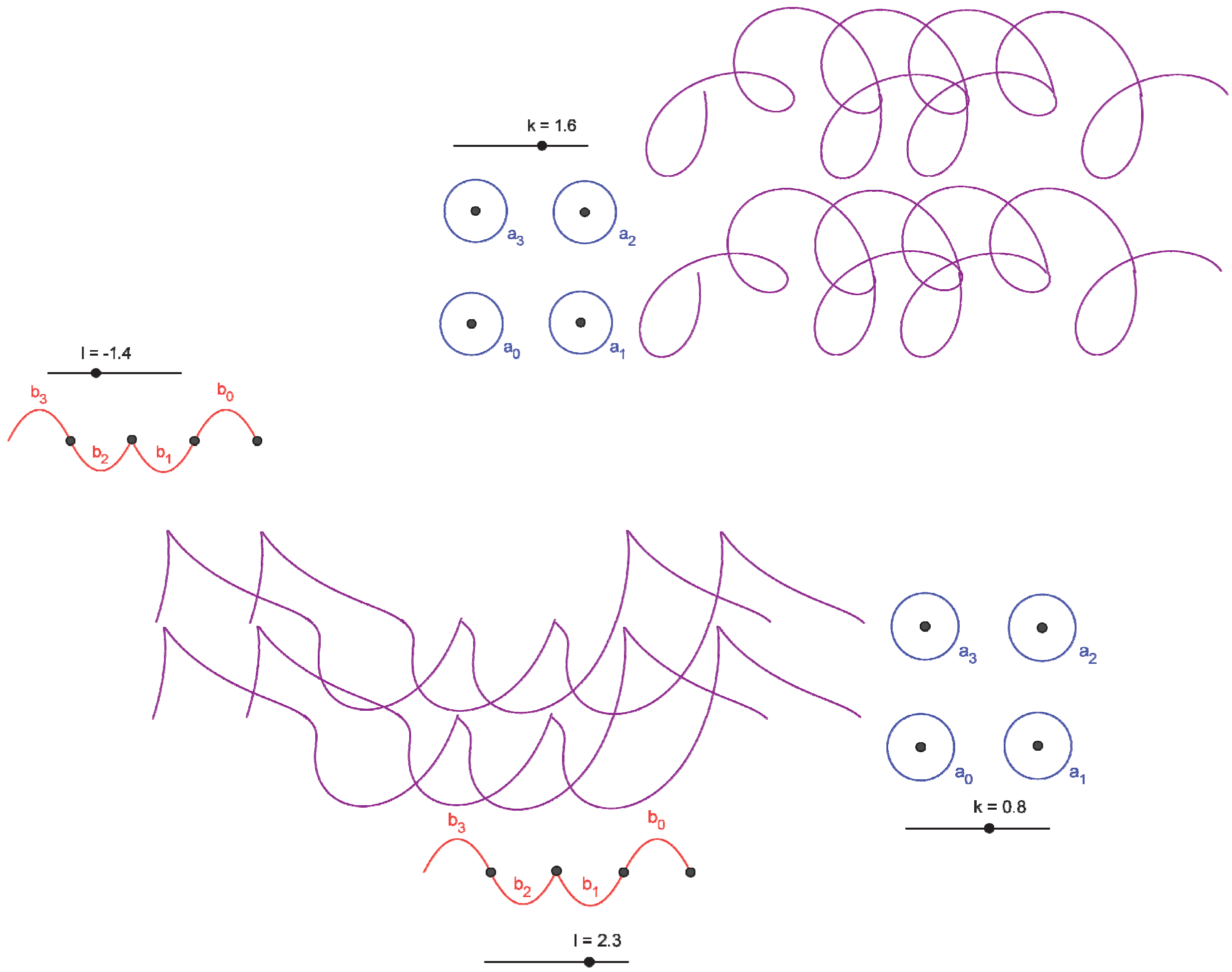



Fig. 7 Minkowski linear combinations of two sets of curves

Unusually interesting geometric structures with certain artistic values can be created using the abstract concept of the Minkowski linear operator as geometric modelling tool. Operand sets of this operator can be chosen both as simple smooth curve segments, or as finite sets of special curve segments distributed in a specific pattern in the space. Several examples are visualized in the fig. 8 as designed by means of $L_{k,l}(A, B)$ for variable constants $k$ and $l$, applied on sets of elementary curve segments, such as circle, parabolic arc, exponential curve, cubic parabola or ellipse and spiral. One-parametric systems of respective multiples of the operand curves by parameter $k$ or $l$ form the net of curves distributed in an unexpected matter. They are forming different planar patterns with specific creative laws that are not straightforwardly predictable. Smooth changes of parameter values enable modelling process of free imagination and design possibilities that is characteristic for artistic creation. Regarding the fact that all presented objects are just simple geometric visualizations based on elementary curve segments realizing abstract algebraic concepts, one can be aware of the magnificent power of mathematical algorithms and laws influencing creative processes, which are still hardly investigated so far.

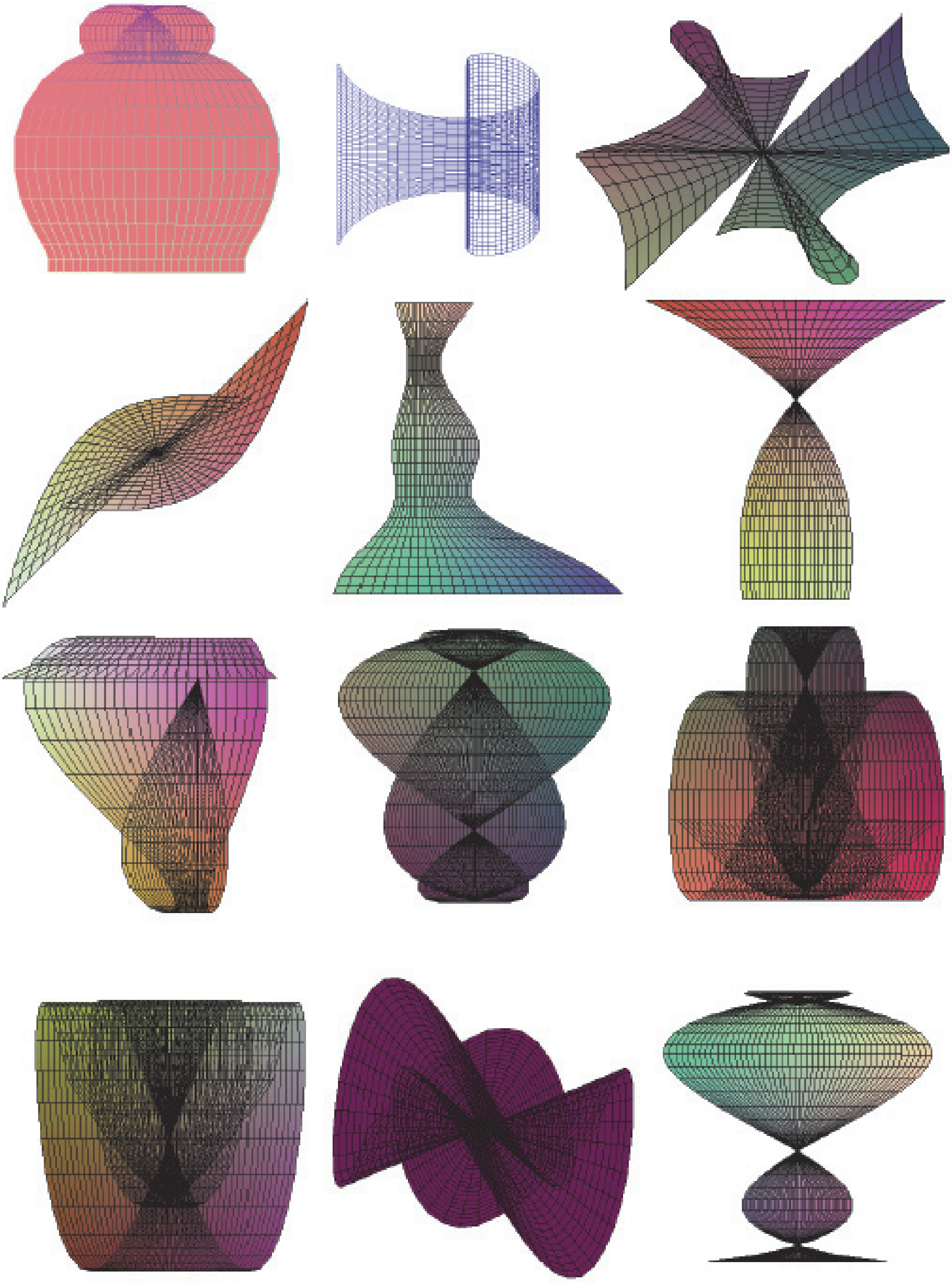


Fig. 8  Artistic visualisations of Minkowski linear operator

Extension to higher dimension can be illustrated on two point sets $A$ and $B$ that are two smooth curve segments in $\mathbf{E}^4$ defined by their vector representations on unit intervals

$$ {}^1\mathbf{r}(u) = \left({}^1x(t), {}^1y(t), {}^1z(t), {}^1w(t)\right), t \in \langle 0,1\rangle \tag{4.9}$$

$$ {}^2\mathbf{r}(t) = \left({}^2x(t), {}^2y(t), {}^2z(t), {}^2w(t)\right), t \in \langle 0,1\rangle \tag{4.10}$$

whose Minkowski linear combination is a curve segment $C$ in $\mathbf{E}^4$ determined in the following way

$$C = L_{k,l}(A, B) = kA \oplus lB = \mathbf{r}^{\oplus}(t) \tag{4.11}$$

while coordinate representation can be obtained as

$$\begin{aligned}\mathbf{r}^{\oplus}(t) &= k\,{}^1\mathbf{r}(t) + l\,{}^2\mathbf{r}(t) = \\ &= \left(k\,{}^1x(t) + l\,{}^2x(t), k\,{}^1y(t) + l\,{}^2y(t), k\,{}^1z(t) + l\,{}^2z(t), k\,{}^1w(t) + l\,{}^2w(t)\right), t \in \langle 0,1\rangle\end{aligned} \tag{4.12}$$

In case of the two operand curve segments $A$ and $B$ parametrization on unit interval for different variables $s$ and $t$, a surface patch in $C$ in $\mathbf{E}^4$ can be determined

$$\begin{aligned}\mathbf{r}^{\oplus}(s,t) &= k\,{}^1\mathbf{r}(s) + l\,{}^2\mathbf{r}(t) = \\ &= \left(k\,{}^1x(s) + l\,{}^2x(t), k\,{}^1y(s) + l\,{}^2y(t), k\,{}^1z(s) + l\,{}^2z(t), k\,{}^1w(s) + l\,{}^2w(t)\right), (s,t) \in \langle 0,1\rangle^2\end{aligned} \tag{4.13}$$

Visualisations of these 4D structures can be partially realised by means of orthographic mappings of the resulting Riemannian manifold $C$ onto all four 3-dimensional orthogonal subspaces ${}_{123}\mathbf{E}^3$, ${}_{124}\mathbf{E}^3$, ${}_{134}\mathbf{E}^3$, ${}_{234}\mathbf{E}^3$ of the space $\mathbf{E}^4$, as illustrated in the fig. 9 for a curve segment or in the fig.10 for a surface patch. In this way, we can expect to develop a certain idea about the form and shape of Riemannian manifolds in $\mathbf{E}^4$ based on their four 3D views, similarly as any object in $\mathbf{E}^3$ can be visualised by its orthographic views onto at least two from three available perpendicular image planes in $\mathbf{E}^3$. Consequently, in order to be able to obtain a unique reconstruction of any 4D object, up to specific singular positions, no more than 3 orthographic views of this object onto 3 different perpendicular 3-dimensional image subspaces in $\mathbf{E}^4$ are necessary. Undoubtedly, a real visualization ability of such an object is far from being realistic on the base of its available 3D views, and certain artistic imagination and fantasy might be necessary to cultivate the 4D space imagination.

a b c d

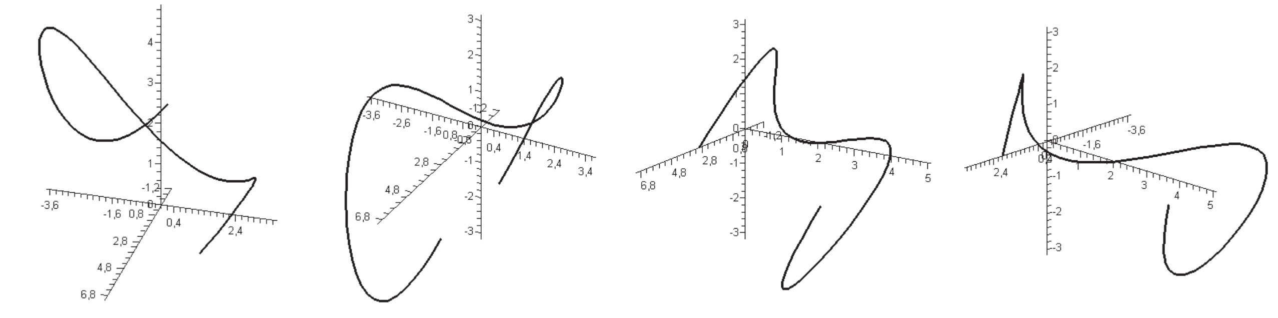


Fig. 9 Orthographic views of curve segment in $\mathbf{E}^4$
a - view in ${}_{123}\mathbf{E}^3$, b - view in ${}_{124}\mathbf{E}^3$, c - view in ${}_{134}\mathbf{E}^3$, d - view in ${}_{234}\mathbf{E}^3$.

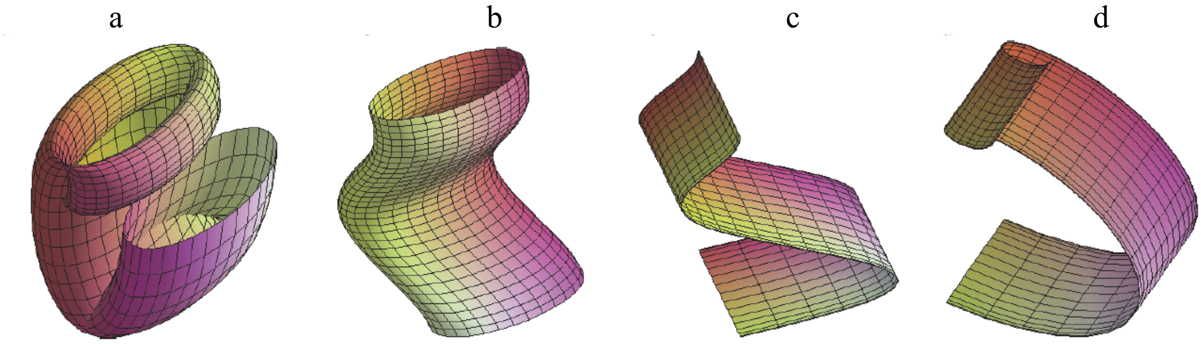



Fig. 10 Orthographic views of surface patch - Minkowski sum of ellipse and conical elliptic helix in $\mathbf{E}^4$
a - view in ${}_{123}\mathbf{E}^3$, b - view in ${}_{124}\mathbf{E}^3$, c - view in ${}_{134}\mathbf{E}^3$, d - view in ${}_{234}\mathbf{E}^3$.

## 5 Conclusions

Abstract algebraic concepts are visualized as abstract geometric forms and figures that can be regarded as certain artistic visualizations of these abstract algebraic relations, concepts and patterns. More dimensional Riemannian manifolds that are generated by Minkowski linear operator in more dimensional spaces and are presented by their orthographic views onto 3-dimensional subspaces represent certain visualizations of creatures from higher dimensions, which can be understood better with adopted artistic imagination and fantasy. To imagine in 4D, we must learn how to read from orthographic 3D views!

**Acknowledgement**

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## References

[1.] GHOSH, P. K.: *A Unified Computational Framework for Minkowski Operations.* In Comput. & Graphics, Vol. 17, No. 4, 1994.
[2.] PETERNELL, M., STEINER, T.: *Minkowski Sum Boundary Surfaces of 3D-Objects*. In Graphical Models, Vol. 69, Issue 3-4 , 2007.
[3.] VELICHOVÁ, D.: *Surface Modelling by Means of Minkowski Sum*. In Aplimat Journal of Applied Mathematics, Vol. II, No. 1, 2009, pp. 165-173.
[4.] VELICHOVÁ D.: *Minkowski Sum in Geometric Modelling*. In Proceedings of the 6th Conference "Geometry and Graphics", Ustroň 2009, Poland, pp. 65-66.

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# NEWS ON SETS OF SPACE-FILLING ZONOTOPES

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**Abstract.** The programmatic base of this contribution is a former conference paper of the author (ICGG 2010 Kyoto [10]). We may cognize spatial and planar tessellations that are new or were not shown up till now. These results may also be used on different fields of art and design as well as in public works.

The 3-dimensional framework (3-model) of any $k$-dimensional cube ($k$-cube) can be produced either based on starting $k$ edges arranged by rotational symmetry or as sequences of strut-chains originated from a separate one whose breakpoints join a single helix. Increasing the number of segments in the strut-chains to large $k$ = n (infinity), continuous helices are created, whose Minkowski sum can be called n-zonotope. Combining $2<j<k$ edges, we can build 3-models of $j$-cubes, as parts of a $k$-cube. The suitable combinations of these zonotope models can result in 3-dimensional space-filling mosaics. The investigated periodical tessellations (up to $k$ =12) always hold the 3-model of the $k$-cube and necessary $j$-cubes derived from it. Such a space-filling mosaic can have a fractal structure as well, since we can replace it with a restructured one, built from multiplied solids. These are composed by addition of 3-models of $k$- and $j$-cubes and are similar to the original ones. The intersections of the mosaics with planes allow unlimited possibilities to produce periodical symmetric plane-tiling. Moving intersection planes result in series of tessellations or grid-patterns transforming into each other which can be shown in various animations.

**Key words:** constructive geometry, hypercube modeling, tessellation, fractal, design

*Mathematics Subject Classification:* 52B10; 52B12, 52B15, 65D17

## 1 Introduction

### 1.1 The Symmetric Base Model of the Hypercube

Modeling of the hypercube has more or less known precedents. Some relating references are listed at the end of the paper and in [6, 10]. The construction's method of the next 3-model is described more detailed in [2, 6, 12].

Lifting the vertices of a $k$-sided regular polygon from their plane, perpendicularly by the same height, and joining with the center of the polygon, we get the $k$ initial edges of the $k$-dimensional cube ($k$-cube) modeled in the three-dimensional space (3-model). From these the 3-models or their polyhedral surface (Figure 1) can be generated by the well known procedure of moving the lower-dimensional elements along edges parallel with the direction of the next dimension [2, 6, 12]. Thus each polyhedron will become a polar zonohedron, more generally a zonotope [6], i.e. a „translational sum” (Minkowski-sum) of some segments [2, 12].

This structure keeps the normal cube's central symmetry and rotational symmetry too. The latter is related to the diagonal joining the starting vertex, referred to the groups of any $j<k$ dimensioned elements. This diagonal is further on called as main diagonal (Figure 1, upper middle). It is also possible to get to the endpoint of the main diagonal from the starting point along easily recognizable strut-chains, whose binding points (the outer vertices of the model) join on one helix each (Fig. 2). The common lead of these helices is the main diagonal. The whole 3-model can be constructed as sequences of strut-chains originated from a separate one whose breakpoints join a single helix. Increasing the number of segments in the strut-chains to large $k$ = n (infinity), continuous helices are created, whose Minkowski sum can be called n-zonotope in which any $k$-dimensional cube's 3-model can be constructed [3, 6].

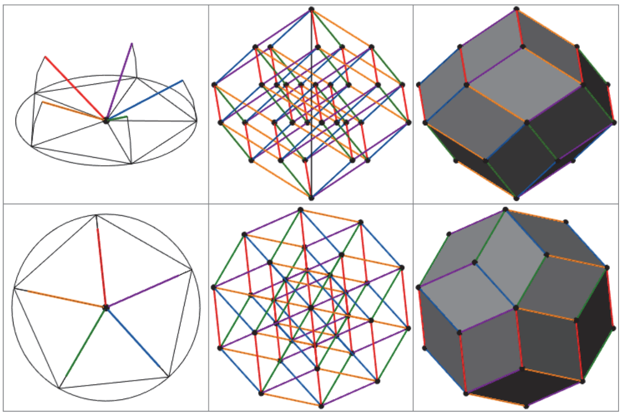

Fig. 1: Our symmetric 3-model of the 5-cube

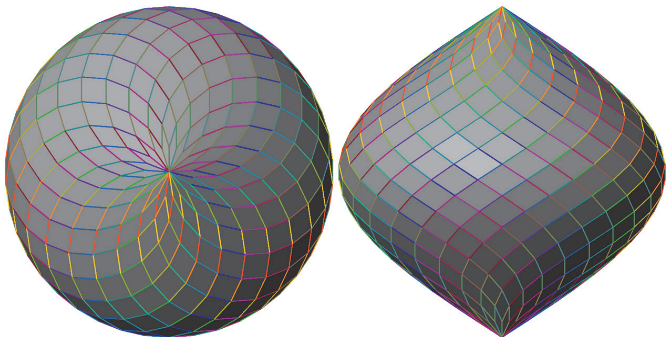

Figure 2: Top and front views of the 3-model of the 21-cube

### 1.2 Systematization of Available and New Results

| Dimension / Method | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|
| Without modification | -<br>2.3 | [10]<br>2, 2.3, 4 | [4]*<br>2.3 | [10]<br>- | [5]*<br>- | [7]*<br>- | - | - | [10]*<br>5.1, 5.2, 6 |
| With each derived $j$-cube | -<br>2.3 | -<br>2.3 | -<br>2.3 | -<br>- | -<br>- | -<br>- | -<br>- | -<br>- | -<br>- |
| With modification | -<br>2.1 | [5] | [2, 10]<br>- | [7]<br>- | -<br>- | [(5), 10]*<br>- | [6]*<br>- | -<br>- | -<br>- |
| $l-1=k$ /<br>$l-o=k$ | -<br>- | [10]<br>- | [(10)]<br>[10] 3 | 2.2<br>- | -<br>- | -<br>[10] | -<br>- | -<br>- | -<br>- |

The above table shows the systematization of our results regarding the modeling of more-dimensional cubes by our way in interest of constructions of space-filling periodical mosaics and plane-tiling patterns gained from these. The references of former results are signed with square brackets. The novelties are inscribed with the numbers of this paper's (sub)sections showing the newer solutions. Round brackets mean that the instances are only mentioned without figures. The starred items have animated planar tessellation series as well, gained from the given spatial mosaics. (See subsection 5.3!) You may find more references about the foregrounds of this paper's topic in [10].

## 2 Space-filling Periodical Tessellations

Combining 2<*j*<*k* initial edges of the above model, we can build 3-models of *j*-cubes, as parts of a *k*-cube (Figures 3a-b). The suitable combinations of these zonotope models can result in space-filling mosaics. Our investigated periodical tessellations (up to *k* =12) always hold the 3-model of the *k*-cube and necessary *j*-cubes derived from it (Figure 4).

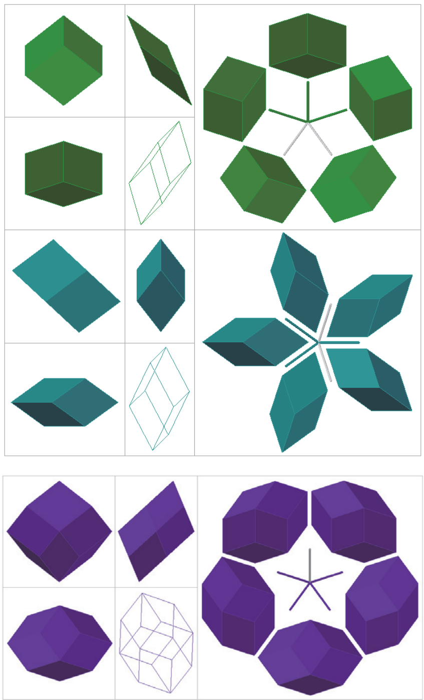

Fig. 3a-b: 3-models of 3- and 4-cubes based on 5 given edges

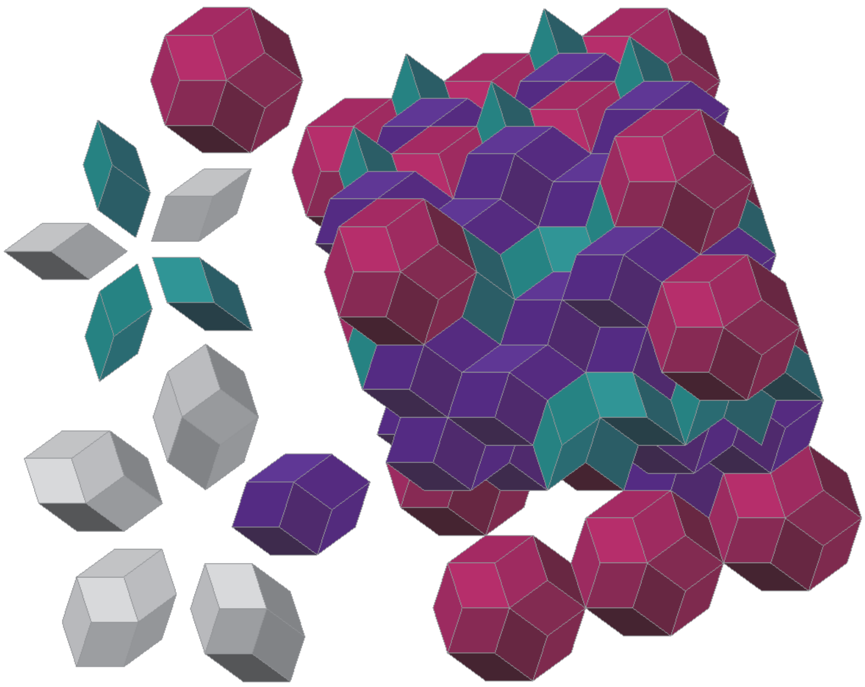

Figure 4: An example of tessellation based on the 3-model of the 5-cube

### 2.1 Mosaics of Modified Models

Due to the former method, it was advisable to originate the construction from a *k*-1 sided polygon in case of odd *k*. The remaining edge is perpendicular to the base plane but its upper endpoint may be in the common plane of the other edges' endpoints. Some sides fall into common planes but will not be identical. This way, we gain a degenerated 3-dimensional axonometric projection of the *k*-cube. This 3-model has similar symmetric properties like in the case of the even *k*+1 and a polygonal contour with even number of sides. Thus in case *k*=5, the hull of the modified model is a rhombi-hexagonal dodecahedron and fill the space itself [5]. The cases of *k* =7 and *k* =9 are described in [5

and 10]. The case of $k$=11 is under research.
The above method is interesting as well if $k$ is even number. We can see the elements and the construction of the space-filling mosaic based on the modified 3-model of the 4-cube in figures 5 and 6, as a simple instance of our topic. The remarkable example with isometric edges for $k$ =6 is the rhombic triacontahedron [2, 9]. In the case of $k$ =10 the 3-model must have more modifications. It is already constructed based on a hexagonal arrangement [6].

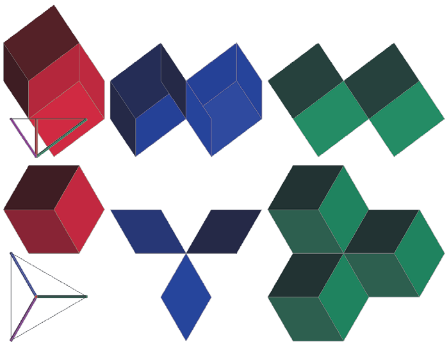

Fig. 5: Elements of the modified 3-model of the 4-cube (front/top)

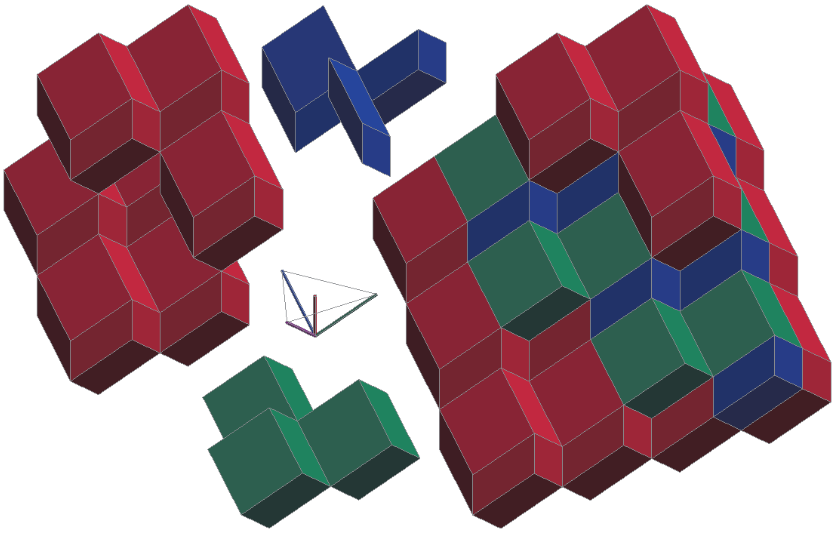

Fig. 6: Space-filling mosaic built from set of the above elements

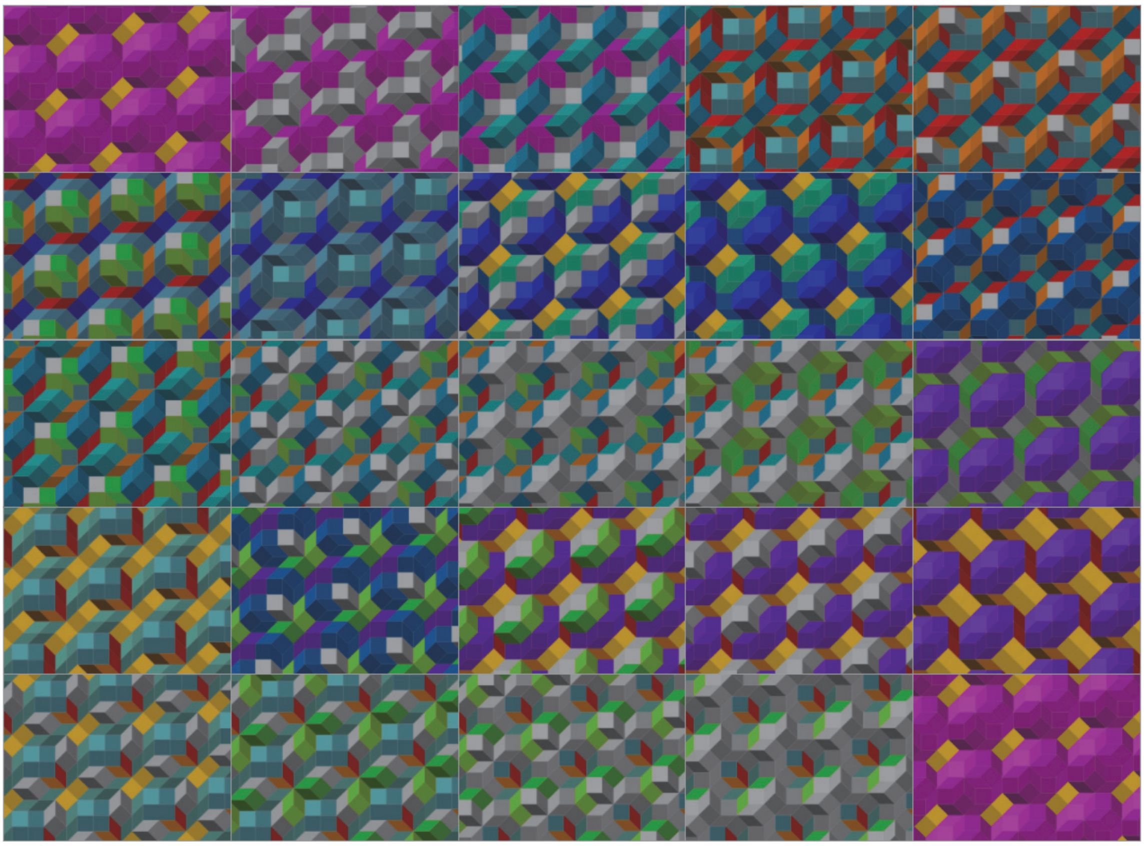

Figure 7: Steps (along meander line) of the tessellation based on the 3-model of the (8-1)-cube

## 2.2 The Base *k*-cube as Part of the (*k*+1)-cube

Nowadays we can create new space-filling mosaics in cases where *k* =5 [1, 2] and *k* =7, deriving the arrangement of the initial edges from the rotationally symmetric 3-models of the 6- and 8-cubes as well, simply omitting the edges parallel to one of the original ones. Other cases of odd *k* are under research. Figure 7 shows the constructive steps of a spatial tessellation based on the (8-1)-cube. This method works in cases of even *k* as well. If *k*=6 for instance, such a spatial mosaic is structured similarly to the tessellation which was described first in [4].

## 2.3 Space-filling Mosaics Based on Models Without Modification

Our periodical tessellations based on the 3-model of the 4-9- and 12-cubes can be already built without any modification as well. The cases of k=10 and k=11 are under research. We can see our newer tessellation for case of *k*=5 in figures 4 and 20. Due to many symmetric properties of the 6-cube's 3-model, we have several space-filling mosaics consisting the derived j-cubes' models in different combinations. The main sections of the compound honeycomb from the newer tessellation of case *k*=12 are showed in subsection 5.1. We may see layers of stones from the same spatial mosaic in subsection 5.2.

Each 3-model of *k*-cubes can be constructed by adding of models of lower-dimensional elements derived from itself [4]. This means that the space-filling mosaics may be restructured according to different points of view. We may enrich, for example, the tessellation of the rhombic dodecahedron models of the 4-cube by this method and by repainting the differently oriented coincident models of the derived 3-cubes. The solution is more difficult if we have the additional condition to use all differently shaped models of *j*-cubes in the spatial tessellation and adjacent stones don't compose models of higher dimensional cubes. The instances for *k*=4, 5 and 6 are shown by plane sections (section 5 and subsection 5.1) in figures 8, 9 and 10.

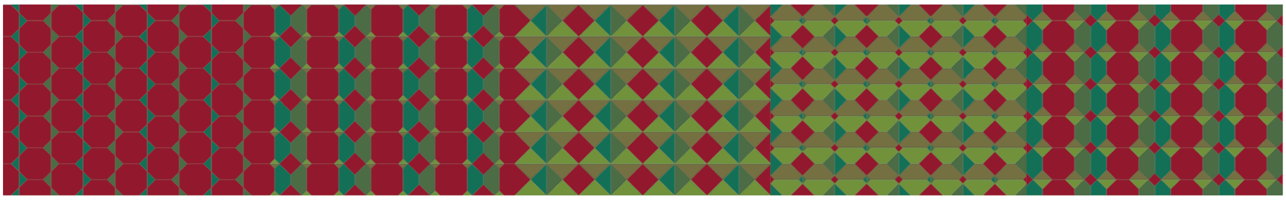

Figure 8

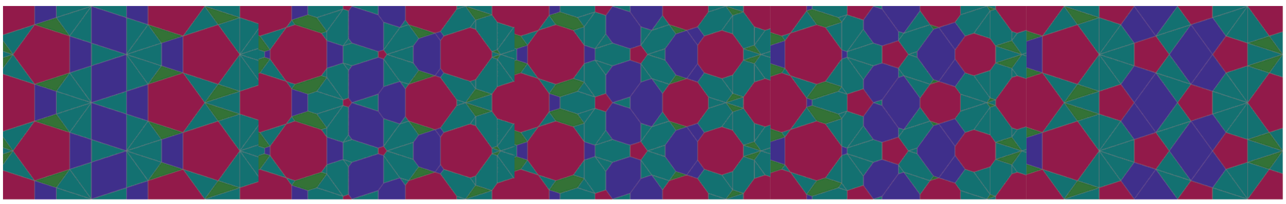

Figure 9

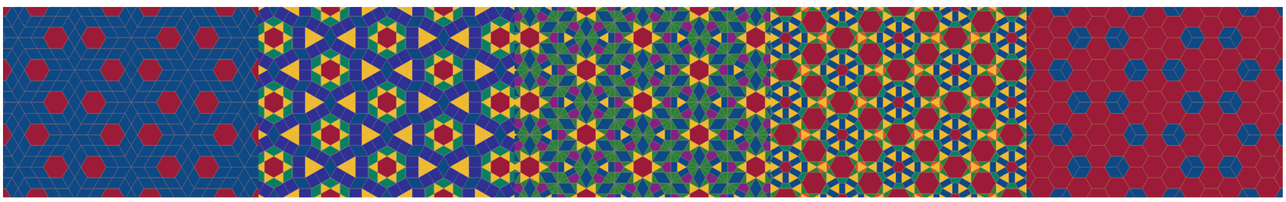

Figure 10

## 3 Sets and Subsets of Space-filling Zonotopes

Generally, we can take the symmetric 3-model of the $l>k$-cube and the lower-dimensional elements described in sections 1 and 2. It is easy to follow: if we find a 3-model of a $k<l$-cube from the above set that can be derived from the former symmetric shape by affine transformations, we can build the earlier tessellations in modified form. It is possible to find other space-filling mosaics as well, created from the subset of the derived elements of the $l$-cube model. Discoveries up to now show space-filling mosaics based on 3-models of the 5- up to 9-cubes, built in modified shapes or structures with selected sets of 343 $j$-elements of the 3-model of the 12-cube [9]. (In consequence of the rotational symmetry, we have to eliminate the differently oriented but corresponding elements.) For instance, the 3-model of the 6-cube can have 80 different shapes on this set. Figure 16 shows in case of $k$=8 that the structure, showed in [4], can be preserved by very differently shaped models as well. This is naturally not true for all subsets. The cases $k$=3 and $k$=4 and some solutions, with sets based on special shapes, are natural. The cases $k$ =10 and $k$ =11 are under research in this respect.

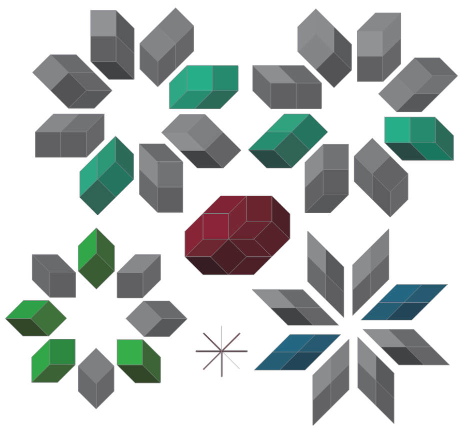

Figure 11: Selecting the usable 3-models of the (8-2)-cube and of derived j-cubes of this (top view)

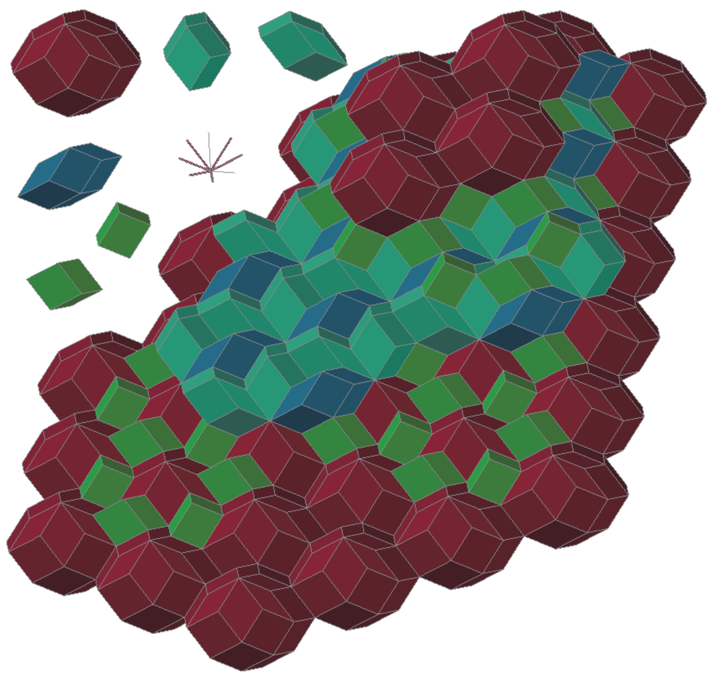

Fig.12: Steps of a tessellation based on a 3-model of the (8-2)-cube ( axonometric projection )

## 4 Fractal Structure in Space-filling Mosaics

Our spatial tessellations can have fractal structure as well, since we can multiply the lengths of the model edges by addition of $k$- and $j$-cubes in order to gain a similar solid to the original one (Figures 13-14). The further similar pairs can be constructed by these compound solids and so on. This means, that we can replace each tessellation with a similar one built from the multiplied solids (Figures 15-16). The models of the $k$- and $j$-cubes can be composed in different ways from models of lower-dimensional $j$-elements [3, 7, 10]. Thus fractals of different rule sets can be created from mosaics of the same base structure. Russel Towle created central mosaics and models with multiplied lengths of edges based on nested zonotopes ordered into the midpoint of the arrangements [1].

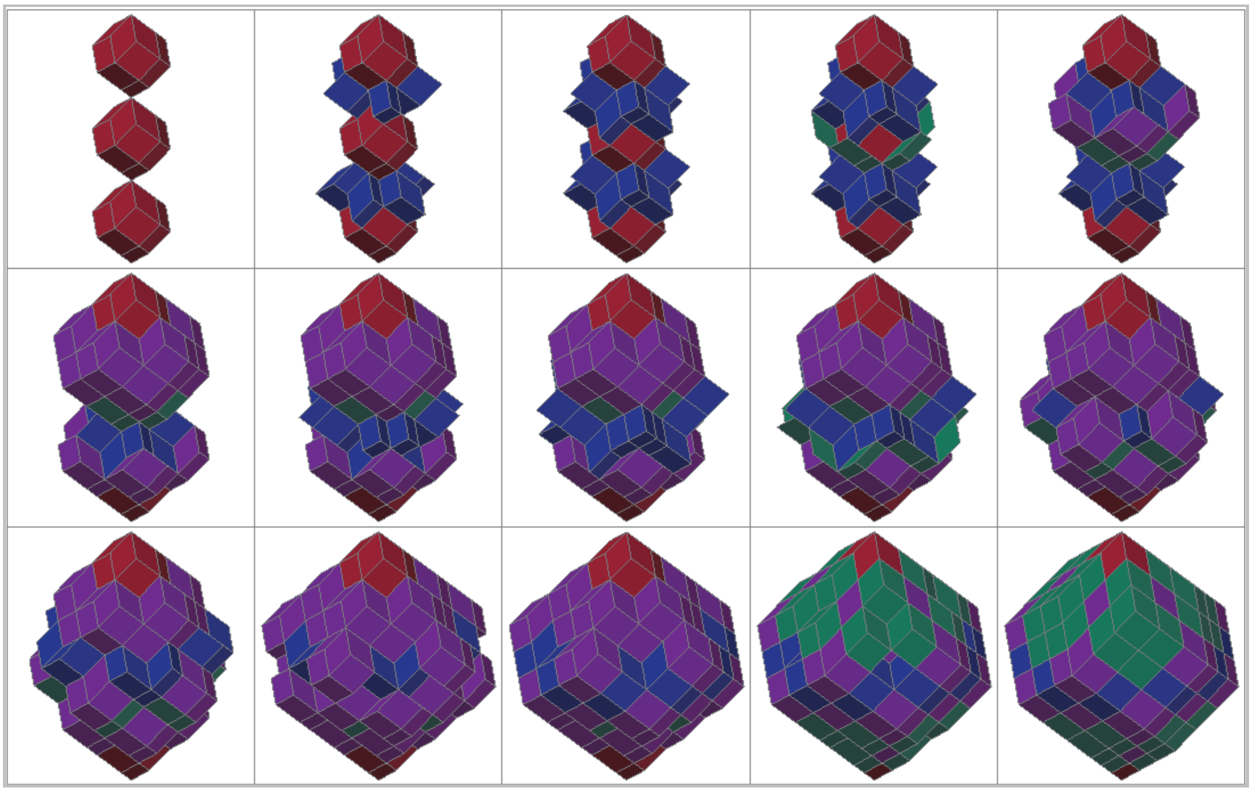

Figure 13: Tripling of the 5-cube's 3-model

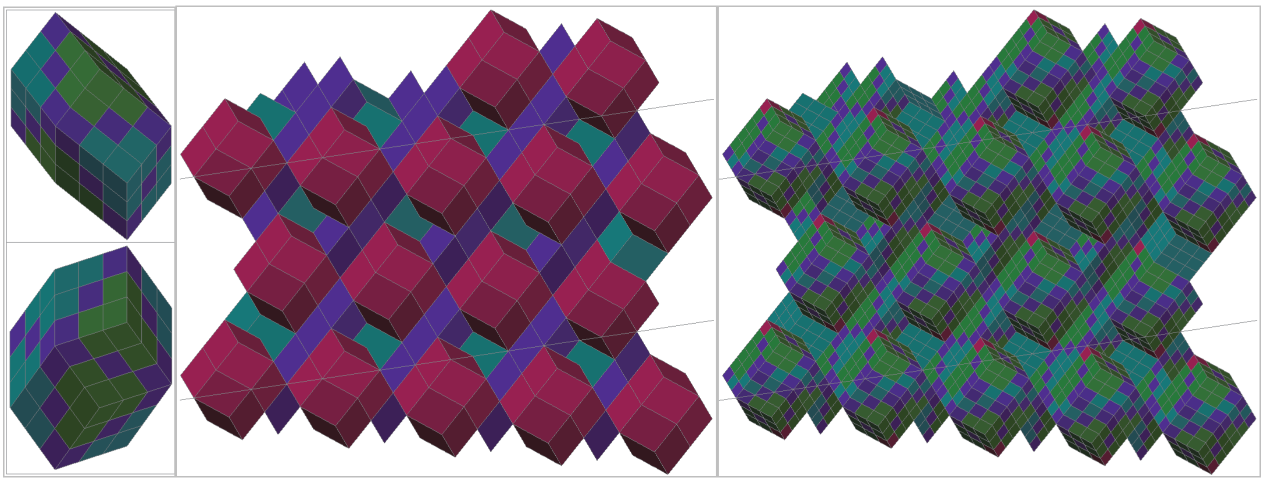

Fig. 14: Tripled 4D part Fig. 15: Mosaic, based on the 5-cube Fig. 16: The same tripled mosaic

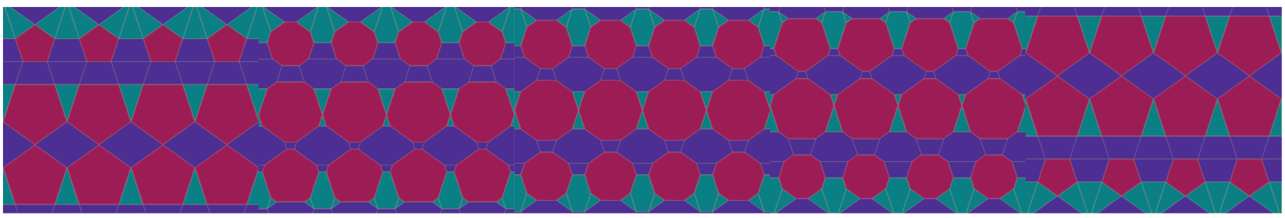

Figure 17: Rotation of a pattern between two main sections of the mosaic showed in figure 15

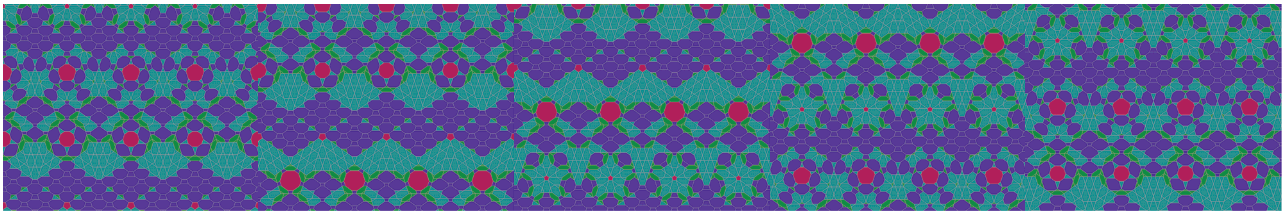

Figure 18: Shift of the same pattern on equidistance planar intersections of the above tripled mosaic

## 5 Plane-tiling patterns

As it follows from the way of our general model's construction, each vertex is held by planes parallel to the initial plane of the construction, therefore a regular plane-tiling appears on these horizontal sections (main sections) of our space-filling solid-mosaic. However these planes can only collect the connecting stones and hold the orthogonal shadows of those. If the intersecting planes are parallel with the oblique sides of the stones, the plane-tilings are well ordered still but we have less regular polygons with odd side-number as well [6]. We can naturally play with the selection of the solids and with the stand and shape of the sectional planes or surfaces. The given pictures can be projected with different methods on different surfaces.

### 5.1 Plane Sections of the Space-filling Mosaics

In figure 19, we can see the symmetrically alternating nineteen different main intersections of the periodically ordered, symmetrically arranged unit tessellations mirrored onto horizontal planes (like a so called compound honeycomb) in our newer space-filling mosaic, based on the 3-model of the 12-cube. The planes of these intersections are perpendicular to the main diagonals of the applied 12-cube models and hold the vertices of the space-filling solids. Generally, all parallel intersecting planes hold the layers of the spatial mosaics, thus we can reconstruct the tessellations on the base of such figures if the planes are spread in appropriate density.

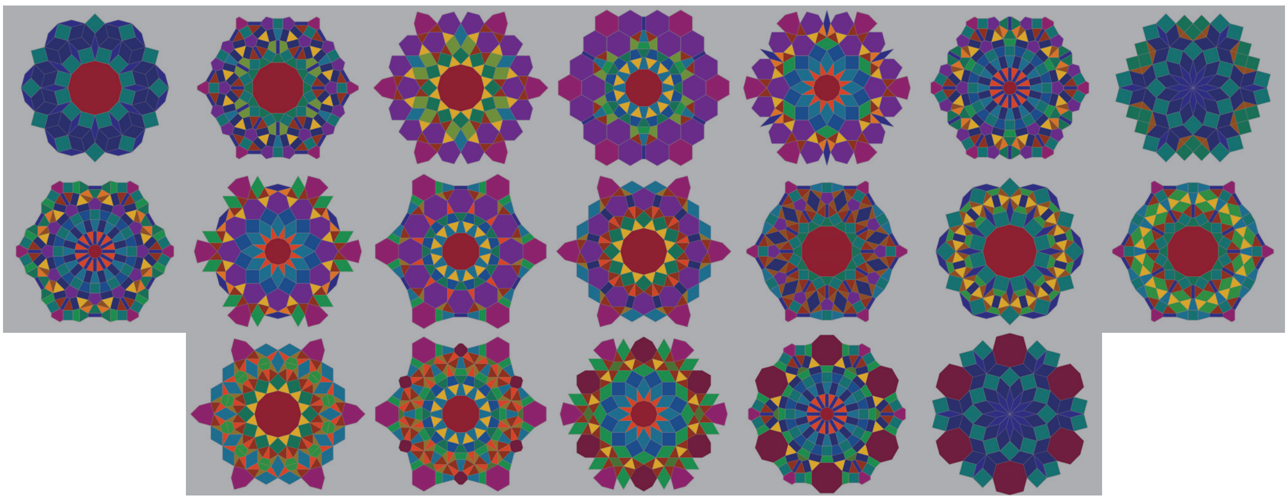

Fig. 19: The 19 main sections of the honeycomb of our newer spatial mosaic based on the 12-cube

We can take only the grid-patterns of the intersections. Different combinations of these result in complex various structures [3 and 7].

### 5.2 Shadows of the space-filling elements

We can select a layer of space-filling stones from the above mosaics. Seeing it simply, these are elements which can be intersected with a common plane. The planar shadows of these layers result in plane-tilings [10]. Figures 20.a-c show three layers of stones from our newer periodical tessellation based on the 3-model of the 12-cube.

Fig. 20.a-c: Three layers of stones from our newer tessellation based on the 3-model of the 12-cube

Other more indirect method is if we take the planar shadows of the $k$-cube's 3-model and these of the derived $j$-cubes as well. This way we gain a set of tiling elements. The method of the distribution becomes free and it is possible to construct fractal like mosaics as well [8, 9, 10].

### 5.3 Animated Plane-tiling

Moving intersection planes result in series of tessellations or grid-patterns transforming into each other which can be shown in various animations. These provide possibilities for exhibitions, publicity work and further usage. You can see different animations of plane-tilings, gained even by oblique planes too, and of combined grid-patterns on this WEB page: http://icai.voros.pmmf.hu

## 6 Art and Design

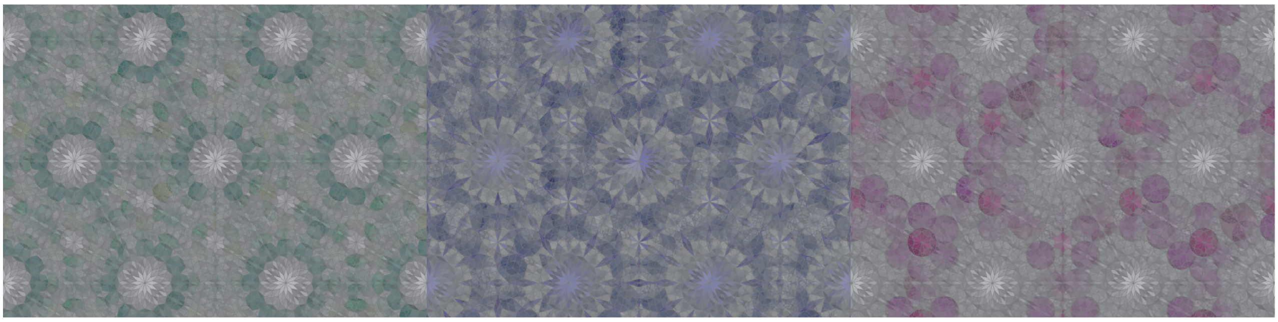

Fig. 21.a-c: Transparent and highlighted stones in our mosaic based on the 3-model of the 12-cube

Planar and spatial symmetry groups are the base of several works in different branches of art. Our symmetric models of the hypercube and the symmetrically arranged periodical tessellations offer several binding points to this field [11]. Constructing the mosaics based on the 3-model of the hypercube and combining the grids of the main sections, we can feel the similarity of the patterns to the structures of the Islamic mosaics [3,7]. The intersections of the compound honeycombs remind us of mandalas. The polyhedral hulls of the models could form domes and cupolas [6]. The edges of the tessellated 3-dimensional models construct spatial lattice girders. Transparency and highlighting of selected elements may result in interesting effects (Figures 21.a-c). We could follow the examples for long, since the approach of the topic described in the paper can generate relations among plane, surface and space based on an elemental problem. The new results can hopefully aid the correspondence among geometry, art and design.

## References


[1.] TOWLE, R.: *Zonotopes,* symmetrical-structures.html, http://zonotopia.blogspot.com, 2008/04

[2.] VÖRÖS, L.:*Reguläre Körper und mehrdimensionale Würfel*, KoG 9, pp. 21-27, 2005 http://master.grad.hr/hdgg/kog/

[3.] VÖRÖS, L.: *A Symmetric 3-Dimensional Model of the Hypercube*, Symmetry: Culture and Science, Volume 17, Nos. 1-2, pp. 75-79, 2006 http://icai.voros.pmmf.hu

[4.] VÖRÖS, L.: *Two- and Three-dimensional Tilings Based on a Model of the Six-dimensional Cube*, KoG 10, pp. 19-25, 2006 http://master.grad.hr/hdgg/kog/

[5.] VÖRÖS, L.: *Two- and Three-dimensional Tiling on the Base of Higher-dimensional Cube Mosaics,* Proceedings of the 7th International Conference on Applied Informatics, Eger, Hungary, January 28 - February 2, 2007, Vol.1, pp. 185-192, 2007

[6.] VÖRÖS, L.: *N-Zonotopes and their Images: from Hypercube to Art in Geometry*, Proceedings: Library of Congress Cataloguing-in-Publication Data : Weiss, Gunter (editor on chief) and ISGG = Proceedings of the 13th ICGG Dresden, August 4-8, 2008 ISBN 978-3-86780-042-6 electronic book / ISGG (ed.) - Gunter Weiss (ed.). - Dresden: ERZSCHLAG GbR, Aue, 2008

[7.] VÖRÖS, L.: *Space-filling Mosaics and (Animated) Plane-tiling Patterns*, Symmetry: Culture and Science, Volume 20, Nos. 1-4, pp. 297-308, 2009 http://icai.voros.pmmf.hu

[8.] VÖRÖS, L.: *Planar Tesselations Based on Shadows of More-dimensional Cubes,* Proceedings-CD of monGEometrija 25th National and 2nd International Conference for Geometry and Engineering Graphics, Beograd, 24-27 June 2010, editor: M. Nestorovic, ISBN 978-86-7924-038-5, 2010

[9.] VÖRÖS, L.: Specialties of Models of the 6-dimensional Cube, Proceedings of Bridges Pécs: Mathematics, Music, Art, Architecture, Culture, editors: G. W. Hart and R. Sarhangi, isbn: 978-0-98460-420-3, issn: 1099-6702,Tessellations Publishing, Phoenix, pp: 353-358, 2010

[10.] VÖRÖS, L.: *Sets of Space-filling Zonotopes with Connection to Art and Design,* Proceedings-DVD of the 14th International Conference on Geometry and Graphics, August 5-9, 2010, Kyoto, ISBN 978-4-9900967-1-7, ©2010 International Society for Geometry and Graphics Article No. 231, 2010

[11.] VÖRÖS, L.: *Art in Shadows of the Six-Dimensional Cube,* Proceedings of Bridges Coimbra: Mathematics, Music, Art, Architecture, Culture, chief ed: R. Sarhangi and C. Sequin ISBN: 978-0-9846042-6-5, ISSN: 1099-6702, Tessellations Publ., Phoenix, pp. 257-262, 2011

[12.] http://en.wikipedia.org/wiki/Hypercube


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# METALLIC SPIRALS

**WINITZKY DE SPINADEL, Vera Martha (ARGENTINA RA)**

**Abstract.** The members of the Metallic Means Family (MMF) are positive quadratic irrational numbers This family was introduced by the author in 1998 [1]. The most prominent member of this family is the well known Golden Mean. Among its relatives, let us mention the Silver Mean, the Bronze Mean, the Copper Mean, the Nickel Mean, etc. Attached to them, we have recently introduced a family of Metallic Spirals [2], from which the "golden spiral" is well known. Then our next aim was to construct a "silver spiral". For that we begun taking in the continued fraction expansion of the Silver Mean, the rational approximate values and connect them with a sequence of gnomonical rectangles. This procedure can be generalized to be applied to every member of the subfamily of the MMF, which have a purely periodic continued fraction expansion.

**Key words**. Metallic Means Family, Golden Mean, Silver Mean, continued fraction expansion,

*Mathematics Subject Classification:* 11A55, 11J70, 11K50, 30B70, 40A15

## 1. Introduction

The Metallic Means Family (MMF) was introduced by the author [1] as the family of positive irrational numbers $\sigma_p{}^q$ which are solutions of the equation

$$x^2 - px - q = 0 \tag{1.1}$$

where $p$ and $q$ are natural numbers.

The MMF is divided into two subfamilies:

I) the positive solutions $\sigma_p{}^1$ of equation (1,1) with $q = 1$

$$x^2 - px - 1 = 0 \tag{1.2}$$

II) the positive solutions $\sigma_1{}^q$ of equation (1.1) with $p = 1$

$$x^2 - x - q = 0\,. \tag{1.3}$$

It is easy to verify that all the members of $\sigma_p^{\,1}$ have a purely periodic continued fraction expansion of the form

$$x = [n, n, ...] = [\overline{n}] \tag{1.4}$$

In fact, if we take $p = 1$, we get $x^2 - x - 1 = 0$, which can be written $x^2 = x + 1$ and dividing by $x \neq 0$ both members, we have the equation $x = 1 + \frac{1}{x}$. Replacing iteratively the value of $x$ we obtain

$$x = [\overline{1}] = \sigma_1^{\,1} \tag{1.5}$$

a purely periodic continued fraction which is equal to $\phi = \frac{1+\sqrt{5}}{2}$ = 0,618..., the well known Golden Mean.
Taking $p = 2$ we find the value

$$\sigma_2^{\,1} = [\overline{2}] \tag{1.6}$$

that is called the Silver Mean $\sigma_{Ag} = 1 + \sqrt{2}$ = 2,414....
For $p = 3$ we have

$$\sigma_3^{\,1} = [\overline{3}] = \frac{3+\sqrt{13}}{2} = 3{,}3027... \tag{1.7}$$

that is the Bronze Mean $\sigma_{Br}$ and so on. This subfamily is called the PPMMF (purely periodic Metallic Means Family).

With respect to the second subfamily, which members are $\sigma_1^{\,q}$, it is easy to verify that they have a periodic continued fraction expansion, not purely periodic, of the form

$$\sigma_1^{\,q} = [m, \overline{n_1, n_2, ...}]. \tag{1.8}$$

Some of the members of this subfamily, denoted by PMMF (periodic Metallic Means Family) are integers. In example, the first one $\sigma_1^{\,2} = [2, \overline{0}] = 2$ is called the Copper Mean and the next integers appear in quite a regular way [2].

The first non-integer Metallic Mean is obtained solving the quadratic equation $x^2 - x - 3 = 0$, which positive solution is

$$x = \frac{1+\sqrt{13}}{2} = [2,3,3,\cdots] = [2,\overline{3}]$$

that is a periodic continued fraction expansion.

All the non-integer Metallic Means of PMMF share the property of being “palindromic” about their centers, except for the last digit of the period, which equals $2m - 1$, as can be noticed in the following values:

$$s_1^4 = \left[2,\overline{1,1,3}\right]; s_1^5 = \left[2,\overline{1,3}\right]; s_1^6 = \left[3,\overline{0}\right] = 3; s_1^7 = \left[3,\overline{5}\right]; s_1^8 = \left[3,\overline{2,1,2,5}\right];$$
$$s_1^9 = \left[3,\overline{1,1,5}\right] : s_1^{10} = \left[3,\overline{1,2,2,1,5}\right]; s_1^{11} = \left[3,\overline{1,5}\right] : s_1^{12} = \left[4,\overline{0}\right] = 4 \qquad (1.9)$$

All the members of the MMF enjoy common mathematical properties that are fundamental in the actual research in every type of design, on the stability of micro- and macro-physical systems, going from the DNA geometrical internal structure to the astronomical galaxies, on the search of universal roads to chaos in the analysis of non linear dynamical systems and so on.

## 2. Construction of sequences of gnomonic figures

There are many references to the "golden spiral", that is, to the spiral associated to the Golden Mean. Following a similar schema for the construction of this spiral, we are going to design metallic spirals associated to other members of the MMF.

We begin with a square and add to it a "gnomon" (term introduced by Aristoteles) which is a figure that when it is pasted to the original figure produces a similar figure to the original. In such a way, a sequence of gnomonic rectangles are obtained, formed by the union of adjacent squares.

We are going to apply this procedure to construct a sequence of gnomonic rectangles.

**A)** Let us begin with a unitary square and add to it a sequence of squares such that the side of every new square is equal to the sum of the two precedent. The original rectangle is formed by the sum 1+1 and then 1+2 gives 3 and so on, following the numerical sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, etc., known as Fibonacci sequence. This sequence has the mathematical property that the ratio of two subsequent terms of it tend to $\phi$. With this sequence of golden rectangles (Figure 2) , it is easy to draw a goleen spiral, as it is shown at Figure 3.

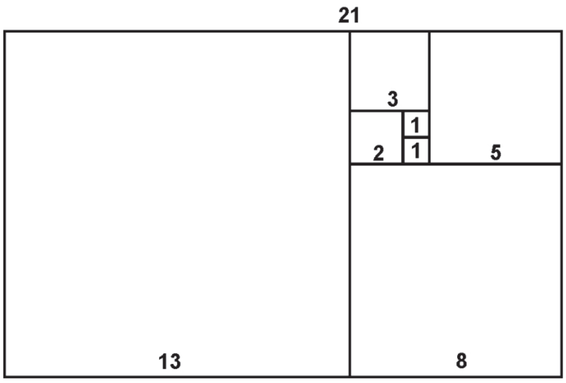



Figure 2

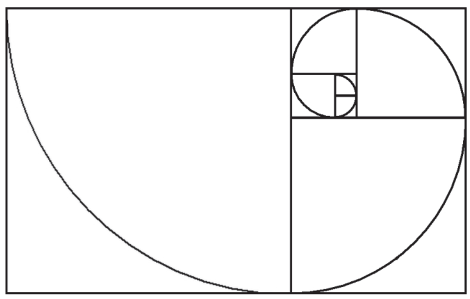


Figure 3

Similarly, we can draw a silver spiral, beginning with a sequence of silver rectangles , in which case the growth will be performed through the sum of two similar squares which sides have the length of the longest one of the preceding rectangle. Beginning with the rectangle 1:2 we add two squares of side 2 and then 1+2+2=5. to the rectangle 2:5, we add now two squares of side 5 and then we have 2+5+5=12, etc. (Figure 4).

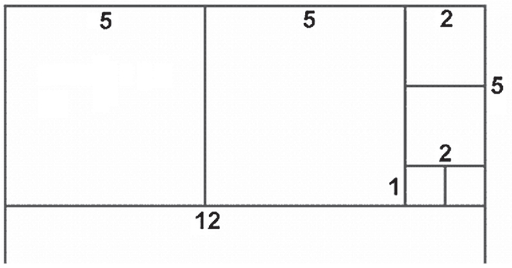



Figure 4

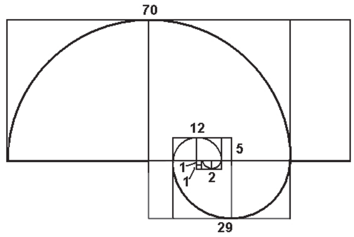



Figure 5

The numerical sequence is in this case 1, 1, 2, 5, 12, 29, 70,… and the ratio between two consequent members of this sequence tends to the Silver Mean, which we shall denote by $\theta$, for simplicity. At Figure 5 the corresponding silver spiral is drawn.

**B)** In this case, we begin with a golden rectangle $1:\phi$ and drawing arcs of circumference of length $\pi/2$ that are inscribed in a sequence of squares, we obtain again a golden spiral, like it is shown at Figure 6. It is easy to prove mathematically, like we have already done at Reference [3], that this spiral is inscribed in a rectangle of surface $\phi$ and the spiral has a total length $L = \frac{\pi}{2}(1+\phi)$.
Take into account that the length of the spiral is equal to the arc multiplied by $1 + \phi$.
Proceeding in a similar way with a silver rectangle, we have drawn a silver spiral shown at Figure 7, arriving to similar algebraic results as in the previous case: the surface of the rectangle is $\theta$ and the length of the spiral $L = \frac{\pi}{2}(1+\theta)$.

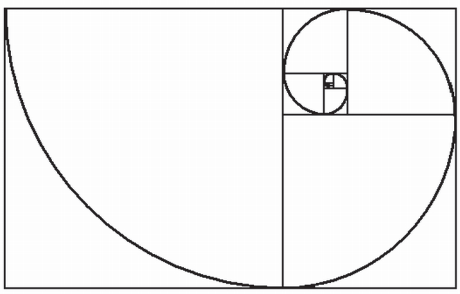


Figure 6

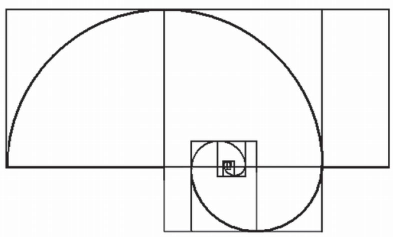


Figure 7

**C)** Following a similar procedure and starting with a bronze rectangle, we can construct a bronze spiral, like it is shown at Figure 8 and Figure 9.

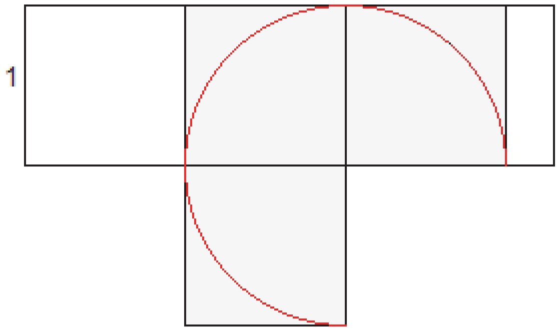



Figure 8

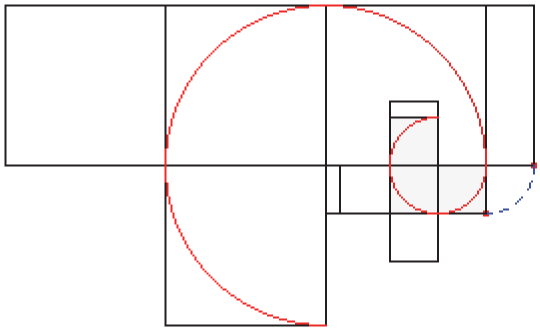
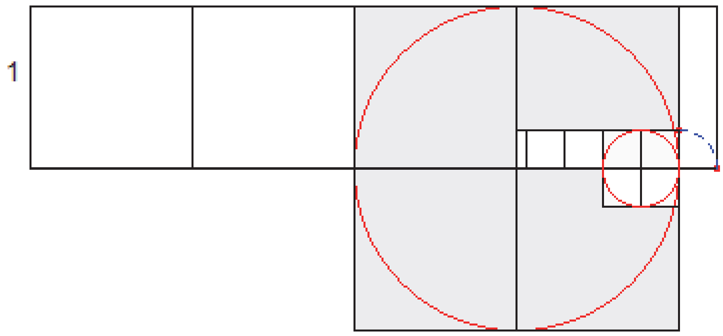


Figure 9

If we repeat the procedure for the fourth member of the PPMMF, $\sigma_4^1 = 2+\sqrt{5} = [\bar{4}]$, the arcs which form the spiral would be 4 quarters of circumferences and the spiral becomes a sequence of tangent circles (Figure 10)



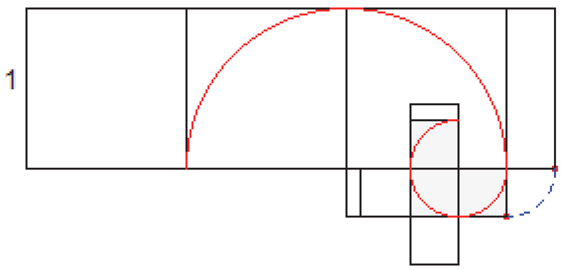

$$L_{\sigma_4^1} = \frac{2\pi}{1-\left(\sigma_4^1-4\right)} = \frac{\pi}{2}\left(1+\sigma_4^1\right)$$

$$S_{\sigma_4^1} = \frac{4}{1-\left(\sigma_4^1-4\right)^2} = \sigma_4^1$$

Figure 10

The construction is also possible for $\sigma_p^1$, $p = 5,6,\ldots$ but obviously the obtained figures are not spirals in a strict sense and we are going to dismiss them.

Notice that if we eliminate the first arc of the bronze spiral, the resultant form that is shown in Figure 11 is a *nickel spiral*, related to the Nickel Mean.




$$\sigma_{Ni} = \frac{1+\sqrt{13}}{2} = \sigma_{Br} - 1$$

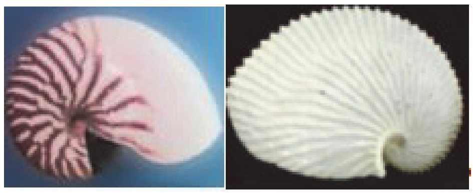
$$L_{\sigma_{Ni}} = \frac{\pi}{2}\left(1+\sigma_{Ni}\right) \qquad S_{\sigma_{Ni}} = \sigma_{Ni}$$

Figure 11

## 2. Spiral designs in Nature

Many sea animals have a shell. Normally, this shell is exterior, like in the case of the organic forms shown in Figure 12, Neverita Josephinia and urchin at the left and Nautilus and Argonaut at the right. Sometimes, this shell is internal or has disappeared, like in the case of the cephalopods (calamari's, octopus, cuttlefish, etc.). The *Nautilus*, is a special sea animal because it is the only cephalopod with an external shell. It was a sea animal very abundant in the Paleozoic Period. Nowadays, there only four types that live in the Pacific Ocean and the Indian Ocean. A wonderful example of the Golden spiral is precisely the Nautilus shell [4] and is the best example of the presence of mathematical elements in Nature, indicating a remarkable harmonic growth.

Neverita Josephinia and urchin



Nautilus and Argonaut

Figure12

However, this well known fact is an exception because if we take a look at the shell of other sea animals, we find similar spirals to that of the Nautilus, more or less rounded but with ratios less than the Golden Mean $\phi$.

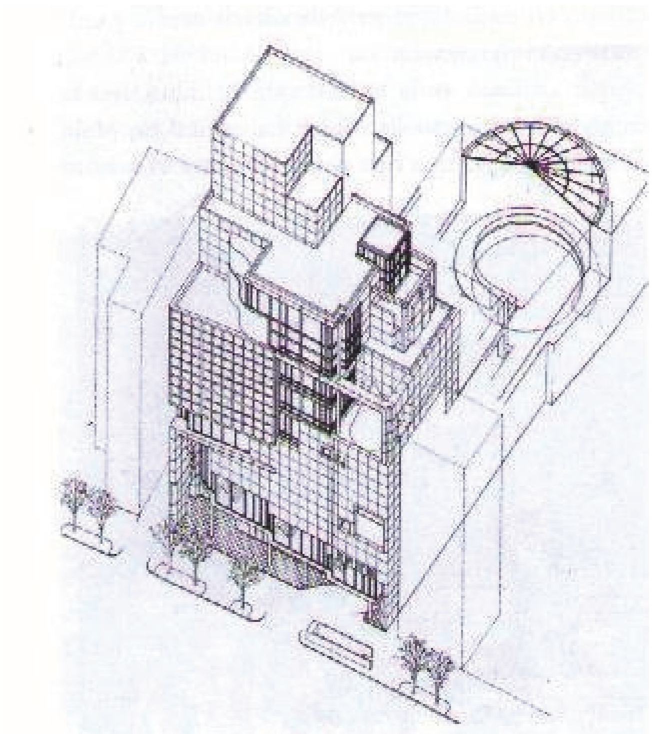
In 2005 (see [5]), C. Falbo, from the California Academy of Sciences of San Francisco, USA, studying the shells of the Nautilus Pompilius, found ratios varying in the range 1,24 to 1,43, with an average equal to 1,33.

The *Argonaut*, also called *paper nautilus*, shown in Fig. 12, is another cephalopod which deserves to be considered. The female develops temporarily a fragile shell to protect its eggs. It is a very good example of fast growth. We do not have a section of this shell, but looking at the figure, it seems to be very similar to some metallic spiral, perhaps a silver spiral…

Finally, in the field of architectonic design, it is interesting to mention the building *Spiral*, designed in 1985 by the Japanese architect Fumihiko Maki in Tokyo, Japan [6]. Maki was honored with the Pritzker Prize in 1993 and in his building S*piral* has used the geometry of this curve that is a marvelous symbol of two concepts: fragment and unattainable centre.

The geometrical figure is an evocation of the spirals found at Kyoto, in the famous *Ginkakuji Temple (Silver Pavilion)* 1338-1573 and in the *Kinkakuji Temple (Golden Pavilion)* 1398, reconstructed in 1955.

Even when these names of Silver and Gold have a religious and historical meaning, they could be an example for design using metallic spirals, as plane curves as well as metallic helicoids.



Axonometría. Edificio *Espiral*, F. Maki.

**References**


[1] Vera W. de SPINADEL, “The Metallic Means and Design”, Nexus II: Architecture and Mathematics. Editor: Kim Willliams. Edizioni dell´Erba, pp. 143-157, ISBN 88-86888-13-9, 1998.

[2] Vera W. de SPINADEL, “From the Golden Mean to Chaos”. 1st ed. 1998 Nueva Librería, Buenos Aires, Argentina, ISBN 950-43-9329-1. 2nd ed. 2004 Nobuko, Buenos Aires, Argentina, ISBN 987-1135-48-3. 3rd ed. 2010 Nueva Librería, ISBN 978-987-1104-83-3.

[3] Vera W. de SPINADEL and Antonia Redondo BUITRAGO, “Sobre los sistemas de proporciones áureo y plástico y sus generalizaciones”, Journal of Mathematics & Design, vol. 9, No. 1, pp. 15-34. ISSN 1515-7881, 2009.

[4] Antonia Redondo BUITRAGO, “Polygons, diagonals and the Bronze Mean”, Nexus Network Journal, Architecture & Mathematics. Editor: Kim Williams, Vol.9, 2007.

[5] C. FALBO, “The Golden Ratio-A contrary Viewpoint”, The College Mathematics Journal, vol. 32, No. 2, 123-134, 2005.

[6] Félix Ruiz de la PUERTA y Juana Sánchez GONZÁLEZ, “La Espiral en la Arquitectura”, Editorial Mairea Libros, Escuela Técnica Superior de Arquitectura, Madrid, España, 2007, ISBN 84-935278-8-2.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# CRITICAL POINTS OF DIFFERENTIAL SYSTEMS IN *MATHEMATICA*

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**Abstract:** In this contribution we deal with some features and aspects of teaching linear differental systems, namely critical points and their classification and properties for some special types of linear systems, with the aid of the programming system *MATHEMATICA*.

**Key words:** autonomous systems, linear differential systems, stable, asymtotically stable, stable and attractive and unstable critical points, proper and impropre nodes, saddle, center and spiral points

## 1. Introduction

In the presented paper we want to show how to apply the programming system *MATHEMATICA* for teaching linear differential systems, especially for investigating critical points of homogeneous linear differential systems of two equations with constant coefficients. We have used the programming system at our department of mathematics for several years in mathematical subjects in basic bachelor programs, as well as in some special subjects of advanced mathematics in master graduate programs.

Many of our students, who learned to work with this system at the begining of their study, use it for the elaboration of their individual works, presented in various competitions, bachelor works, diploma projects and others. We have many positive experiences ([2]-[5]) and reactions, not only from students.

Utilizing very good visual and graphic abilities, mainly in the higher versions of this programing system, we use it for a little unconventional procedures in solving various problems.

Initially we started to work with the version *MATHEMATICA* 2, then we gradually introduced to education new upgraded versions and we utilized all available innovations appropriate and applicable to our study. At present time we use the version *MATHEMATICA* 7.

One of advantages of this programming system, utilized in what follows, is that it allows to draw „effortlessly“ a greater number of integral curves, it means graphs of investigated particular solutions. With the aid of these graphs students are able to verify properties of the found solutions, especially their behaviour in a neighbourhood of critical points.

## 2. Preliminaries

First let us consider a general system of two differential equations of the form

$$x' = F(x, y)$$

$$y' = G(x, y)$$

where the unknown functions $x$ and $y$ depend on an independent variable *t*. Such a system, in which the independent variable does not occur explicitely is called an **autonomous system**. Solutions of an autonomous system can be investigated in the *xy*-plane. This approach, among other benefits, includes considerations on stability of solutions.

Any solution *x(t), y(t)* represents a plane curve *C,* called a solution curve, or trajectory of the system. If *t* is time and *C* the path of a moving point, then the sence of increasing time is the sence in which the point moves in the progressing time along *C*. The main topic of this contribution is applying the programming system *MATHEMATICA* for investigation of solutions behaviour in a neighbourhood of so called **critical points**, it means points, at which both functions *F* and *G* assume the value zero.

There are defined different types of critical points, they can be classified in different ways, for example with respect to their stability, as **stable, asymtotically stable, stable and attractive and unstable** critical points, or with respect to the geometric shape of the trajectory in their neighbourhood, as **node, saddle, center and spiral** points.

Since it is not our purpose to study the problem of critical points in general, we are not going to specify here precise individual definitions of different types of critical points. A more complete theory may be obtained for example by [1].

It can be proved (see [1]), that in most practical cases the problem of identifying the type of an isolated critical points can be done by linearizations, therefore, under some requirements and conditions, by studying the following type of linear systems with constant coefficients:

$$x' = a_1 x + b_1 y$$

$$y' = a_2 x + b_2 y$$

It is well known, that solutions of such a linear system depend on the type of its **eigenvalues**, which means on roots of the quadratic equation

$$\lambda^2 - \left(a_1 + b_2\right)\lambda + a_1 b_2 - a_2 b_1 = 0\,,$$

This equation is called the **characteristic equation** of the system. In our contribution we restrict ourselves to the case if discriminant of characteristic equation is not positive, therefore

$$a_1^{\,2} + b_2^{\,2} - 2a_1 b_2 + 4a_2 b_1 \le 0\,,$$

hence, that the charactristic equation has a couple of complex conjugate roots, or a double real root. In other words, given linear system has a couple of complex conjugate eigenvalues, or a double real eigenvalue.

In what follows we solve differential systems by means of the command **DSolve** and then, combining commands **Table, ParametricPlot, Evaluate** and **VectorPlot** we draw several integral curves, in other words trajectories of given system, included in correspondig direction field.

## 3. Linear systems with complex conjugate eigenvalues

If the characteristic equation of the given system has a couple of complex conjugate roots, then the type of its critical points depends on the value of the real part of complex eigenvalues. This can be zero, positive or negative.

Consider the following system.

$$x' = y$$
$$y' = -x$$

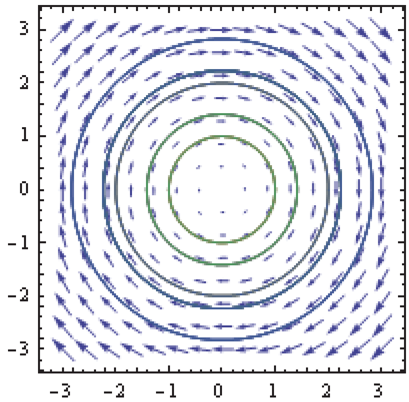
```
In[1]:= sys = {x'[t] == y[t], y'[t] == -x[t]};
        DSolve[sys, {x[t], y[t]}, t]

Out[2]= {{x[t] → C[1] Cos[t] + C[2] Sin[t], y[t] → C[2] Cos[t] - C[1] Sin[t]}}
```

Since this system has a couple of purely imaginary eigenvalues $r_{1,2} = \pm i$, it means the real part is zero, therefore its critical point is a stable, or „neutraly stable" center.

```
In[3]:= p = Table[{c1*Cos[t] + c2*Sin[t], - c1*Sin[t] + c2*Cos[t]}, {c1, -2, 2, 1},
           {c2, -2, 2, 1}];
        pa = ParametricPlot[Evaluate[p], {t, 0, 2 Pi}];
        vp = VectorPlot[{y, -x}, {x, -3, 3}, {y, -3, 3}];
        Show[vp, pa]
```

Out[6]=



The following two systems have also complex eigenvalues, but with non-zero real parts.

.

First let us have the system.

$$x' = -7x + y$$
$$y' = -2x - 5y$$

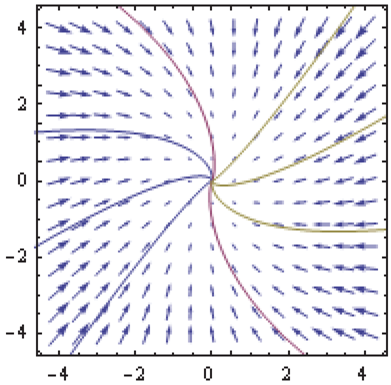
```
In[7]:= sys = {x'[t] == -7 x[t] + y[t], y'[t] == -2 x[t] - 5 y[t]};
        DSolve[sys, {x[t], y[t]}, t]

Out[8]= {{x[t] → e^(-6 t) C[1] (Cos[t] - Sin[t]) + e^(-6 t) C[2] Sin[t],
          y[t] → -2 e^(-6 t) C[1] Sin[t] + e^(-6 t) C[2] (Cos[t] + Sin[t])}}
```

This system has a couple of complex cojugate eigenvalues $r_{1,2} = -6 \pm i$, therefore the real part is negative, therefore its critical point is a stable or „asymptotically stable" spiral point.

```
In[9]:= p = Table[{e^(-6 t) c1 * (Cos[t] - Sin[t]) + e^(-6 t) c2 * Sin[t],
        -2 e^(-6 t) c1 * Sin[t] + e^(-6 t) c2 * (Cos[t] + Sin[t])}, {c1, -1, 1, 1}, {c2, -1, 1, 1}];
    pa = ParametricPlot[Evaluate[p], {t, -Pi / 4, Pi / 4}];
    vp = VectorPlot[{-7 x + y, -2 x - 5 y}, {x, -4, 4}, {y, -4, 4}];
    Show[vp, pa]
```

Out[12]=




And finally consider the system:

$$x' = 2x - y$$
$$y' = x + 2y$$

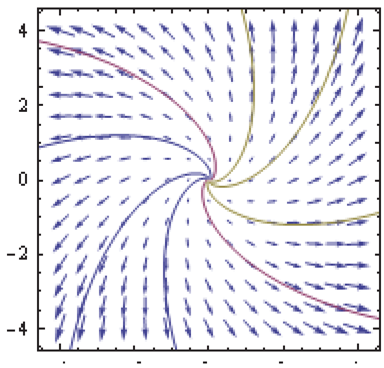
```
In[13]:= sys = {x'[t] == 2 x[t] - y[t], y'[t] == x[t] + 2 y[t]};
     DSolve[sys, {x[t], y[t]}, t]
```

Out[14]= $\{\{x[t] \to e^{2t} C[1] \, Cos[t] - e^{2t} C[2] \, Sin[t], \; y[t] \to e^{2t} C[2] \, Cos[t] + e^{2t} C[1] \, Sin[t]\}\}$

In this case $r_{1,2} = 2 \pm i$, therefore the real part of complex conjugate eigenvalues is positive, therefore the critical point is an unstable spiral point.

```
In[15]:= p = Table[{e^(2 t) c1 * Cos[t] - e^(2 t) c2 * Sin[t], e^(2 t) c2 * Cos[t] + e^(2 t) c1 * Sin[t]},
        {c1, -1, 1, 1}, {c2, -1, 1, 1}];
     pa = ParametricPlot[Evaluate[p], {t, -Pi / 2, Pi / 2}];
     vp = VectorPlot[{2 x - y, x + 2 y}, {x, -4, 4}, {y, -4, 4}];
     Show[vp, pa]
```

Out[18]=



## 4. Linear systems with double real eigenvalues

If the characteristic equation of the linear system has a double real root, then the type of its critical point depends on the number of its eigenvectors. If there exist two eigenvectors (in the case of so called „split systems“), then the critical point is a proper node, which is unstable, if the eigenvalue is positive and stable or „asymptotically stable“, if it is negative.

Let us have the system.

$$x' = x$$
$$y' = y \, .$$

```
In[19]:= sys = {x'[t] == x[t], y'[t] == y[t]};
         DSolve[sys, {x[t], y[t]}, t]
Out[20]= {{x[t] → e^t C[1], y[t] → e^t C[2]}}
```

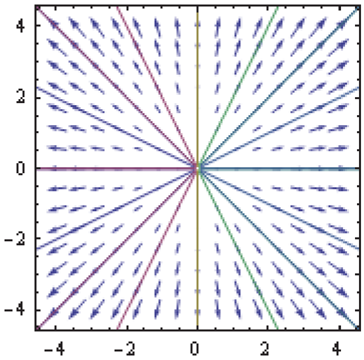
This system has a double real eigenvalue $r_{1,2} = 1$ and two eigenvectors $\vec{v}_1 = (0,1)$ and $\vec{v}_2 = (1,0)$, what can be confirmed by the command **Eigensystem**:

```
In[21]:= Eigensystem[{{1, 0}, {0, 1}}]
Out[21]= {{1, 1}, {{0, 1}, {1, 0}}}
```

Hence the critical point is an unstable proper node.

```
In[22]:= p = Table[{e^t c1, e^t c2}, {c1, -2, 2, 1}, {c2, -2, 2, 1}];
         pa = ParametricPlot[Evaluate[p], {t, -3, 3}];
         vp = VectorPlot[{x, y}, {x, -4, 4}, {y, -4, 4}];
         Show[vp, pa]
```

Out[25]=




Now let the double eigenvalue be negative:

$$x' = -2x$$
$$y' = -2y$$

```
In[26]:= sys = {x'[t] == -2 x[t], y'[t] == -2 y[t]};
         DSolve[sys, {x[t], y[t]}, t]
Out[27]= {{x[t] → e^(-2 t) C[1], y[t] → e^(-2 t) C[2]}}
```

This system has a double real eigenvalue $r_{1,2} = -2$ and again two eigenvectors $\vec{v}_1 = (0,1)$ and $\vec{v}_2 = (1,0)$:

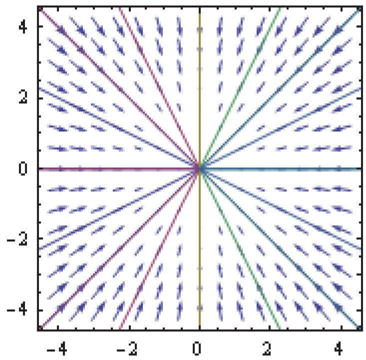
```
In[28]:= Eigensystem[{{-2, 0}, {0, -2}}]

Out[28]= {{-2, -2}, {{0, 1}, {1, 0}}}
```

Thus the critical point is a stable or „asymptotically stable“ proper node.

```
In[29]:= p = Table[{e^(-2 t) c1, e^(-2 t) c2 }, {c1, -2, 2, 1}, {c2, -2, 2, 1}];
         pa = ParametricPlot[Evaluate[p], {t, -3, 3}];
         vp = VectorPlot[{-2 x, -2 y}, {x, -4, 4}, {y, -4, 4}];
         Show[vp, pa]
```

Out[32]=



And finally, if the characteristic equation of the linear system has a double real root and only one eigenvector, then the critical point is an improper node, which is again unstable, if the eigenvalue is positive and stable or „asymptotically stable“, if it is negative.
First let the double eigenvalue be positive:

$$x' = 3x + y$$
$$y' = -x + 5y$$

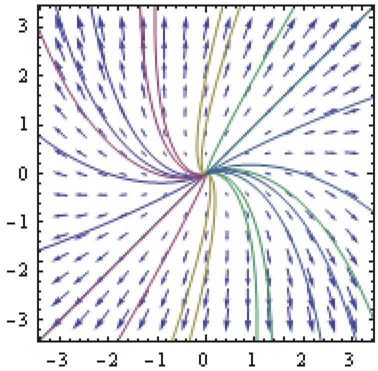
```
In[33]:= sys = {x'[t] == 3 x[t] + y[t], y'[t] == -x[t] + 5 y[t]};
         DSolve[sys, {x[t], y[t]}, t]

Out[34]= {{x[t] → -e^(4t) (-1 + t) C[1] + e^(4t) t C[2], y[t] → -e^(4t) t C[1] + e^(4t) (1 + t) C[2]}}
```

This system has a double real eigenvalue $r_{1,2} = 4$ and only one eigenvector $\vec{v} = (1,1)$. Thus the critical point of the system is an unstable improper node.

```
In[35]:= p = Table[{ -e^(4 t) (-1 + t) c1 + e^(4 t) t * c2, - e^(4 t) t * c1 + e^(4 t) (1 + t) c2}, {c1, -2, 2, 1},
    {c2, -2, 2, 1}];
  pa = ParametricPlot[Evaluate[p], {t, -2, 2}];
  vp = VectorPlot[{3 x + y, -x + 5 y}, {x, -3, 3}, {y, -3, 3}];
  Show[vp, pa]
```

Out[38]=




.

As the last is the system with a double negative eigenvalue and one eigenvector.

$$x' = -2x - y$$
$$y' = x - 4y$$

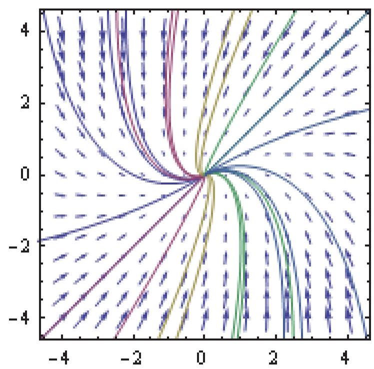
```
In[39]:= sys = {x'[t] == -2 x[t] - y[t], y'[t] == x[t] - 4 y[t]};
  DSolve[sys, {x[t], y[t]}, t]

Out[40]= {{x[t] → e^(-3 t) (1 + t) C[1] - e^(-3 t) t C[2], y[t] → e^(-3 t) t C[1] - e^(-3 t) (-1 + t) C[2]}}
```

Given system has a double real eigenvalue $r_{1,2} = -3$ and one eigenvector $\vec{v} = (1,1)$. Hence the critical point is a stable or „asymptotically stable“ improper node.

```
In[41]:= p = Table[{ e^(-3 t) (1 + t) c1 - e^(-3 t) t * c2, e^(-3 t) t * c1 - e^(-3 t) (-1 + t) c2}, {c1, -2, 2, 1},
    {c2, -2, 2, 1}];
  pa = ParametricPlot[Evaluate[p], {t, -2, 2}];
  vp = VectorPlot[{-2 x - y, x - 4 y}, {x, -4, 4}, {y, -4, 4}];
  Show[vp, pa]
```

Out[44]=



## Conclusion

In this article we wanted to present one possible application of the programming system *MATHEMATICA* for teaching and investigating critical points of linear differential systems, representing a special case of autonomous differential systems. It is very useful and practical not only because of reducing lengthy and labourious numerical computations, but mainly because of superior ability to visualize the obtained results.

## References

[1] KREYSZIG E.: *Advanced engineering mathematics,* John Willey & Sons, New York, 1988

[2] DOBRAKOVOVÁ J.: *Ako definovať integrál*, XX.Mezinárodní kolokvium o řízení osvojovacího procesu, Vyškov (2002), 81-84.

[3] DICSOOVÁ A., ZÁHONOVÁ V.: *Počítačom podporovaná výuka matematiky,* XVIII. Mezinárodní kolokvium o řízení osvojovacího procesu, Vyškov (2000), 48-52.

[4] ZÁHONOVÁ V.: *Výučba integrálneho počtu funkcie jednej reálnej premennej s podporou programu Mathemetica,* Zborník 25. konferencia VŠTEZ,-Matematika v inžinierskom vzdelávaní, Trnava (1998), 192-197.

[5] ZÁHONOVÁ V.: *Možnosť využitia programového systému Mathematica pri výučbe matematiky v novokoncipovanom štúdiu na SjF STU v Bratislave,* Aplimat, Bratislava (2002), 449-452.

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# TEACHERS FACING CONCEPTUAL NODES OF QUANTUM MECHANICS

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**Abstract.** In the framework of IDIFO Project, aimed at providing a Master in Physics Education, we discuss problems related with the "conceptual nodes" that Teachers encounter when introducing Quantum Mechanics at School. These problems are related both with interpretation and with mathematical formalism. We discuss in particular the outcomes of a questionnaire that was distributed among participants, with various hints for further research on the subject and possible applications to other didactical contexts in Science in general and in Mathematical Physics in particular.

**Key words.** Quantum Mechanics, Foundations, Education, Measurements

*Mathematics Subject Classification:* Primary 97B50, 8101; Secondary 81P15, 81P40

## 1 Introduction

Within the IDIFO Project aimed at forming Teachers on Modern Physics a number of activities for Quantum Mechanics (QM) have been set up: 1) on-line education with cultural and conceptual materials, with didactical proposals; 2) laboratory activities on crucial experiments for interpreting QM; 3) discussion workshops on main concepts and didactical re-elaboration and/or planning; 4) conferences and round tables to confront ideas; 5) in-depth seminars to analyze the cases proposed under a problematic form, as angles of reflection on didactical paths in QM, especially centered about: Quantum Logic, Quantum Formalism, historical aspects as well as *"conceptual nodes"*.

In this last context we have set up a questionnaire with opens answers, with the aim of constructing a *"fil rouge"* as a reference for discussions on QM included in the aforementioned activities, in the presence of a group of Teachers (22) involved in the IDIFO Master. The points raised by the questionnaire have been proposed as a reference and a guide for discussion and planning of didactical materials in this field, as well as a grid to elaborate a final report for each student. The questionnaire has been elaborated on the basis of a series of deep discussions concerning contents and goals of the various workshop activities indicated above [1]. We present here the basic elements starting from which we have developed the planning of the questionnaire, the case analysis about answers given by course students and conclusions, especially concerning:

- typical answers of course students;
- disciplinary and didactical nodes that have been emphasized;
- indications on how to form Teachers on the teaching and learning of QM in High Schools.

## 2 Planning the Questionnaire

In planning the questionnaire we have set the goal of constructing an open tool for collecting teachers' concepts about the main nodes of QM and the way in which one can tackle with them in a class with students. We have taken into account for different reference points:

A) A didactical perspective on the debate on conceptual foundations of Quantum Theory as a general reference on disciplinary nodes to be taken into consideration (D'Espagnat 1976; Cohen-Tannoudji et al. 1977; Sonego 1992; Girard 1997; Steer 2002; Newton (2002); Pospiech et al. 2008)

B) The questionnaire and the questions proposed therein in researches about learning of QM, as a framework on contexts investigated and modalities under which they have been explored (Fischler, Lichtfeldt 1992; Niedderer, Daylitz 1999; Johnston et al. 1998; Singh 2001; Cataloglu, Robinett 2002; Muller, Wiesner 1999, 2002)

C) The main proposals based on research on teaching/learning of QM in High School in Continental Europe and in the Anglo-Saxon Colleges, as a reference on the nodes they had to deal with (Zollman 1999; Phs.Educ 2000; AJP 2002)

D) Researches on the formation of Teachers. On one side those centered on the integration of disciplinary contents and pedagogical contents (Shulman 1986; Michelini 2004) as well as their evaluation (Ceylon, Bagnio 2006; Hanley et al. 2008; Schuster et al. 2009). On the other hand those aimed at didactical innovation, in particular in QM (Olsen 2002; Sperandeo 2004; Michelini et al. 2004; Stefanel et al. 2004; Asikainen 2005; Justin et al 2005), for the choice of nodes to be further investigated and the typology of questionnaire to be proposed.

These have been the background for individuating the following aspects, that have to be considered for analysing conceptual profiles of teachers about QM: 1) measure in QM: indetermination of physical processes and uncertainty in measures; probabilistic/statistic interpretation of a measurement process, properties of a system and their measurement; 2) properties of a system and the notion of a "quantum state", its representation, pure states and statistical mixing; 3) intrinsic properties and dynamical properties, superposition principle in classical and quantum form, evolution of the state in QM; 4) Quantum Logic and classical Logic; 5) the role of formalism in the theory; 6) historical cross-points, such as the role of Thermodynamics in the birth of QM and the debate wave/particle; 7) quantum interference.

## 3 The Investigation Tool

The investigation tool we adopted is a questionnaire ("open interview") that includes a first section to collect personal data as well as a short presentations of the scope it wants to finalize; and two sections respectively centered around: A) conceptual nodes of QM, with heading: "***Elements on which focusing a reflection about teaching and learning of QM"***; B) on scopes and relevant aspects for a didactic of QM, with heading: "***QM at School".***

### 3.1 Questions of Part A: Elements onto which Focusing the Reflections of Teachers

[1] These discussions have been conducted by the authors of the present paper in collaboration with R. Giannitrapani, L. Marinatto and G. Pospiech, who have contributed in various hands-on activities of the workshop.

Proposals for a didactical work require that contents are dominated. Literature calls this *"conceptual knowledge"*, that in turn implies the necessity of having a clear frame for the founding kernels of knowledge topics considered, as a background to enucleate didactical tools and modalities. We have therefore enucleated the following five points, as kernel questions to the "conceptual knowledge" of the Teachers:

**QA.1** Which are the elements that characterize/identify the quantum behavior of a system?

**QA.2** Properties of a system: does knowing or not knowing them imply the existence of such properties?

**QA.3** How does the role of measurement change when passing from Classical Physics to QM?

**QA.4** The result of measurement: its predictability and the objective nature of measured properties.

**QA.5** The domain of QM: can we apply QM to macroscopic systems?

QA.1 – The first question investigates the aspects of in-tension and ex-tension of a quantum system/behavior under the conceptual frame of a Teacher. We are in particular interested in exploring how the two possible evolutions of QM of a system are individuated and distinguished: i) the unitary evolution of an unperturbed system, deterministic/reversible/causal, based on *Schrödinger Equation*; ii) those established by the collapse postulate, intrinsically stochastic/indeterministic, that one has in the measurement process. This question implicitly explores the modalities under which the dichotomy deterministic/indeterministic can be taken into account, the complementarity expressed by the duality wave/particle, i.e. by the *Principle of Indetermination*. Conceptual nodes for this question are: the identification of the wave function or the state vector as a physical entity; the missing understanding that QM predicts a "double evolution"; the stochastic behavior of microscopic systems in interaction with a measurement apparatus, attributed to causal perturbations rather than to the intrinsically stochastic nature of such interactions.

QA.2 – The second question focuses on the properties of systems and the correlation knowledge/existence of properties. We explore the modalities with which we have taken into appropriate account the possibility (or not) to attribute a property to a quantum system, to the meaning of properties in Quantum Physics (taking for granted that we accept the possibility of attributing one) and the meaning of the impossibility of attributing such a property to a quantum system without having previously obliged it to pass a measurement process.

QA.3 – The third question is directly correlated with the second one, as it explicitly concerns the comparison between the role of a measurement process in QM or in Classical Physics. It is crucial to recognize a quantum measure process as the preparation of the state of a system, i.e. as a place where the dynamical properties of that system are "produced". Under a realistic perspective it is important to recognize the intrinsically irreversible character of measurements in QM and their role in "creating" the state properties of the system that undergoes measurement, with respect to the role that in Classical Physics a measurement has (i.e., just registering a pre-existing state).

QA.4 – The fourth question deepens this last point in regard to the non-epistemic nature of probability in QM, in comparison with the epistemic one it has in Classical Physics. Its goal is to produce an exploitation of the objective/non-objective nature of the properties of quantum systems. Associated to Question 2, it offers the possibility to enucleate the following typically unresolved nodes: attributing to a system properties that where pre-existing to measurement, without explicitly and coherently assuming an approach to hidden variables; the missed understanding of *Entanglement*, in the sense that entangled sub-systems might exist to be separately considered and still maintain their unitarity and distinguishability, i.e. the conviction that such properties can be attributed to sub-systems of an entangled system.

QA.5 – The fifth and last question concerns the domain of applicability of QM. We explore the awareness about the absence in QM of a limit of applicability, with the consequent legitimation to analyze in quantum terms also a macroscopic system. This implies the necessity of attacking the problems of "macro-realism", of entanglement of a macroscopic system, such as the *Schrödinger's Cat Paradox* or the coupling between a (quantum) system that undergoes measurement and a (classical) measuring device. One implies also the problem of the relation between classical and quantum interpretations of processes, i.e. the "continuity" from Classical to Quantum Physics.

### 3.2 The Questions of Part B: QM at School

The knowledge that Teachers have is transformed into professional skills (PCK - Shulman 1991) whence some problems related with didactical aims are solved, through a reflection process that preludes to the choice of didactical tools and modalities. Such a reflection concerns two kinds of choices: those concerning the role of the topics in the disciplinary, social and didactical contexts, and those concerning the formulation (the *"rationale"*) of the proposal. The second part (B) of the questionnaire tackles with this dimension and it is formed by the following three questions:

**QB.1** Why teaching QM?

**QB.2** Fundamental concepts that cannot be excluded from a didactical proposal in QM. Explain the reasons of the choices.

**QB.3** Which aspects should be privileged (formal, historical, logical, conceptual, applicative)?

These three questions aim at collecting the convictions of teachers about the motivations to teach QM in a (High) School, the fundamental notions and the aspects that have to be given a main role in a coherent and effective didactical proposal.

We explore the awareness of Teachers about the importance of QM in the current view on microscopic world, its role of paradigmatic theory, the role that QM can have in building theoretical and formal thought. In parallel one is interested in understanding how Teachers enucleate the formative contribute of QM and possibly the role hidden in retracing the birth of the theory and its contributions to epistemological and philosophical debate. Conceptual knowledge requires enucleating new views of basic concepts, in comparison with a classical frame, such as the very notion of "quantum state" and the "superposition principle", as well as new concepts, such as entanglement and non-locality. QB.2 aims at individuating not only the choices regarding basic notions, but also a synthetic view of the conceptual framework of QM.

One should therefore settle in the perspective of analyzing the main concepts indicated, as investigated in QB.3, that offers the occasion to recognize the conceptual profiles of Teachers and much probably (although indirectly) their formative necessities.

## 4 Context and Criteria for Analysis

We suggest here a qualitative analysis of six among the twenty-two questionnaires distributed to course students of the Master IDIFO. They have been selected for their completeness and representativity of answers when compared to a preliminary exploration of elements that have emerged, with reference to those a priori selected for the survey. We indicate the peculiar aspects that emerged from Teachers in this sample and the specificity characterizing the single profiles. It is therefore a case analysis that gives significant indications on conceptions that Teachers had matured in the central phase of their formative path.

## 5 Case Analysis

Here we discuss the outcomes of the questionnaire and point out a few hints that emerge from its checking.

### 5.1 Part A - QA.1 Which are The Elements that Characterize/Identify the Behavior of a Quantum System?

Quantum behavior: which are the elements that characterize and identify it? All teachers notice as peculiar of a quantum behavior:

- A1) "indeterminism" and stochasticity *("the intrinsic **indetermination** that physical systems show"*, *"Probably the main characteristics of quantum systems is their stochastical behavior"*)
- A2) "non locality", usually associated with Entanglement.

Aspects that are explicitly indicated are: *"a probabilistic description to describe phenomena"*, *"the principle of superimposition"*, *"the incompatibility of observables"*. They are associated with the behavior of systems (3/6), without any specification (2/6) or referred to the description of phenomena (1/6): *"the necessity of using a probabilistic description to describe phenomena"*. An accent is always put onto the measurement process, even without making it explicit or specifying it (6/6). The unitary evolution of unperturbed systems, on the contrary, has never been cited in the answers.

### 5.2 Part A - QA.2 The Properties of a System: Knowing them or not, does it Imply the "Existence" of such Properties?

All Teachers claim that a property does not exist before being measured *("a property does not exist until when it is measured"*; *"In QM the properties of a system are born with measurement"*; *"After measurement one can claim that the system possess such a property, while this is not possible before the measurement"*). In one case it is made more explicit that *"besides the exceptional case of a system that happens to be already in a well defined self-state"*. Two ways of thinking are peculiar:

- Comparing ideas: *"In Classical Physics objects have fixed properties that measurement makes evident. In QM, instead, before making a measurement we cannot anticipate that a system has a certain property"*;
- Recognizing measurement as an associated human act *"The very term Knowledge presumes that one has already made a measurement referring to the properties of a system"*.

Some of the expressions that emerged form the basis for reflections and discussions aimed to negotiation and sharing in the formation of Teachers: *"According to Bohr's interpretation, until a physical magnitude does not undergo measurement **it is not well determined**"*, *"Before making a measurement we cannot claim that an object possess a property, but we can only speak of **the probability that it possesses such a property**"*, *"The term knowledge presumes that one has already made a measure **relative to a property of the system**"*. The boldface parts of these answers contain the ambiguity about properties pre-existing to the measurement. These are subtle aspects, that could however create problems in students' learning, since they keep together and continuously move the theoretical reference from a standard interpretation of QM to one with hidden variables (Bell 1987). If, as it begins to emerge from some investigation, the most spontaneous approach to Quantum Physics is of "realistic kind" (Baily, Finkelstein 2010), with reference to theories with hidden variables (Michelini 2008), it is thence important to pay a particular attention to the language used to propose them to the students of the Teachers.

### 5.3 Part A - QA.3 The Measurements: how the Meaning and the Role of Measurement Change from Classical to Quantum Physics?

Teachers give a single kind of answer, with some specifications on the style of "comparison between ideas": *"In Classical Physics measurement of a property records a characteristic of the system that was pre-existing to the measurement itself. In Quantum Mechanics, on the contrary, the measurement plays an active role, in the sense that it contributes to determine the quantity that is measured"*, *"In Classical Physics the operation of a measurement gives indications on the value of a magnitude that the system possessed an instant before the measuring. In QM this is no longer true, the operation does not provide information on pre-existing properties of the system but it is rather the operation itself to make the system acquire a property"*. Not missing (4/6) the emphasis on the stochastic nature of the outcome of a measurement *("to pass from a superposition state to a well defined state, even if the choice of which self-state will be reached by the system is essentially of probabilistic nature"*; *"the result of a measurement is of stochastic nature"*). The active role of the measurement process in transforming a superposition state into a self-state is seldom made explicit (1/6), nevertheless it is present in almost all answers as a substantial element in the process of measurement.

### 5.4 Part A - QA.4 The result of a Measurement: its Predictability and the Objective Nature of the Properties being Measured

There are three classes of answers about predictability, often disjoint:

- the impossibility of making deterministic predictions *("the result of a measurement is not predictable in Quantum Mechanics"*; *"the value obtained in a measurement cannot be predicted in a deterministic way"*);
- the stochastic distributions of results *("if we make infinitely many copies of a state and perform infinitely many measures of the same observable, we never obtain the same result, but a distribution of results"*);
- the possibility of making only probabilistic predictions *("it will just be possible to calculate the probability of the outcome of a measurement"*).

In just one case the last two aspects have been considered *("If the system is not in a self-state of the magnitude one wants to measure, the result of the measurement will not be a priori determinable, but it will just be possible to calculate the probability of the outcome of the measurement itself"*).
It emerges again the need of establishing a shared language through which make explicit the peculiarity of "quantum indeterminacy", taking mainly into account the fact that all claims made, without further specification, could be referred also to a classical context of measurement: measures repeated, in fact, give always a distribution of results within an interval of confidence.
Two are the classes of ideas about the objectivity of the properties of a system:

- the pre-determination of the system in a self-state of the observable through a measurement *("If the system is in a self-state of the observable that one wants to measure, one can certainly claim that the result of the measurement… (will be the corresponding eigenvalue) – but knowing the state of the system presumes that I have already and previously made a measurement!"*).
- denying objectivity (2/3) *("We can no longer speak of objective properties of a system"*, *"The measured properties have no objective nature but self-create at the moment of the measurement"*).

No one of the answers clarifies, up to the very end, the extent of the claims; the second one of these classes completely excludes a realistic position, that is instead implicit in the first. This problematic appears as a node to be tackled with, together with a discussion on the diverse nature of intrinsic properties, that allow for instance to define the nature of a particle with respect to others, as well as of dynamical properties, that are correlated to the system when it is in a well defined state.

### 5.5 Part A - QA.5 The Domain of QM: can we Apply QM to Macroscopic Systems?

The domain of applicability of QM divides Teachers into two main positions: A) non-existence of limits for the application of QM *("There are no scale limits to the applicability of QM")* (4/6); B) QM is applicable only to the microscopic world *("No! QM can be applied only to microscopic systems")* (2/6). The arguments in favor of the position A have an operational basis, such as – for example – the recent experiments on diffraction and interference, realized with macromolecules, or they are justified on the basis of a scarce use of QM the macroscopic domain *"macroscopic systems are too complex so that it is preferable to describe them by means of Classical Mechanics rather than through complicated quantum calculations"*. Doubts and uncertainties characterize the position B: *"Which are the dimensions under which a system is subjected to the laws of QM?"*; *"Why a macroscopic system can be described by Classical Physics?"*; *"Why macroscopic systems obey to laws that are different from those valid for microscopic systems?"*. The awareness that QM does admit neither limitations in principle, nor domains in which it is not empirically clear whether applying it or not, should be strengthened in a formative process, by discussing the problem of micro-realism and the separability system/measurement device (quantistic the first, classical the second).

### 5.6 Part B - QM at School - QB.1 Why should we Teach QM at School?

The cultural and paradigmatic value of Quantum Theory, its "ideas", that are structurally different from those of Classical Physics, that allows one to explore the limits of applicability, are the main motivations, according to classical literature on the subject (Pospiech et al. 2008). More specifically, three motivations are presented by every answer:

- the "cultural value of QM";
- the fact that QM is one of the "fundamental theories of current Physics" ;
- its necessity as a tool to "explain all microscopic phenomena".

It is also often added (4/6) that: *"It is interesting since it introduces absolutely new ideas, that are not intuitive with respect to Classical Physics and common sense"*; *"It provides a new vision on Nature that is fundamentally different from the one introduced by Classical Physics"* and in one case it is also remarked that QM allows to stress the limits of a theory *"In the specific case Classical Physics is not able to explain the behavior of microscopic systems and therefore it cannot be applied to them"*.

### 5.7 Part B - QM at School - QB.2 Basic Notions that Cannot be Renounced to in a Didactical Proposal on QM. Motivate the Choices

The Linear Superposition Principle is assumed as a non-disposable base by every answer: *"The principle of superimposition is essential to understand superimposed states and as a consequence of the need of a probabilistic description"*. It is also added (4/6) that Indeterminism or the Principle of Indetermination is a new way to look at reality.

The most frequent indications concern the key concepts of Quantum Theory and are coherent with what is indicated into the answers of the first part of the questionnaire. This seems to prove the clearing of contents and the relative didactic, that has emerged in Teachers without a specific formation about it at the end of a course on the Dirac approach to QM (Battaglia et al. 2009). On the contrary to what is usually assumed in a didactic centered on disciplinary prerequisites, it seems that a preparation of teachers based on conceptual reflections starting from didactical proposals may give good fruits for mastering contents. One has to pay attention also to the self-referential position of a Teacher who believes that a choice of "non-renounceable concepts" depends on his freedom to choose the didactical set-up: one of the answers, in fact, has been *"The choice of basic concepts that have to be considered as non-renounceable depends on the followed path. The choice of an approach and of the concepts that one considers to be non-renounceable depends on didactical considerations made by the Teacher himself"*.

### 5.8 Part B - QM at School - QB.3 Which Aspects Should be Privileged (Formal, Historical, Logical, Conceptual, Applicative)?

All Teachers claim for themselves the right and the "competence" about their choices: *"These are choices that depend also on personal preferences"*; *"We cannot give a univocal answer, since more than ever before the choice belongs to the Teacher"*, without however making criteria explicit. It seems that Teachers want to give a major role to personal choices rather than to those that are suggested by lines of research. However, all teachers strongly emphasize concepts *("I would privilege a conceptual treatment"*) and choose the foundational ones for QM. The same Teachers at the beginning of their formation had indicated the following among the topics to be considered: the dualism wave/particle; the Principle of Indetermination and the quantization of variables (Battalion 2008): the formation has produced in them an important transformation in the attitude towards QM.

The position about formalism is more variegated, as it is exemplified by the following statements: *"It is important to introduce as much as possible also a formal structure"*; *"Concerning formalism I believe that it should be limited to the essentials"*, *"I claim that it is rather unlikely to be able to introduce the notions of Hilbert Space and Operator"*.

The position concerning the way of dealing with historical aspects is also variegated: *"In my opinion the role of History should be less significant, since it risks to create a confusion by superimposing more modern ideas to obsolete concepts"*; *"The aspects to be privileged are the historical ones (to show how one has reached such a Theory) and the conceptual one (to illustrate the really peculiar and distinctive aspects of the Theory)"*. The applicative aspects have been indicated only in a very generic way by just one Teacher *("I believe that it is important to present also some applications of QM"*).

## 6 Final Considerations and Conclusive Indications for the Formation of Teachers

The answers to questions on the foundations of QM make evident the need of including into the formation also the discussion about some concepts in comparison with their classical role. It emerges, in particular, the need of balancing the relevance given in QM to the process of measurement in comparison with Classical Physics, with recognition of the founding character of the Principle of Superimposition and of the linearity and unitarity of evolution. The analysis of the peculiarity of measurement in QM, making explicit its active role together with entanglement, make evident the distinctive elements of QM with respect to the foundational assumptions of Classical Physics. These are specific aspects. What makes QM a Theory and not only a set of conjectures and rules should emerge from the analysis of its basic constructive elements: the Superposition Principle, that guarantees its linearity; the modalities that determine the possible outcomes of measurements and the a priori probability of the same outcomes; the unitary nature of the unperturbed evolution of quantum systems and the irreversible nature of any measurement process; the modalities under which a system is "quantized".

Introducing a basic formalism is important not only to show its applicative power but mainly to show that it allows a compact and unitary description of founding contents together with their meaning, such as for instance the non-epistemic Indeterminism, that characterizes measurement processes, as well as the probabilistic predictions associated with them, that are intrinsically included into the linear superposition principle itself.

Negotiating and sharing meanings, that is important in each formative process, becomes in the case of QM a conceptual “must”, since in the language that is commonly accepted to describe phenomena and processes there are implicit assumptions of causal and deterministic nature. Lexical ambiguities and allusions, as it emerged sometimes from the answers to the questionnaire, assume a particular valence in the case of QM, since the descriptive categories of processes are not always “classical”. The request for coherence, in particular in comparison with a specific interpretative framework (either orthodox or with hidden variables) is an indispensable premise to the construction of coherent schemes, as it was already stressed by Bell (1987).

A specific language should be built to characterize measurements in QM and diversify it from measurements in Classical Mechanics. It is not sufficient to recall its stochastic nature, that is indeed a specific characteristic of any measurement process. It is necessary to specify also the intrinsic indeterminism and not-epistemic of the outcome of any quantum measurement process together with the non-classical character of Probability in Quantum Mechanics, that emerges whenever it is no longer possible to apply Bayes Theorem to calculate probabilities and makes necessary to take quantum interference into account. The objectivity/non-objectivity of the properties of a quantum system is a further node to be dealt with in the formation of teachers, together with the discussion on the different nature of the properties that we can call “intrinsic” for a quantum object, that in turn allow to define its very nature.

The framework in which QM is applicable without any limitation of principle, the problem of micro-realism and the problem involved in the need of predicting the separability of the system observed from the measuring device, do constitute open nodes in the researches about foundations (of QM) so that it becomes even more stringent the need for Teachers to explore those contexts that make more evident their meanings.

The integration between conceptual contents and didactical perspectives, a main objective in the formation of Teachers and, at the same time, an open problem at least in the initial phase, seems to be enough realized. The indications that emerge address us towards a formation of teachers on QM that plans to reflect on the epistemology of the discipline as well as a discussion on criteria needed to identify the different disciplinary approaches to QM, on a structural setting, of contents, of foundational concepts, of appropriate methodologies. The “didactical reconstruction” of contents (Duit 2006) should be accompanied by an investigation of methodologies adopted in didactic research (to better plan didactical interventions, and to analyze the difficulties that students encounter when dealing with the founding notions of QM) and also by planning occasions to translate into an operative scheme approaches of “Inquiry” type (McDermott, Shaffer 2000), problem solving (Watts 1981) as well as critical details (Viennot 2002). The formation has to be centered around the integration of the different levels of knowledge about contents (CK) and the “Pedagogical Content Knowledge” - PCK (Schulman 1986; Michelini 2004; Sperandeo 2004; Michelini et al 2004a,b).

A structural role in the research on Physics Education about the formation of Teachers may constitute a founding element that allows Teachers to overcome their self-referentiality (that was stressed by some of the answers to the second part of the questionnaire). This is therefore, in the specific frame of formation in teaching/learning of QM, an element that cannot be renounced in view of innovation. The questionnaire that we have discussed above is in fact an open tool, that can be exported also in other contexts and is in particular usable as a pre/post test to analyze the notions that Teachers have at the beginning and at the end of a formative path, easily transformable to be proposed also as a questionnaire for students. In fact it has been used not only for the Teachers involved in the activities of IDIFO, but also with students of Summer Schools on Modern Physics that have been organized for students of the last two years of High Schools in Udine in the years 2007 and 2009 (Gervasio et al. 2010, Cassan et al 2010), also presented in English within a specific Workshop for researchers in the occasion of the Conference “Girep 2008” (Pospech et al. 2008).

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**References**

[1.] AJP, Am. J. Phys. 2002, Special Issues 70 (3)

[2.] ASIKAINEN, M., 2005, *A study of students' learning processes on a new quantum physics course for preservice and inservice teachers*. Proceeding of the Esera Summerschool 2005, in press

[3.] BAILY C. and FINKELSTEIN N.D. (2010) *Teaching and understanding of quantum interpretations in modern physics courses*, Phys. Rev. St Phys. Educ. Res. 6, 010101.

[4.] BATTAGLIA R. O., CAZZANIGA L., CORNI F., De AMBROSIS A., FAZIO C., GILIBERTI M., LEVRINI O., MICHELINI M., MOSSENTA A., SANTI L., SPERANDEO R.M., STEFANEL A. (2010) *Master IDIFO (Innovazione Didattica in Fisica e Orientamento): a community of Italian physics education researchers for a community of teachers as a model for a research based in-service teacher formation model on modern physics*, in L. Rogers, Community and Cooperation, selected paper from GIREP-EPEC & PHEC Conference 2009, University of Leicester-Girep, Leicester.

[5.] BELL J.S. (1987) *Speakable and Unspeakable in Quantum Mechanics*, Cambridge: Cambridge Univ. Press

[6.] CATALOGLU E., ROBINETT R.W. (2002) *Testing the development of student conceptual ad visualization understanding in quantum mechanics through the undergraduate career*, Am. J. Phys. 70 (3) 238-251

[7.] CASSAN C., COLOMBO M., MICHELINI M., MOSSENTA A., SANTI L., STEFANEL A., VERCELLATI S., VIOLA R. (2010) Scuola Estiva di Fisica Moderna per studenti di scuole secondarie superiori, LFNS

[8.] Cohen-TANNOUDJI C., DIU B., LALOË F., 1977, *Quantum Mehanics*, Paris: Hermann and Wiley

[9.] D'ESPAGNAT B. (1976) *Conceptual foundation of Quantum Mechanics,* 2nd ed., Menlo Park: Benjamin.

[10.] DUIT R. (2006) *Science Education Research – An Indispensable Prerequisite for Improving Instructional Practice*, ESERA Summer School, Braga, July 2006 (http://www.naturfagsenteret.no/esera/summerschool2006.html)

[11.] EYLON B., BAGNO E. (2006) *Research-design model for professional development of teachers: Designing lessons with physics education research*, Phys. Rev. St Phys. Educ. Res. 2, 020106.

[12.] EINSTEIN A., PODOLSKY B., ROSEN N. 1935, *Can Quantum Mechanical description of Physical Reality Be Considered Complete?*, Physical Review, 47, 777-780

[13.] FISCHLER H., LICHTFELDT M. (1992) *Modern physics and students' conceptions*, Int. J. Sci. Educ. 14 181–90

[14.] GERVASIO M., MICHELINI M., MOSSENTA A., SANTI L., SCIARRATTA I., STEFANEL A., VERCELLATI S., VIOLA R. (2010) *Attività di laboratorio nell'ambito della scuola estiva di fisica moderna dell'università di Udine*, XLVIII Congresso AIF

[15.] GHIRARDI G.C. (1997) *Un'occhiata alle carte di Dio*, Milano: Il Saggiatore, p.33; *Sneaking a look at God's Cards*, Princeton University Press USA, 2004

[16.] GHIRARDI G.C., GRASSI R., MICHELINI M. (1997) *Introduzione delle idee della fisica quantistica e il ruolo del principio di sovrapposizione lineare*, La Fisica nella Scuola, XXX, 3 Sup., Q7, p.46-57.

[17.] HANLEY P., MARINGE F., RATCLIFFE M. (2008) *Evaluation of Professional Development: Deploying a process-focused model*, International Journal of Science Education, 30 (5), 711 — 725.

[18.] JOHNSTON I.D., CRAWFORD K. and FLETCHER P.R., 1998, *Student difficulties in learning quantum mechanics*, International Journal of Science Education, 20, No 4, 427-446

[19.] JUSTI R.S., SOUZA V.C.A., FERREIRA P.F.M. (2005) *Analogies for the atom: students' and teachers' (mis)understandings*, R. Pinto ed. Cresils, Barcelona

[20.] McDERMOTT L.C., SHAFFER P.S. (2000), *Preparing teachers to teach physics and physical science by inquiry*, Phys. Educ. 35(6) 411-416

[21.] MICHELINI M. ed. (2004) *Quality Development in teacher Education and training*, Second International Girep seminar, Udine, Forum

[22.] MICHELINI M. (2008) *Approaching the theory of quantum mechanics: the first steps towards a coherent synthesized interpretation with a supporting formalism*, in "Frontiers of Physics Education", Jurdana-Sepic R. et al eds., Girep-Epec book of selected contributions, Zlatni: Rijeka 93-101

[23.] MICHELINI M., STEFANEL A. (2008) *Learning Paths Of High School Students In Quantum Mechanics*, in Jurdana-Sepic R., *Frontiers of Physics Education*, Girep-EPEC Conf. 2007, sel. Contrib., Zlatni: Rijekas, 337-343

[24.] MICHELINI M., STEFANEL A. (2010) *High school students face QM basic concepts*, Santoro G. ed New Trends in Science and Technology Education, Book of Selected Papers, Clueb Bologna

[25.] MICHELINI M., RAGAZZON R., SANTI L., STEFANEL A. (2000) *Proposal for quantum physics in secondary school*, Michelini M, Sartori C, 1998, Esperienze di laboratorio didattico in una struttura di raccordo, Università & Scuola, III, 1/R pp. 18-29.

[26.] MICHELINI M., RAGAZZON R., SANTI L., STEFANEL A. (2004), *Implementing a formative module on quantum physics for pre-service teacher training,* in "Quality Development in the Teacher Education and Training", Girep book of selected papers, M. Michelini ed., Forum, Udine, pp. 429-435

[27.] MICHELINI M., STEFANEL A. (2008) *Learning Paths Of High School Students In Quantum Mechanics*, in Jurdana-Sepic R., in "Frontiers of Physics Education", Girep-EPEC Conf. 2007, sel. Contrib., Jurdana-Šepić R., Labinac V., Žuvić-Butorac M., Sušac A. Eds, Rijela, Zlatni, ISBN: 978-953-55066-1-4, pp. 337-343

[28.] MÜLLER R., WIESNER H. (1999) *Students' Conceptions of Quantum Physics*, in Research on Teaching and Learning Quantum Mechanics, Zollman 1999, loc. Cit,

[29.] MÜLLER R., WIESNER H. (2002), *Teaching quantum mechanics on an introductory level*, Am. J.Phys. 70 (30), 200-209

[30.] NEWTON R.G. (2004) *What is a state in quantum mechanics?*, Americam Journal of Physics 72 (3)

[31.] NIEDDERER, H., DEYLITZ, S., 1999, *Evaluation of a new approach in quantum atomic physics in high school.* In D. Zollman (Ed.), Research on teaching and learning quantum mechanics (pp. 23-27): National Association for Research in Science Teaching.

[32.] OLSEN R.V. (2002) *Introducing quantum mechanics in the upper secondary school: a study in Norway*, International Journal of. Sience Education, 24, 6, pp. 565-574.
[33.] Phys Educ. (2000) Special Issues 35 (6).
[34.] POSPIECH G., MICHELINI M., STEFANEL A., SANTI L. (2008) *Central Features of Quantum Theory in Physics Education*, Discussion Workshop B, in Frontiers of Physics Education, Girep-EPEC Conf. 2007, sel. Contrib., Jurdana-Šepić R., Labinac V., Žuvić-Butorac M., Sušac A.. Eds, Rijela, Zlatni, ISBN: 978-953-55066-1-4, pp. 85-87
[35.] SHULMAN L.S. (1986) *Those Who Understand: Knowledge Growth in Teaching*, Educational Researcher, Vol. 15, No. 2. (Feb., 1986), pp. 4-14.
[36.] SCHUSTER D., COBERN W.W. , APPLEGATE B. , SCHWARTZ R.S., UNDREIU A. (2009) *Assessing Pedagogical Content Knowledge of Inquiry Physics Teaching*, AAPT Winter Meeting, Chicago 2009; Schuster D, Undreiu L, Undreiu A (2009) Interactive PS Tutorials Through Visual Programming, Compadre Document Delivery, http://www.compadre.org/PER/document/ServeFile.cfm?ID=8121&DocID=754
[37.] SINGH C. (2001) *Student understanding of quantum mechanics*. Am. J. Phys., 69 (8), 885-895
[38.] SONEGO S. (1992) *Conceptual foundations of quantum theory: a map of the land* , Ann. de la Fond. L. de Broglie 17, 405; Sonego S. (1993) *errata:* ibid. 18, 131.
[39.] SPERANDEO R.M. (2004) *The pre service physics teacher education model implemented by the FFC research project involving & Italian Universities: Guidelines and preliminary results*, in M. Michelini, Quality Development in teacher Education and Training, Udine: Girep-Forum, 89-95.
[40.] STEFANEL A. (2008) *Impostazioni e percorsi per l'insegnamento della QM nella scuola secondaria*, Giornale di Fisica 49(1), pp. 15-33.
[41.] STEFANEL A., MICHELINI M., RAGAZZON R., SANTI L., (2004), *Blended activity on quantum mechanics knots for pre-service teachers*, in proc. Girep Conference – Ostrava (Cz), M. Melchova et al eds. Girep-Univ. Ostrawa, pp.206-208
[42.] STYER D.F., BALKIN M.S., BECKER K.M., BURNS M.R., DUDLEY C.E., FORTH S.T., GAUMER J.S., KRAMER M.A., OERTEL D.C., PARK L.H., RINKOSKI M.T., SMITH C.T., AND WOTHERSPOON T.D. (2002) *Nine Formulations of Quantum Mechanics, American Journal of Physics* 70, 288--297
[43.] VIENNOT L. (2002) *Enseigner la physique*, De Boeck.
[44.] ZOLLMAN D. (Ed.) (1999) *Research on teaching and learning quantum mechanics* (pp. 23-27): National Association for Research in Science Teaching

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# IDIFO3 - TEACHERS FORMATION ON MODERN PHYSICS AND MATHEMATICAL FOUNDATIONS OF QUANTUM PHYSICS: A CROSS SECTIONAL APPROACH

**FRANCAVIGLIA Mauro, (I), LORENZI Marcella Giulia, (I), MICHELINI Marisa, (I), SANTI Lorenzo, (I), STEFANEL Alberto, (I)**

**Abstract.** We present a review on the Master IDIFO on Physics Education, aimed at building a Master Platform on-line for the Education of Modern Physics. We discuss the general setting, the structure of the Master, its main goals and achievements, with special emphasis on: didactics of Quantum Mechanics (à la Dirac), Art & Science

**Key words.** Education, Learning, Art & Mathematics, Art & Physics

*Mathematics Subject Classification:* Primary 97B50, 8101; Secondary 81P15, 81P40

## 1 Introduction

Italy's main response to the fall in motivation with regard to scientific studies has been the *Progetto Lauree Scientifiche* (PLS) (Scientific Degrees Project), promoted by the coordination of the Science Faculties of Italian Universities and organised in three areas: Mathematics, Physics and Chemistry ([1]).

In the field of PLS ([2]), the Italian Research Units in Physics Education ([3]) worked together to offer a Project focused on Didactic Innovation in Physics Education and guidance (*Innovazione Didattica In Fisica e l'Orientamento - IDIFO*) including a Master, for "in service teacher education" on modern Physics, as a result of researches carried out in this field. The IDIFO project

[1] Each university project has developed various activities for students and teachers, belonging to one or several areas. National coordination was organised according to discipline, with meetings arranged between all subjects involved (local level, by subject and on national scientific committee levels).

[2] Interuniversity national project "scientific degrees" (DM prot.no. 262/2004 5th August) - "Orientation and training of physics teachers".

[3] By Unità di Ricerca in Didattica della Fisica (Research Unit in Physics Education) we refer to the research groups in physics didactics in this University, which in the previous years have had projects approved and financed in the area of Progetti di Rilevante Interesse Nazionale (PRIN) (Projects of Relevant National Interests) and specifically through the PRIN046-Fis21.

was presented first in 2006 by the University of Udine, as an initiative promoted by nine Physics Education Research (PER) Units. A second edition was presented and activated in 2009 with 15 partners. IDIFO3 project was presented by the Physics Education Research Unit of Udine University on behalf of the eighteen partners shown in Figure 1.

Piano Lauree Scientifiche 2010-2013
Orientamento e Formazione degli Insegnanti – Area Fisica

**IDIFO3**
**Innovazione Didattica in Fisica e Orientamento**
Responsibile: Marisa Michelini
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**Project site**
http://www.fisica.uniud.it/URDF/laurea/index.htm
http://www.fisica.uniud.it/URDF/laurea/pls3.htm
**e-learning platform**
http://idifo3.fisica.uniud.it/idifo3/render.userLayoutRootNode.uP

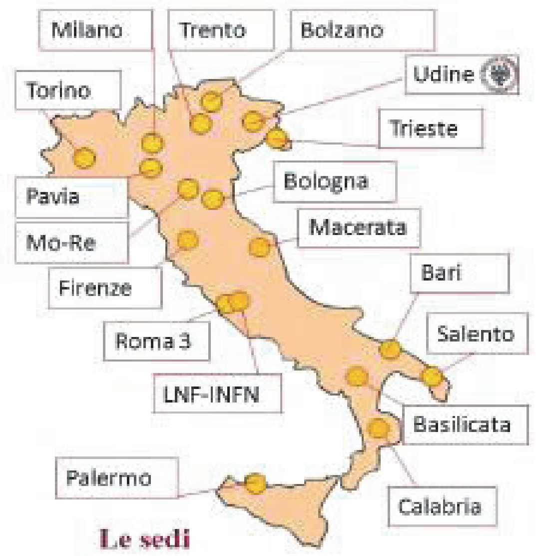



Figure 1: **IDIFO3 Partners**

IDIFO3 offers differentiated actions of educational innovation, science learning laboratories and teachers training. It exploits the results and teaching materials of researches in Physics education, as well as those developed in the projects IDIFO1 and IDIFO2, which focused on the Physics of the XX century (Quantum and matter Physics, Relativity, and statistical area) and the formative orientation (Problem Solving) for the training of teachers and the experimental and educational laboratory activities. The skills acquired in distance education for teachers on topics of modern Physics and the dimension of training in the perspective of educational research are brought into play in the two years Master M-IDIFO3 and in the CP-IDIFO3 specialization Annual Course, activated upon reaching the planned number of enrolled students and started in January 2011.

The activities in IDIFO3 are of 8 types, grouped into the following 4 areas:

A. Educational and experimental workshops in presence;
B. Training teachers at a distance and in presence;
C. Games Experiments Ideas Exhibit: exhibition, informal education, conceptual CLOE laboratories;
D. Participation and sharing of cultural diffusion and education projects in Science:
   a. Mosem2: LLP European Project on Superconductivity;
   b. LabGEI approved and funded by the MIUR within the L. 6/2000 for cultural diffusion;
   c. “Sicuramente”, for road safety education, approved and funded by the Friuli Venezia Giulia Region.

We present here the Master IDIFO3, which implements a model of teacher training based on the active involvement of the same teachers, with a setting of developing formal thought and the connection among Computer Science-Mathematics-Physics on modern Physics topics.

## 2 The IDIFO3 Master for Teachers

University of Udine Post graduate Master's Degree in "*Innovation in Physics Teaching and Orientation"* (M-IDIFO3) was activated for the Academic Years 2010/2011 and 2011/12 as a joint initiative of the Research Units in Physics Education shown in Figure 1.

The aim of the course is to create an experienced teacher in:

1. teaching of modern Physics (especially Quantum Physics, relativistic Physics, with elements of Astrophysics and Cosmology);
2. use of information and communication technologies (ICT) to overcome the conceptual nodes in Physics;
3. training to theoretical thinking in Physics and to the experimental work on crucial experiments for the foundation of quantum and relativistic thinking;
4. design and construction of Physics in context experiences;
5. learning activities based on reading qualified popularization articles about Scientific Research ("Asymmetries" by INFN);
6. laboratory teaching with Inquiry Learning, problem solving and PEC strategies;
7. design and implementation of materials and activities for the orientation training in Physics;
8. analysis of learning processes in educational innovation.

The courses offered in the Master consists of over 135 credits (cts) to be organized in the training profiles of 60 cts according to general criteria and the learning needs of individuals. Distance education was enriched by laboratory courses in presence, following the PLS procedures implemented in the different cooperating locations. The Master training plan should include an examination for each module of 3 cts provided in the individual training plan of 60cts, including the PSOF (which must be followed by all) and the final thesis. Practical activities of educational experimentation with children are planned and supported under the 90% of the modules. The e-learning platform used in the Master has been prepared for the specific task of teacher training starting from the American U-Portal. It has been improved several times during its construction. The integration of the changes made in an organic structure is required for its reuse. The frequency of at least 70% of the scheduled hours for the learning activities of the course was compulsory and was controlled with traditional procedures for the activities in presence and IT procedures for the activities at a distance.

The thesis must document educational experimentation in presence (or at a distance with secondary school students or other teachers in training on the topics of the modules) for at least 36 hours, including at least 16 in the same class. The thesis will be discussed with a Commission appointed by the Council of the Master.

A number of teachers (25) were awarded a scholarship to cover part of the tuition fees, each of which has chosen 60cts for his/her training profile. Other 12 were enrolled in the Associated specialization Course. In addition, about 50 teachers attended single courses.

## 3 Features of the Proposal for Teacher Education

The IDIFO3 Master implements an innovative idea of continuing teacher training, which is characterized by the following characteristics.

### 3.1 Cultural Area

The choices of the cultural area considered to be relevant and the priorities for the continuing teacher training, as well as for current cultural policies, are related to the Physics of the last Century and its new way of thinking about the interpretation of the world, which involves proposing new bases for the culture of all, integrating and overcoming traditional and perception-based views for explanations and understanding models.

### 3.2 Mathematics and Science Relationship

Discussing about these issues means to conceive a new relationship between Mathematics and Science. The Physics theories of the last Century radically changed the ways of thinking within the discipline and raised the question of the role and the meaning given to the formalism in order to account for the phenomena. For example, Quantum Mechanics according to Dirac's framework reverses the traditional perspective of research of the formalism that translates conceptual interpretations, as in the case of Newtonian Mechanics, of the mesoscopic models in theories of fluids and the laws of the Fourier phenomenon. The planned training of teachers offers a course specifically dedicated to "The role of Geometry in modeling Physics", but offers insights on the relationship between Mathematics and Physics in many other courses, such as the following:
- Cosmology from antiquity to Einstein;
- Geometric Approach to Relativity;
- Spacetime-relativistic dynamics;
- Approaches to Relativity: Einstein and Minkowski interpretations compared;
- The new way of thinking of Quantum Physics and the Dirac formalism;
- Conceptual nodes of Quantum Mechanics;
- Education proposals in Quantum Physics: a comparative analysis;
- Quantum Physics: educational proposals related to Field Theory;
- Rutherfod Backscattering Spectroscopy in class.

### 3.3 Information and Communication Technologies and new Media as Teaching Tools

The update of the curriculum is also proposed in terms of the use of multimedia tools in the curriculum of Physics. Specific proposals related to the educational workshop with wireless sensors and the realization of real-time graphics for fostering the link between observation of the phenomenon and its representation with the evolution of the graphs of significant magnitudes. These are mostly educational workshops in which data collection is accompanied by activities with computer modeling and computer simulation, so that the students are the protagonists of the interpretation of the phenomena developing formal models for calculating the results of which can be subjected to comparison with the experimental data. To synchronous and asynchronous online learning environments three courses have been dedicated to:
- Learning in computer networks;
- Installations and live performances;
- New media.

At the end of this paper we will give some examples of these modules (examples 1 and 2).

### 3.4 Physics in Context and Science Popularization

A subject area of "Physics in Context" has been included, in order to connect school learning and the everyday life, to make the school offer to young people practical skills in addition theoretical knowledge in Physics. Thus some laboratories deal with the Physics of Cooking, of Sports, or Theater and Science shown as opportunities for cultural diffusion and popularization. The opening of school to Science Popularization needs a great support and IDIFO3 has implemented this connection with the direct participation of the world of basic research and INFN (National Institute for Nuclear Physics), which promotes a biennial conference on these issues, as well publishing "Asymmetries", a journal very useful for teachers

### 3.5 Teacher Formation on Subject Related Guidance and Orientation

Previous researches on the adaptation of the *Popular Problem Solving* by Mike Watts in differentiated contexts for *Problem Solving* activities for *Orienting* purposes (PSO) allowed to develop a methodology for the subject related orientation and formation, which involves the subject's personal involvement in games challenges of *open problems*. Two courses which are common to all the curricula provide in-depth examination of the role of the teacher in orientation, the idea of an institutional network to build a culture of orientation and lay the foundations for the design. The working phase consists of: A) a reflection on the epistemological status of Physics: the characters that constitute it as a discipline and the foundations nuclei, to form a base of useful information to design a module of PSO in Physics; B) the presentation, the reports, of the characteristics of the PSO methodology and training methods for the design; C) the development of a PSO individual project; and: D) its test in classes.

## 4 Research in Physics Education at the Base of Teacher Education

The whole IDIFO3 project is based on the results of researches in Physics Education in the educational processes, that in teacher education takes on four dimensions: 1) a source of knowledge of spontaneous interpretative models, learning processes and difficulties of children; 2) resource for curricular proposal for educational intervention modules; 3) source of tools, methods and strategies to be applied in educational planning; 4) methodological reference for the development of a teaching expertise in the framework of the Pedagogical Content Knowledge (PCK - Shulman 1989). The 80% of the course contains proposals for prototype educational activities validated by the research and many of the laboratories involve reworking of educational proposals from researchers, analyzed from a scholarly perspective. The experimental teaching of courses designed by teachers in training is analyzed in the perspective of and with tools of educational research.

## 5 Modular Design of the Didactic

Figure 2 shows the course structure and organization in macro 4 (FM, RTLM, FCCS, OR): 41 courses of 3 cts that allow the structuring of different organic paths at different levels of understanding and with different titles achieved: a) single course; b) specialization course (15 cts - 4 courses and a work project experienced in school); c) Master / equivalent to Post graduate Master's Degree / (60 cts - 16 courses chosen by the teachers on training courses + 2 courses on formative guidance and counseling for all + a thesis of 6 cts including a pilot experimentation in classroom).
This structure allows each teacher to take courses that match their training needs with a gradual commitment corresponding to the specific needs, without losing coherence and consistency with the objectives of the training.

The activities approach is cross-sectional and provides for the co-design of educational interventions with teachers, their testing and analysis. Metacultural, experiential and situated action research models of teachers training are integrated. Cross-sectional and broad cultural issues offer new important opportunities for learning and provide the context for understanding the broader role of Science in different contexts. Significant examples of this are the Physics context subjects, the topics of Quantum Mechanics and superconductivity, courses on Cosmology, Geometry, Theatre and Science - Installations and Live Performances, offered by two of us (MGL and MF). The argumentative activities proposed in the laboratories integrate scientific leaning styles and methods with mathematical and philosophical ones, thus broadening and enriching perspectives. Finally, PSO Problem Solving for orientation is proposed as a model for the counselling training.

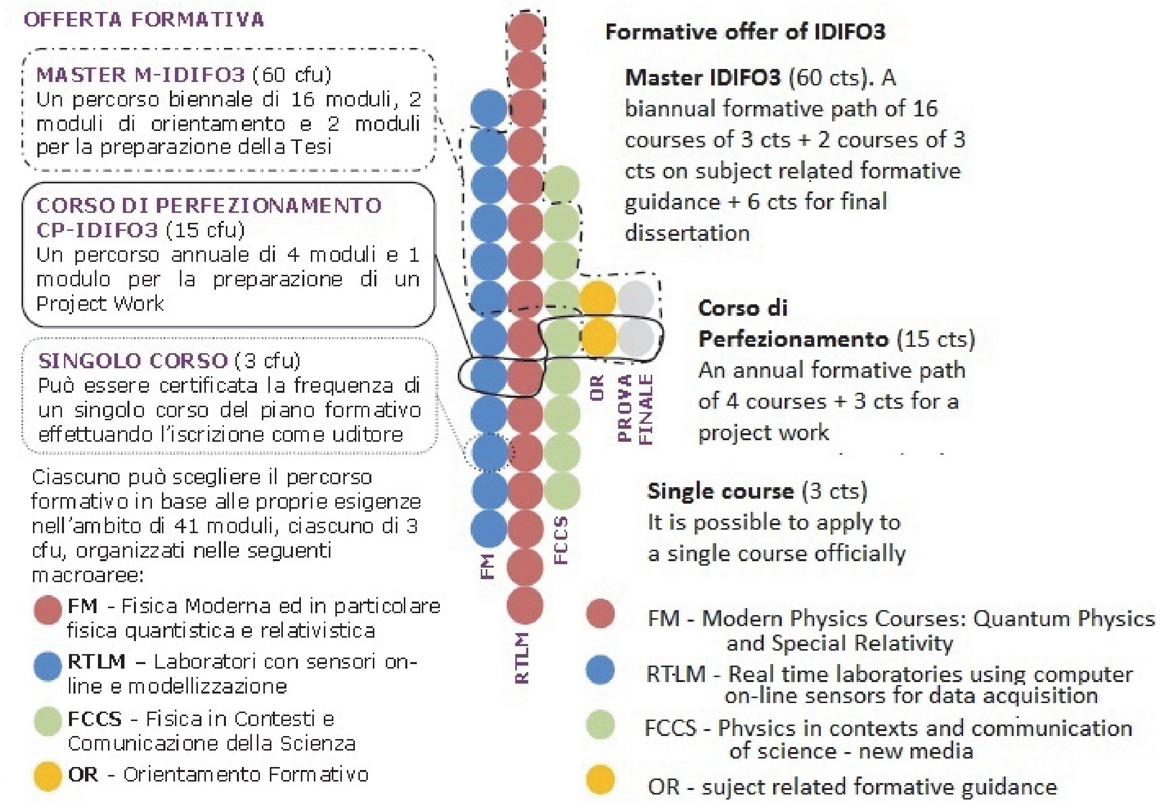



Fig. 2 - FORMATIVE OFFER OF IDIFO3. Each participant decide their own formative path, chosing the courses he need under 41 of 3 cts each offered in 4 macroareas.

## 6 A Couple of Examples

We shortly discuss here a couple of examples.

### 6.1 Example 1: Dirac's Approach to the Teaching of Quantum Mechanics Module

As a first example of a formative module on the online learning platform, Dirac's approach to the teaching of quantum mechanics (QM) is considered here. This module focused on the analysis of teaching materials and original works from the research project on the teaching/learning of QM in secondary school, resulting from preceding studies [4]. It was addressed to 22 participants.

4 See M. Francaviglia, M Lorenzi, M. Michelini, L. Santi, A. Stefanel, TEACHERS FACING CONCEPTUAL NODES OF QUANTUM MECHANICS, in these Proceedings.

The module on the approach to Dirac's Quantum Mechanics. The module is divided into three main phases:

Phase A - A1-A2 online courses: based on the foundation concepts of Quantum Mechanics following Dirac's approach and on the presentation and discussion within a forum of the nodes on which the aforementioned project is developed together with its worksheets. At the beginning of this teaching a questionnaire on the contents and pedagogical skills in QM has been delivered to the participants.

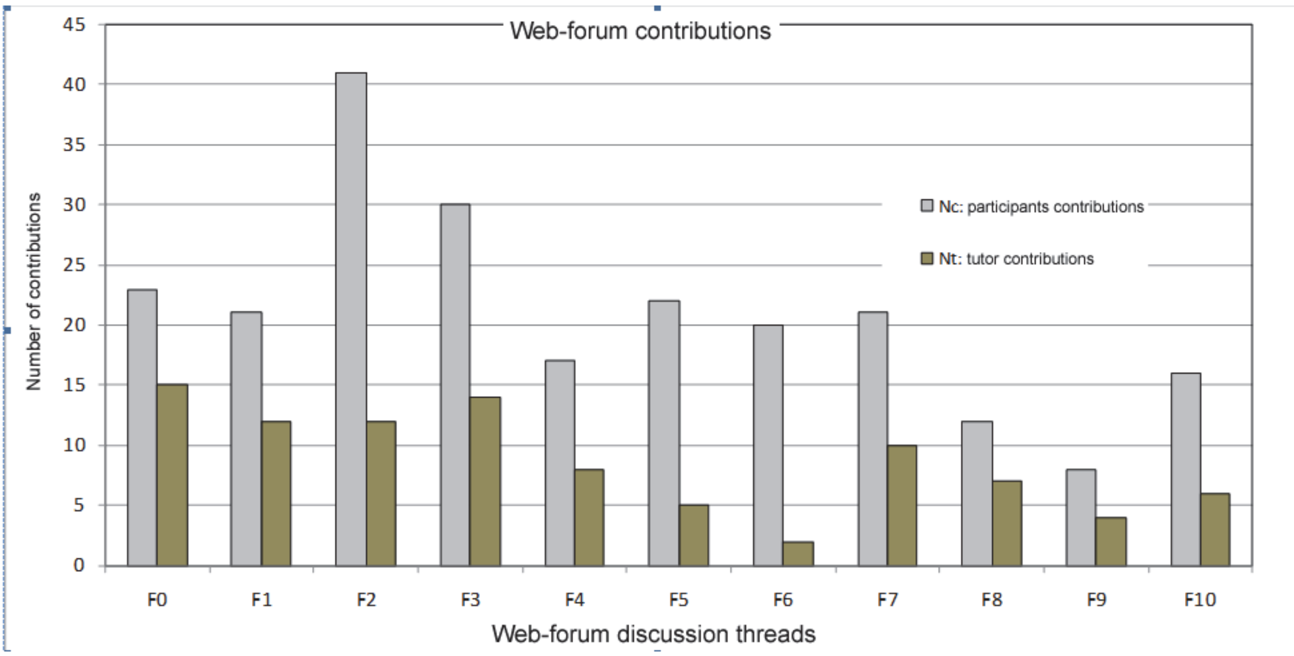
Phase B - Meeting in presence with teachers in order to talk about the proposal and the nodes emerged and those unresolved in the online discussions.

Phase C - Course B online didactic laboratory, aimed at designing a micro module focused on the research proposal analyzed in the previous two phases. These projects in some cases have led to experiments in the classroom, implemented as part of the Master IDIFO training activities.

The documents offered for discussion in phase A were the reference articles of the proposal [5], the tutorials of the same proposal, edited and processed on purpose for the discussion. The discussion on the forums of the online courses has been proposed in different threads, one for each of the nodes on which the educational proposal object of analysis and discussion of the course focuses.

Figure 3 shows the frequencies of the contributions of the 22 participants (NC) and of the teacher / tutor of the course (Nt) to the various web-forum threads of discussion about the A1 online course. The trends of the discussions clearly emerge in the peak frequencies of the interventions, for example on the threads F2, about probabilistic interpretation of quantum processes, and F3, on how a dynamic property can be attributed to a quantum system (i.e. polarization in the case of photons) and the meaning of this attribution.

The dynamics going on in the first thread is particularly complex for the different sub-threads initiated by the students (at least 5 top ones) often intertwined, both for the length of the different posted interventions, often simultaneously dealing about multiple nodes. The sub-threads of discussion, sometimes, have been animated by a rich interchange dialogue between two or more students, concluded by the intervention of other students, who suggested the analysis of other aspects of the discussion.




5 ibidem

Figure 3 Frequencies of the contributions of the 22 participants to the courses (Nc) and of the tutor. F0 – Test; F1 – Phenomenology of Polarization; F2 – Probabilistic interpretation; F3 – Dynamical properties of a quantum system; F4 – Interpretative hypothesis; F5 – Quantum systems and trajectories; F6 – Incompatibility and mutual elusiveness; F7 – States and vectors; F8 – observables and opererators; F9 – generalization; F10 – non-locality.

The dynamics of the discussion in the second thread has been more focused on the proposed theme, compared to the previous thread, but also less rich in interventions, in sub-threads of discussion (three main ones), and in the effective interaction in the online discussion, both in the proposed ideas. The main values of the discussion on the online platform were: A) becoming familiar with the nodes of QM; B) promoting the discussion and sharing of their views on the themes; C) discussion on the educational issues related to specific discussed nodes.

The online discussion, which was complemented by the discussion in presence of the related educational proposal, has enabled the expertise in the management of innovative approaches to teaching QM. This trend, which had resulted in education projects often experienced in classroom teaching, was monitored using different tools. From the initial questionnaire, a good knowledge of the contents emerged, even if the vision of the QM-related teaching was more relying on approaches focused on Quantum Physics, rather than on QM.

The Uncertainty Principle and the wave-particle dualism are the theoretical elements that constituted the ultimate goal of the hypothetical learning paths. The online debate, some elements of which were here presented, has gradually shifted the attention to the fundamental issues of QM:

This attention has resulted in professional expertise with the latest design discussed and shared on the network with the intermediate and final projects that have resulted in the project work and the final thesis on which it based its assessment of the Master.

## 6.2 Example 2: Theatre and Science - Installations and Live Performances

These two modules are examples of how Cross-sectional and broad cultural issues offer new important opportunities for learning and counselling and provide the context for understanding the broader role of Science in different contexts. In particular, learning Science in informal context could be fruitful for young students, and teachers should be aware of different experimental projects carried out all over the World. Our attention was focused on the use of artistic languages, thus related to emotions, in order to explain or simply popularize Science. Theater, for example, has been over the Centuries strictly related to Science (concepts of Space and Time and the creation of the plays, in which they are modified in various ways, to fit the dramatic action) and Technology (set and scenery, lighting, stage machinery, etc.).

In addition, there are a lot of plays inspired by scientific themes, or by the lives of great scientists, or both, and many experimentations around scientific subjects. Moreover, if in traditional theatre spectators are passive attending to a show, live performances, interactive installations, happenings, and other new forms of performance, involve the participation of the public at different levels of interaction. This is what is also at the basis of informal centers for learning and education, such as Science Festivals and “hands on” Science Centers. What is possible on a reality level, had also been reproduced in virtual worlds, open to new and more and more interactive and stimulating ways of socialization and collaborative leaning (Second Life, video-games, interactive storytelling, cross-media, m-learning, etc.), that present unimaginable potentialities to teach/popularize Physics and Science in general.

Participants of the course were invited to examine real examples, explaining them the way they were designed in order to achieve their educational aim. In addition, they were invited to create their own project, at least at a design level. Many of the participant teachers were not aware of these kind of experimentations, and were by themselves amazed and at the same stimulated to add more creativity to their teaching strategies and proposals.

## 7 Final Considerations

The commitment of participants required was very high, much higher than the standard one for a Master's degree. The students on the other hand have been of great cultural and professional level, deeply interested in becoming skilled professionals in the subject dealt with. Some have failed to support the workload required to work in parallel with their school-teacher appointments and have left school (3), others have asked for extensions and recoveries (6). A post-graduate specialization course was established, which undertook part of the Master to offer to those in need the opportunity to conclude a shorter training course and welcome a couple of delayed requests for entry.

Apart from the excessive workload, the adopted training model is quite effective and suited to the needs in its integration of cultural, disciplinary, teaching and professional aspects. It integrates a meta-cultural training with an experiential and situational one, offering to everybody the opportunity to design a personalized project development, corresponding to peculiar needs and motivations. We could conclude that the Master seems to be an effective training methodology for training teachers to educational innovation. The possibility of concluding the training at different levels and with flexible durations will be taken into consideration for next editions.

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**References**

[1.] AAVV (1995) *Teaching and Learning Towards the Learning Society*, European Commission White Paper, Bruxelles.
[2.] ANDERSON L.W. Ed. (1995) *International Encyclopedia of Teaching and Teacher Education* (II Edition) - Elsevier Science Ltd. Oxford_UK.
[3.] BARROWS H.S., TAMBLYN R.M., *Problem Based Learning. An approach to Medical Education*, Vol.1, Springler, New York (1980).
[4.] BEDNAR A.K., CUNNINGAM D., DUFFY T.M., PERRY J.D. (1991) *Theory into practice. How do we link?*, in Instructional technology. Past, present and future, J.C. Angelin ed., Englewood, Colorado, Libraries Unlimited, 1991;
[5.] M.D.MERRILL, Constructivism and instructional design in T.M. Duffy, D.H.Jonassen ed., *Constructivism and the technology of instruction*, Hillsdale, New Jersey, Erlbaum (1992).
[6.] BOSIO S., MICHELINI M., SCHIAVONE T., VOGRIC F., *Problem solving per l'orientamento in ambito disciplinare: metodica esempi, formazione degli insegnanti,* Modelli e Strumenti per l'Orientamento Universitario, in una struttura territoriale di orientamento, CRUI - Università di Udine, Forum 1999, p. 346
[7.] BOSIO S., CAPOCCHIANI V., MICHELINI M., VOGRIC F., CORNI F., *Problem solving actrivities with hands on experiments for orienting in science,* Girep Book on Hands on experiments in physics education, G. Born, H. Harries, H. Litschke, N. Treitz Eds. for ICPE_GIREP_Duisburg University, Duisburg, 1998
[8.] BOSIO S, CAPOCCHIANI V., MICHELINI M., PUGLIESE Jona S., SARTORI C., Scillia M.L., Stefanel A., *Playing, experimenting, thinking: exploring informai learning within an exhibit of simple experiments, in New Way for Teaching,* Girep book, Ljubljana 1997

[9.] CARAVITA S., HALLDEN O. (1995) *Reframing the problem of conceptual change*, Learning and Instruction, 4, p.89

[10.] DUTTO M.G., MICHELINI M., ScHiavi S. (2000) *Reinventing in-service teacher training: research grants for teachers, in Practitioner Research - International trends and perspectives*, V. Trafford and F Kroath ed.

[11.] EULER M. (2004), *The role of experiments in the teaching and learning of physics*, in Redish E.F., Vicentini M. (eds.), *Proceedings of the International School of Physics 'Enrico Fermi', Varenna, Course 'Research on Physics Education'* – July 2003, Italy, Amsterdam: IOS Press, pp.175-221

[12.] MICHELINI M., STRASSOLDO M. (1999) *Modelli e Strumenti per l'Orientamento Universitario, in una struttura territoriale di orientamento*, CRUI – Univ. Udine, Forum 1999

[13.] MICHELINI M., *Universty of Udine: orientation Experience.* Forum Europèen de l'Orientation Acadèmique (Fedora) - VI Int. Conference book 1997

[14.] MOSSENTA A., MICHELINI M. (2007), Role play as a strategy to explore spontaneous interpreting models of Electric properties of matter, in Proceedings of *International GIREP Conference Modeling in Physics and Physics Education,* T. Ellermajer & Ed Van den Berg eds, *Amsterdam, The Netherlands. At* http://www.girep2006.nl/

[15.] RESNICK M., WILENSKY U. (1998), *Diving into Complexity: Developing Probabilistic Decentralized Thinking Through Role-Playing Activities*, Journal of the Learning Sciences, *7*(2), 153-171.

[16.] SHULMAN L.S., (1987), *Knowledge and teaching of the new reform*, Harvard Educational Review, 57, 1-22

[17.] WATTS M. (1991) *The Science of Problem Solving - A Practical Guide for Science Teachers*, ed. Cassell Educational Limited, Londra.


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# ICT AND ASSESSMENT OF RESULTS OF EDUCATION

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**Abstract.** Introduction of information technologies into teaching is a worldwide trend at all school levels. In many countries, introduction of ICT into teaching goes hand in hand with curricular reforms, whose aim is a change in the content taught and the teaching methods used with the objective of developing students' ability to use information technologies in their learning processes effectively. Despite the fact that the use of ICT in the education process clearly boosts students' motivation and attention, its effects on the results of education are not so unambiguous and provable. One of the reasons why the use of ICT in education process does not correlate with students' achievement in comparative tests may be that the learning process changes under the influence of the use of ICT, which is not reflected in comparative tests. The paper deals with the question to what extent the assessment of educational results complies with the objectives of education, all this in the highly specific area of use of modern technologies.
The author uses his experience from teaching pre-service teachers of mathematics.
The goal of this paper is to initiate the discussion on the use of ICT in testing. Must it necessarily be regarded as cheating? Is it possible to test efficiently the ability to solve a problem with the help of the available means?

**Key words.** ICT, assessment, tested curriculum

*Mathematics Subject Classification:* Primary 97U50; Secondary 97B50.

## 1 Introduction

Introduction of information technologies into teaching is a worldwide trend at all school levels. In many countries, introduction of ICT into teaching goes hand in hand with curricular reforms, whose aim is a change in the content taught and the teaching methods used with the objective of developing students' ability to use information technologies in their learning processes effectively. One of the competences that a student should master at school is the ability to make an efficient use of ICT in his/her education. In particular, to work meaningfully with information sources, to analyze the found information critically, but also to process real data or model real situations. The role of ICT in education is reinforced in curricular documents; students are expected to use ICT efficiently and with the help of ICT to improve their achievement in all areas of education.

However, despite the fact that the use of ICT in education process clearly boosts students' motivation and attention, its effect on the results of education are not so unambiguous and provable. The effectiveness of the use of ICT in education is influenced by a number of factors. One of them is that what students actually learn at schools is by no means limited only to what is demanded in curricular documents. There are many factors that shape students' attitudes and knowledge and are not officially formulated ([1]). These factors became known as the hidden curriculum about 40 years ago. Hidden curriculum, in general terms, is "some of the outcomes or by-products of schools or of non-school settings, particularly those states which are learned but not openly intended."

Hidden curriculum and its impact on pupils have been subject to studies of many educators and psychologists. The original concept has gradually been narrowed especially to its social implications, political underpinnings, and cultural outcomes ([2]).

This article discusses those effects on students that are not included in official curricular documents and therefore can be regarded as belonging to the hidden curriculum in the broader sense. Since the paper discusses the influences that are related to the teaching content, it moves within the framework of division of curricula to official, taught, tested, learned and hidden ([3]). It focuses primarily on the covert influence of the tested curriculum ([4]). Some aspects of this issue will be demonstrated on examples from pre-service mathematics teacher training (see [5] and [6]) at Charles University in Prague.

## 2. The tested curricula

Testing both at class or school level and on national level is an inseparable part of education. The content and form of testing indirectly determines what a student learns but also how he/she prepares for this testing (Sambell K. and McDowell L. 1998). It can be expected that a student primarily focuses on what will be tested and tries to find a strategy that will lead to the highest possible achievement in the test with the least effort. This hypothesis implies that a student uses in his/her studies and revision ICT only in the extent in which it brings benefit in the subsequent testing.

## 3. What is the role of ICT in testing?

This is a fundamental question if one is to state the influence of the tested curriculum on actual use of ICT at schools. The tested curriculum not only determines how a student prepares for testing but also influences the taught curriculum, in other words how a teachers plans his/her lessons and what teaching methods he/she selects. Both these curricula do not necessarily correspond to the official curriculum.

The current common practice (at least in the Czech Republic) does not allow students to use ICT when tested. However, even if a teacher lets his/her students use ICT, they will definitely not be allowed to use ICT in the school leaving test – the state maturita exam (Czech secondary school leaving exam). They are not even allowed to use graphic calculators. In consequence it might seem that teaching students to solve problems efficiently with ICT is counterproductive as it can result in students' lower achievement in national comparative exams where the use of ICT is forbidden. Paradoxically, the official documents demand that some topics (e.g. functions) should be taught with the help of graphic calculators or appropriate software but the use of these very aids is strictly prohibited when testing the outcomes of this teaching.

The actual use of ICT in teaching is closely connected to the question of the extent in which students are allowed to use ICT in comparative tests and how results achieved with the help of ICT can be assessed.

Undoubtedly, most experts and researchers use ICT in their work: when processing results of their research, when searching information sources, or when discussing the solved problems with colleagues. None of the results are regarded as secondary only because they were achieved with the use of information technologies. In other words in real life everybody is free to make use of all possible information sources and technologies. This is in contrast to school reality where the use of ICT in testing is very limited (with the exception of on-line tests). It is also a common practice that a solution found with the help of ICT is regarded as secondary and less valuable (e.g. in home rounds of mathematical olympiad) or as cheating. The reasons for these teachers' attitudes to the use of ICT spring from several levels. Let us mention here personal, factual and conceptual levels.

### 3.1 Personal level

Teachers are afraid to use ICT in teaching because they lack experience with its use. If the teacher himself/herself cannot use ICT efficiently, he/she usually does not allow his/her students to use it ([8]). However, this cannot be the only and universal reason. The fear of the use of ICT in testing could be heard also in the agenda of a section focusing on the use of ICT in mathematics education on CERME 2010 conference, i.e. a discussion of experts who cannot be suspected of lack of expertise and insufficient knowledge of ICT.
The true reason might be that very few teachers have personal experience with use of ICT in testing. Despite the fact that computers have been used in teaching for more than 50 years, very few teachers have ever sat a test in which they would have been allowed to use ICT unless the test had been directly targeted at the use of ICT. They can hardly imagine the progress of such a test, how to pose and structure its questions and how to evaluate its results.
Another problem may be that mathematical software is at this stage so developed that almost any problem from secondary school mathematics and a considerable proportion of problems at university level may be solved without any knowledge of the needed solving procedures. Teachers therefore fear, and quite legitimately, that students might restrict to a mere entering of the data from the assignment to a computer and subsequent interpretation of the results. This takes us to the factual level of this problem.

### 3.2 Factual level

Is it possible at all to evaluate students' knowledge if, while working on the test, they have access to ICT and information on the Internet?
Setting a test in which students are allowed to use the means of ICT is as a rule much more difficult than setting a standard test. If students are granted access to electronic information sources, it is almost impossible to use factual questions such as "Who was Francesco Petrarca?", "When was the oldest European university established?", or "Define continuity of a function in a point". It is a question of a few seconds to find answers to these questions and therefore they do not test the student's knowledge and comprehension of the topic at all. The author of the test must pose such questions, answers to which cannot be "googled". This can either be achieved by absolute concretization of the question, for example "State which architectonic elements on our municipal house are typical for baroque", or to link several topics, e.g. "What are the inherent differences in the behavior of a continuous function on an open and close interval". In case of these questions it is not enough to find the definitions of cornerstone concepts; they must also be well applied. However, setting and evaluation of such tests is extremely difficult. Community lifestyle of contemporary students and their use of social networks also prevent the teacher from repeated utilization of a relatively limited number of questions of this type. Experience of the author and his

colleagues shows that students share questions from the test and use sample answers as a study text. Therefore the effect is contrary to the initial aim – deeper understanding of the topic.
Another problem is that students while tested might, using the internet, communicate with other persons who then solve the problems instead of the student. In such case the test does not assess the student's knowledge but his/her social skills.

### 3.3 Conceptual level

This level is connected the whole conception of education. It is necessary to clearly and explicitly define what knowledge and skills students must master directly, without use of computer technologies, and where this knowledge may be supported or even substituted by ICT, or from which stage it is beneficial or possible to allow students to use ICT. Knowledge must not be replaced by the ability to look for information and information cannot be interlinked without any prior knowledge. Teaching without the use of ICT often results in formalism, students must learn by heart large quantities of information that are easily accessible on the internet and lack deeper understanding of the issue. On the other hand, teaching with the use of ICT might result in semi-literacy when the student is able to answer a question, or find the answers but has no deeper understanding of the meaning of the answers. This can be illustrated on the example of a university lecturer, who, in a discussion on an international conference focusing on the use of ICT in mathematics education proposed the following problem as suitable for solution with the help of computers: "Reduce the expression". This implies that his students are able to solve expressions with exponents and logarithms, but might be totally ignorant of the definition of both of these functions, or do not know that these functions are inverse. Understanding concepts has thus been substituted by the ability to enter the functions into a computer and to run calculations.

## 4. Experience from Charles University

Curricular changes in pre-service mathematics t
eacher training on Faculty of Education, Charles University in Prague lead to boost of subjects focusing on ICT, introduction of the use of ICT to didactics of mathematics and to some other mathematical subjects. In consequence to these changes also students' assessment and evaluation in state exams have been changed. In the written part of the exam students are now allowed to use any information sources (their own notes, hard copies of textbooks, computers with data retrieved from the Internet). All these materials must be prepared before the exam commences. The students are informed about the topics that will be included in the test a few weeks in advance (e.g. Algebra – Relations, Mathematical analysis – Progressions, Geometry – Conic Sections). Students then have time to prepare relevant study texts, install applicable software and download related information sources. During the exam students are not allowed to use the internet. The reason for this restriction is that students used the internet for communication with third persons. From the conceptual point of view it is not a problem to allow the students to use the internet and monitor that they use it only for work with published information, not for communication with other people. However, so far the technical aspects of this type of monitoring make it impossible to put it in practice. The students are familiar with this testing method from the course of their studies as they are allowed to work with information sources similarly during exams in selected subjects (in some cases even using internet access).

### 4.1 Experience with testing

Despite initial distrust and objections of some of the trainers, this form of the written part of the state exam turned out to be very good. What could be observed was a gradual change in posing and assessment of the problems, as well as a gradual change of students' approach to information technologies.

### 4.2 Changes in assignment and evaluation

The permission to use information sources resulted in changes both in the assignment and evaluation of the problems. The influence of this permission was most marked in assignments in mathematical analysis and computational problems in algebra where the level of difficulty of problems increased by introduction of parameters that prevent direct solution of the problem with the help of CAS programs. The emphasis shifted to justification of one's solution. Students thus may use computer technologies to find a solution but must then be able to justify the correctness of the found solution, or to interpret correctly the solving procedure used by the computer.
All problems are posed in such a way that they may also be solved without the use of computer programmes. It is up to the student to decide whether to use ICT or not. This decision often depends on their ability to use ICT efficiently.

### 4.3 Students' approach

In the first tests students used computers very rarely (only 20 % – 30 % of students), they only downloaded study texts in digital form. Usually they did not make much use of these materials; some students brought their computers but did not even turn them on in the course of the exam. In the first year, only one student had installed and was using a programme from CAS category. In the following years the number of students using computer in the test was increasing, nowadays almost all students work with their notebooks. The way in which students use their computers has also changed over the years. Nowadays they have installed programmes for dynamic geometry and computer algebra systems (CAS). In some cases they also have downloaded offline versions of selected websites (most often Mathematical Assistant on Web). They are able to make effective use of these programs in problem solving.
Students also responded to the new trends in assessment and evaluation in which it is no longer enough to find a solution but it is also necessary to justify the solving procedure. Originally, when use of computers was introduced to testing and exams, we came across a few cases when students only presented in their paper the result calculated by the computer. They tried to persuade the trainer that such a solution was sufficient. Nowadays they fully respect the fact that justification of the solving procedure is necessary and perceive solution computed by the computer without giving the correct solving procedure as incomplete. We could also observe that the longer the students are allowed to use computers in the exams, the better their ability to use them in solution of problems is.

## 5. Conclusion

The tested curriculum considerably influences how students prepare for exams and what the results of teaching are. The role of ICT in education is limited also by the extent to what students are allowed to use it in testing. This practical use of ICT cannot be adequately substituted by tests focusing directly on "computer literacy". The experience from Faculty of Education, Charles University in Prague shows that under certain conditions it is possible to carry out testing in which

students are allowed to use ICT. Such testing results in students' own efforts to use ICT in such a way that is of practical benefit.
The nature of mathematics and mathematical software are specific. It is therefore not easy to answer how the above described experience can be generalized and applied in other subjects. It can be presumed that the importance of the role of ICT in education will grow and sooner or later all students will (albeit legally or illegally) use ICT also during testing. To what extent this use should be legal, how tests should be set and what knowledge, skills and competence might be thus tested is a subject for further research and discussion. Undoubtedly the change in testing must also be reflected in the teaching content, as the taught and the tested must correlate. This is another topic for a broader discussion.
However, teachers and educators must be ready for such a discussion. Personal experience that pre-service teachers get at Charles University is a solid ground that will enable them to assess the role of ICT and to evaluate correctly the solutions achieved with its help.

**Acknowledgement**

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**References**

[1.] MARTIN, J. "What Should We Do with a Hidden Curriculum When We Find One?" *The Hidden Curriculum and Moral Education*. Ed. Giroux, Henry and David Purpel. Berkeley, California: McCutchan Publishing Corporation, pp. 122–139, 1983.
[2.] APPLE, M. and KING, N. "What Do Schools Teach?" *The Hidden Curriculum and Moral Education*. Ed. Giroux, Henry and David Purpel., California: McCutchan Publishing Corporation, Berkeley, pp. 82–99, 1983.
[3.] LONGSTREET,W.S. and SHANE, H.G *Curriculum for a new millennium*, Allyn and Bacon, Bostom, 1993.
[4.] JOUGHIN G. "The hidden curriculum revisited: a critical review of research into the influence of summative assessment on learning" *Assessment & Evaluation in Higher Education*, Vol. 35, Iss. 3, pp. 335-345, 2010.
[5.] ADAMEC, M.; JANCARIK, A.; JANCARIKOVA, K., STEHLIKOVA N. ICT in the Education of Future Teachers in *Aplimat 2009: Proceeding from the 8-th International conference Aplimat*, pp. 637-643, 2009
[6.] JANCARIK, A. Increasing the Effectivity of Teaching Mathematics At Universities in Aplimat 2005 : Proceeding from the 4-th International conference Aplimat: pp. 305-308, 2005
[7.] HALDER S. and CHAUDHURI S. "Psychological Challenges Towards Implementing ICT in Teaching-Learning", in *Proceedings of the 5th International Conference on e-learning*, Univ Sains Malaysia, Penang, pp. 140-148, 2010.

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# STUDENTS' AUTOEVALUATION IN THE FIRST YEAR OF SCIENCE STUDY FIELDS ESPECIALLY IN MATHEMATICS PHASE 2

**KONEČNÁ Petra, (CZ), HABIBALLA Hashim, (CZ)**

**Abstract.** The paper sketches results of educational research among students of the first grade in Faculty of Science, University of Ostrava. It analyses mathematical knowledge upon several criteria mainly secondary school types and study fields. It shows differences in several mathematical topics according to secondary school attended.

**Key words.** mathematical education, topics, frame curricula.

*Mathematics Subject Classification:* Primary 97A06; Secondary 97R06.

## 1 Introduction

In the frame of the curricular reform in the Czech Republic there are developing and implementing frame curricula (RVP) preceded by more strict standard curricula. In the scope of particular schools they are called school curricula (SVP), which are derived from RVP. First RVP for secondary schools was accepted in 2007 and from 2009 first groups of schools starting to teach according to school curricula. According to the project of implementing RVP to school curricula, the last phase proceeds in 2012.
In comparison with old curricula although they describe compulsory and optional topics and they are more flexible than old ones. It leads to high imbalance of students' mathematical knowledge from school to school.

### 1.1 Research and Objectives

We have decided to observe students' basic knowledge in mathematics starting from 2010 from above mentioned reasons. It affects students of the first grade studying all courses on Faculty of Science University of Ostrava.

We observe following 8 mathematical topics:

- sets and numerical fields (SET)

- propositional logic and proofs (LOGIC)
- functions (FUNC)
- in/equations (EQUAT)
- fundamentals of the differential and integral calculus (MA)
- combinatorics (COMB)
- probability and statistics (STAT)
- analytical geometry (GEOM)

Each topic is represented by four or five notions. Every student can evaluate knowledge on the scale 1 – 5, where 1 is no knowledge and 5 is maximal knowledge. It has been used the method Analysis of Variance (ANOVA). We analyzed results upon the secondary school attended and study fields[1].

Particular types of schools are the following: G - Gymnasium (preparation for university studies), SPS - industrial school (preparation for practice in several technical branches), SOS - integrated school (specialized branches for practice), OA - secondary business school (economically oriented for practice), SOU - secondary education for practice in a trade.

The questionnaire aim is to monitor students' own perception of basic knowledge according to the needs of particular fields of study on the Faculty of Science. Therefore we do not directly test the knowledge, but students' self-evaluation. The results are compared annually to observe overall trend. They are also compared upon input requirements of particular study fields.

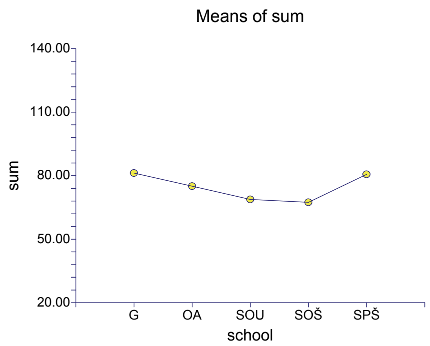
### 1.2 Research in 2010

In 2010 we compared results of the research with curricula at secondary schools [3]. We predicted that differences between secondary schools providing preparation for practices (SPS, SOS, OA, SOU) and secondary schools providing preparation for university studies (G) will be especially in the topic LOGIC and next differences would be rather at an intensity of knowledge. All school leavers should have worse knowledge in topics GEOM and MA. In the contrary, they should have strong knowledge in EQUAT, FUNC and STAT.

Our assumptions were confirmed partially. Expected better results in the topic LOGIC was confirmed at G, but very good results were observed at SPS too. Statistically significant differences were identified in other topics. On the basis of Bonferroni (All-Pairwise) Multiple Comparison Test we could separate schools to four groups (Table 1). Likewise strong consciousness in the topic MA does not correspond to the fact, that this topic is not obligatory at secondary schools already since year 1999. On the contrary, in the long term fixed subjects at secondary schools STAT have week position in the consciousness of respondents.

We also verified that that student’s knowledge depended upon the study field.

[1] They are marked field’s codes in the chapter 2.1.




*Picture 1: Results of Anova 2010.*

| | school | Different From |
|---|---|---|
| 1. | G, SPŠ | SOU, SOŠ |
| 2. | OA | SOŠ |
| 3. | SOU | G, SPŠ |
| 4. | SOŚ | G, SPŠ, OA |

*Table 1: Division of schools according to Bonferroni Multiple Comparison Test (2010)*

## 2 Research in 2011 and comparing with results in 2010

In 2011 we continued in a research of the same type. The sample included 527 respondents.

At first again, we divided answers into two classes, the class 1 = "I've never hear about it" and the class 2 = „other answers". We obtained a similar outcome as in the year 2010, i.e. results did not correspond with curricula at secondary schools [1], [2], [4].

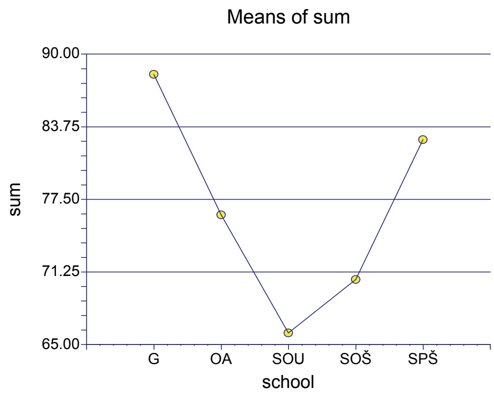
| | | SET | LOGIC | FUNC | EQUAT | MA | COMB | STAT | GEOM |
|---|---|---|---|---|---|---|---|---|---|
| G | 243 | 88 | 91 | 98 | 100 | 92 | 97 | 84 | 94 |
| OA | 37 | 57 | 51 | 97 | 100 | 86 | 97 | 51 | 86 |
| SOŠ | 151 | 63 | 42 | 88 | 97 | 66 | 90 | 60 | 85 |
| SOU | 22 | 55 | 36 | 86 | 91 | 59 | 68 | 45 | 82 |
| SPŠ | 74 | 81 | 73 | 96 | 100 | 92 | 99 | 82 | 92 |
| | | | | | | | | | |
| celkem | **527** | 76 | 69 | 94 | 99 | 83 | 94 | 73 | 90 |

*Table 2: Percentage of students's knowledge (2011).*

Consequently we used the method ANOVA to observe differences between schools (in overall results and for every particular topic) and between study fields.

### 2.1 Data analysis and results in 2011

Results of ANOVA showed statistically significant differences between schools in overall results - Fratio = 28.16 , i.e. Prob. level < 0,000001.




*Picture 2: Results of ANOVA with respect to schools.*

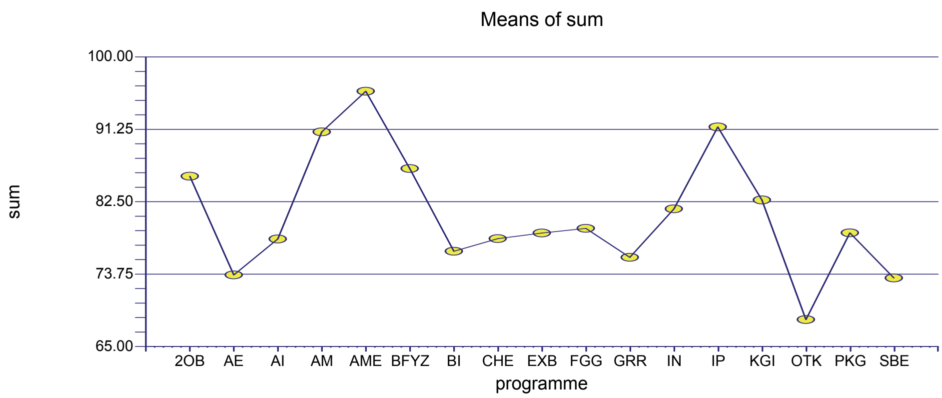
For multiple comparing and identification of a different group of schools we used Bonferroni (All-Pairwise) Multiple Comparison Test.

| Group | Count | Mean | Different From Groups |
|---|---|---|---|
| SOU | 22 | 66 | SPŠ, G |
| SOŠ | 151 | 70.59602 | SPŠ, G |
| OA | 37 | 76.16216 | G |
| SPŠ | 74 | 82.63513 | SOU, SOŠ |
| G | 243 | 88.25103 | SOU, SOŠ, OA |

*Table 3: Results of Bonferroni (All Pairwise) Multiple Comparison Test with respect to schools.*

The best results had school leavers of G, succeeded SPŠ. On the other side there was the group of school leavers SOU and SOŠ. Statistically significant differences were observed in overall results and for every particular topic too. The most significant differences were in topics LOGIC and SET.

In the end we're verified, whether knowledge of student depend upon the study field. We used ANOVA and Bonferroni (All-Pairwise) Multiple Comparison Test and we obtained following results.




*Picture 3: Results of ANOVA with respect to study fields.*

| Group | Count | Mean | Different From Groups |
|---|---|---|---|
| OTK | 24 | 68.25 | 2OB, AM, IP, AME |
| SBE | 29 | 73.27586 | IP, AME |
| AE | 31 | 73.64516 | IP, AME |

| | | | |
|---|---|---|---|
| GRR | 35 | 75.77143 | AME |
| BI | 2 | 76.5 | |
| AI | 66 | 77.98485 | AME |
| CHE | 24 | 78.04166 | |
| EXB | 31 | 78.70968 | |
| PKG | 41 | 78.7317 | AME |
| FGG | 33 | 79.27273 | |
| IN | 30 | 81.63333 | |
| KGI | 10 | 82.7 | |
| 2OB | 113 | 85.56637 | OTK |
| BFYZ | 4 | 86.5 | |
| AM | 13 | 90.92308 | OTK |
| IP | 21 | 91.52381 | OTK, SBE, AE |
| AME | 19 | 95.8421 | OTK, SBE, AE, GRR, AI, PKG |

*Table 4: Results of Bonferroni (All Pairwise) Multiple Comparison Test with respect of schools.*

## 2.2 Comparison of results - 2010 vs. 2011

If we compare results of Bonferroni (All-Pairwise) Multiple Comparison Test in years 2010 and 2011, we can observe increasing difference between G and other school types and also decreasing level of OA.

| 2010 | school | different from |
|---|---|---|
| 1. | G, SPŠ | SOU, SOŠ |
| 2. | OA | SOŠ |
| 3. | SOU | G, SPŠ |
| 4. | SOŚ | G, SPŠ, OA |

*Table 5: Division of schools according to Bonferroni Multiple Comparison Test (2010).*

| 2011 | School | different from |
|---|---|---|
| 1. | G | OA, SOU, SOŠ |
| 2. | SPŠ | SOU, SOŠ |
| 3. | OA | G |
| 4. | SOU, SOŚ | G, SPŠ |

*Table 6: Division of schools according to Bonferroni Multiple Comparison Test (2011).*

If we compare measured annual results we can observe positive trend in knowledge evaluation - students have better notion fixation (table 7). However when we compare the knowledge on the two-valued scale (no knowledge vs. certain knowledge), it can be concluded that no significant difference is found. It can be unfortunately observed decreasing knowledge in mathematical analysis and logic. Especially low logic knowledge may be important disadvantage for students' efforts to enroll university studies because several entry examinations may consist of logical task (table 8).

| | MNOOBR | LOGIKA | FUNKCE | ROVNICE | MA | KOMB | PASTA | ANAGE |
|---|---|---|---|---|---|---|---|---|
| **2010** | 2,19 | 2,05 | 2,6 | 2,88 | 2,38 | 2,52 | 2 | 2,3 |
| **2011** | 2,29 | 2,06 | 2,71 | 3,01 | 2,38 | 2,63 | 2,09 | 2,41 |

*Table 7: Average mark of knowledge 2010 and 2011.*

| | SET | LOGIC | FUNC | EQUAT | MA | COMB | STAT | GEOM |
|---|---|---|---|---|---|---|---|---|
| 2010 | 71 | 73 | 95 | 99 | 88 | 86 | 71 | 89 |
| 2011 | 76 | 69 | 94 | 99 | 83 | 94 | 73 | 90 |

*Table 8: Percentage of students's knowledge 2010 and 2011.*

## 3 Conclusions

The research results in 2011 have shown statistically significant differences in knowledge of students of different school types. There can be also seen increasing gap between G and other types of schools. We can also state the increasing fixation of knowledge but not the knowledge itself. There are small fluctuations corresponding to the secondary education e.g. in mathematical analysis but there are also ones not to be advocated by the RVP. General conclusions can however done upon long-term observations.

**Acknowledgement**
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## References

[1.] *Curricula for Gymnasium*. MŠMT, Fortuna, Praha, 1999. ISBN 80-7168-659-X.
[2.] *Framework Education Programme for Secondary General Education* [online]. VÚP, Praha, 2007. ISBN 978-80-87000-11-3.
[3.] KONEČNÁ P., HABIBALLA, H.: *Student's Autoevaluation in the First Year of Science Fields Especially Mathematics Abilities.* In Mathematics XVI, Scientivic Issue Jan Długosz University in Częstochowa, pp. 219-224, 2011. ISBN 978-83-7455-209-7.
[4.] *Manual for School Education Programmes at Gymnasium* [online]. VÚP, Praha, 2007. ISBN 978-80-87000-13-7.
[5.] HABIBALLA, H.; KMEŤ, T. Theoretical branches in teaching computer science. In International Journal of Mathematical Education in Science and Technology. 6/2004(35), pp.829-841, Taylor and Francis : Great Britain, ISSN 0020-739X.

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# THE USE OF SIMULATED RANDOM SAMPLES FOR TEACHING STATISTICS

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**Abstract.** This paper contains examples of exploitation simulation studies during researching properties of normality tests, test of first-order autocorrelation, and also of interval estimates of parameters of the exponential regression model using three various estimate methods.

**Key words:** Simulated random sample, normality tests, tests of the first-order autocorrelation, linearized regression model.

*Mathematics Subject Classification:* Primary 62-01; Secondary 97K80.

## 1 Introduction

Almost every teacher of mathematical statistics was asked to answer the question: "Which statistical test should be applied when there are a few alternatives?" The teachers mostly accept recommendations of the professional literature. We have decided to use results of our own simulation studies and present here the conclusions we have come to. When we have been selecting the tests and methods the properties of which we have wanted to explore by means of simulated random samples, we have been used our long time experience in teaching statistics at both the Faculty of Science, Masaryk University in Brno, and Faculty of Mechanical Engineering, Brno University of Technology.

## 2 Normality tests

In Marek Haičman bachelor thesis (tutor – Marie Budíková, thesis successfully defended in 2011) are compared Shapiro-Wilk test (S-W test), Liliefors test, and Anderson-Darling test (A-D test). In accordance with the assignment of his work, Haičman deals with estimates of probability of type I and type II errors. The MATLAB system was used for the simulations.

Note: Fig. 1 and 2 are taken from [6], so labels are in Czech.

### 2.1 Estimate of probability of the type I error

From the standardized normal distribution 100,000 random samples of ranges from 5 to 1,000 were generated. All three tests (significance level – 0.05) were applied on these samples. A relative frequency of the cases when an unjustified rejection of the true null hypothesis occurred was determined. This relative frequency is considered an estimate of the probability of the type I error $\alpha$. The dependency of the estimate $\alpha$ (vertical axis) on the sample size is shown on the Fig. 1 while the values on the horizontal axis (range of random samples) are calculated using logarithms.

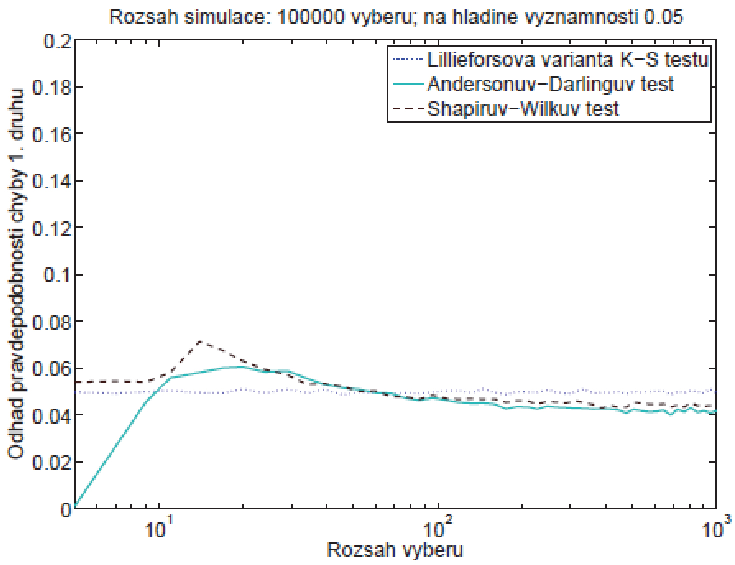


Fig. 1

The Liliefors test produces an estimate $\alpha$ independent of the sample size and remains on 5 %.
The S-W test has the highest estimate $\alpha$ for the sample size 60, then drops under 5 % and does not exceed 5 %.
The A-D test has the lowest estimate $\alpha$ for the sample size up to 10 elements and above 60 elements.

### 2.2 Estimate of probability of the type II error

For this research following distributions were selected: continuous uniform distribution on the interval (0,1), normalized exponential, log-normal with parameters 0 and 1, and Student's with one, three and five degrees of freedom. For each of these distributions 100,000 random samples were generated with of the ranges from 5 to 1,000. All three tests were applied on this samples and a relative frequency of the cases when the test did not reject the false null hypothesis was determined. This relative frequency is considered the estimate of probability of the type II error $\beta$.

Fig. 2 shows a dependency of the estimate $\beta$ (vertical axis) on the sample size (ranges of random samples on the horizontal axis are calculated using logarithms) for the exponential distribution.

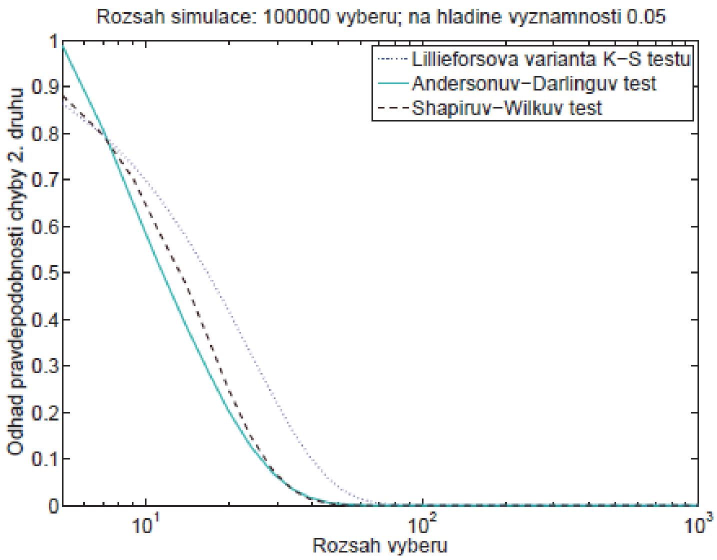


Fig. 2

The results of the simulations for all the above described distributions may be summed up in the following way: the Liliefors test is convenient to be used for small samples, roughly up to 10 elements. The S-W test and A-D test rarely make errors for the samples with rather big size (over 60). However, errors occur mainly for very big samples from Student's distribution which is quite difficult to differentiate from the normal distribution.

## 3 Two tests of the first-order autocorrelation

The Durbin-Watson test (D-W test) is used in the linear regression model for testing the following hypothesis: the random deviations of the exact model are non-correlated random variables against the alternative hypothesis that there is a positive or negative first-order autocorrelation between them. It is assumed that the random deviations come from the distribution $N(0, \sigma^2)$.

Critical values $d_L(\alpha), d_U(\alpha)$ are tabulated for significance level $\alpha$, the number of observation (n) and the number of regression parameters p (without constant).

For the testing of the positive autocorrelation at significance $\alpha$, we proceed the following way: if $D \in (d_U(\alpha), 2)$ we do not reject null hypothesis and if $D \in (0, d_L(\alpha))$ we accept alternative hypothesis. If $d_L(\alpha) \le D \le d_U(\alpha)$, then we can accept no decision (we call that the test is silent).

For the testing of the negative autocorrelation at significance $\alpha$, we proceed the following way: if $D \in (2, 4 - d_U(\alpha))$ we do not reject null hypothesis and if $D \in (4 - d_L(\alpha), 4)$ we accept alternative hypothesis. If $4 - d_U(\alpha) \le D \le 4 - d_L(\alpha)$, we accept no decision.

The other explored test - we call AR(1) test - is applied when it is assumed that the stationary time series is governed by the autoregression first-order model AR(1) with an unknown correlation coefficient $\rho$. The null hypothesis says that there is no linear dependency between two adjacent members of the time series, i.e. $\rho = 0$, while the alternative says the opposite. The null hypothesis is rejected on the significance level $\alpha$ when $|\hat{\rho}|$ lies outside the $100(1-\alpha)\%$ confidence interval for the correlation coefficient $\rho$.

### 3.1 Description of simulations

Exploring the power of these two tests, we simulated data governed by the model of the regression line. 50,000 numbers for each $n \in \{10, 15, \ldots, 45, 50\}$ were generated from the continuous uniform distribution on the $(10, 20)$ interval and distributed in ascending order based on their size: $x_1 \le x_2 \le \ldots \le x_n$ in the MathCad system. Then we applied on the each $x_i$ the linear transformation $\beta_0 + \beta_1 x_i$.

Next, a sequence of autocorrelated numbers with a correlation coefficient $\rho$ (computed according equation $\varepsilon_i = \rho\varepsilon_{i-1} + u_i, i = 2, \ldots, n$, where $u_2, \ldots, u_n$ are noncorrelated random variables with normal distribution N(0,1) and $\rho$ is unknown correlation coefficient, $\rho \in \{0.2; 0.3; \ldots; 0.9\}$. This sequence was added to the $\beta_0 + \beta_1 x_i$ sequence as the correlated random effects that provided data $y_1, y_2, \ldots, y_n$.

The least-squares method was used to calculate estimates $b_0, b_1$ of the parameters of the regression line $y = \beta_0 + \beta_1 x$. The residuals $e_i = (\beta_0 + \beta_1 x_i) - (b_0 + b_1 x_i)$, $i = 1, \ldots, n$ was subjected to the Durbin-Watson test and AR(1) test. The probability of the type II error $\beta$ was estimated as a relative frequency of the cases when the test did not recognize the existence of the autocorrelation, i.e. it did not reject the false null hypothesis $H_0 : \rho = 0$. However, the power $(1 - \beta)$ of the test is more interesting and we will discusse it in the text below.

### 3.2 Results of simulations

The results of the simulations are shown in the Fig. 3 and 4. On the horizontal axis, there are sizes of samples, while the vertical axis shows estimates of the power of the test. The estimates of both the tests for various correlation coefficients are then differentiated by various types of lines.

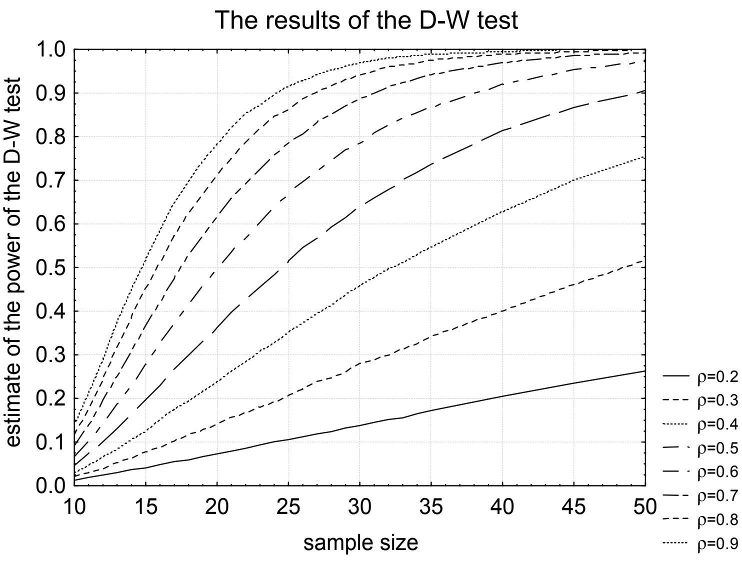


Fig. 3

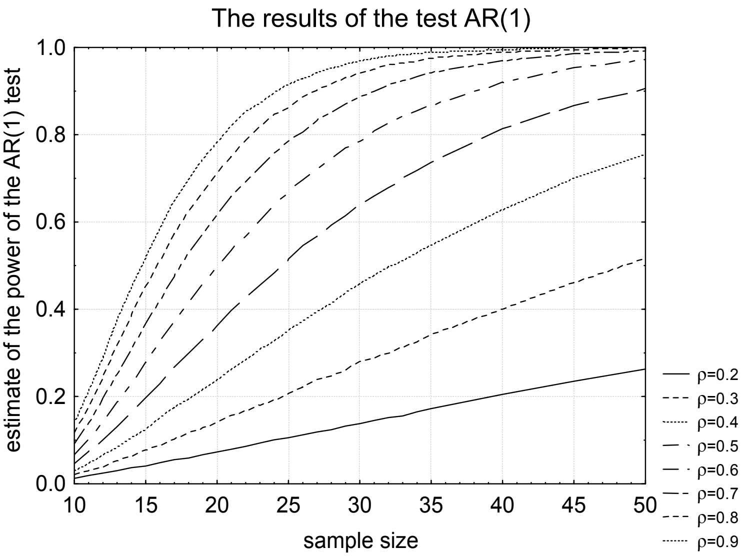


Fig. 4

It is clear from both the graphs that for the low values of the correlation coefficient ( $\rho = 0.2$ or $\rho = 0.3$ ) the estimate of the power of the test is very low, especially for small sizes of the simulated samples. As the correlation coefficient and size of the random sample grow, the estimate of the power of the test grows too.

Comparing the estimates of the Durbin-Watson test and of the test of the first-order autocorrelation, we can see that the Durbin-Watson test shows higher estimates of the power of the test, especially for the small samples. This can be explained by the fact that the Durbin-Watson test is determined for testing the autocorrelation of residuals of the regression model while the other test is more general.

## 4 Comparison of the quality of the three methods of estimates of parameters of the exponential regression function

For the estimates of parameters of a linearizable regression model, the least-squares method is usually used. However, this method does not usually provide the really minimal sum of the squares, as the use of a non-linear transformation violates the normality of the random deviations and homogeneity of their variances.

In this case, it proves to be good to estimate the parameters of the linearized model by the weighted least-squares method or by the iterated weighted least-squares method. Another option to estimate the parameters is to use the iteration method suitable for solving non-linear equations. We decided to explore the properties of all these three methods of estimation on the simplest linearizable non-linear model $y = \exp(\beta_0 + \beta_1 x)$.

### 4.1 Description of simulations

Let us consider a model $y = \exp(\beta_0 + \beta_1 x)$ with parameter values of $\beta_0 = 6.19411$, $\beta_1 = 0.0205$ (the exponential curve goes through points (1, 500) and (71, 2100)). Let us select 15 (alternatively 29) values of the independent variable x, i.e. $x_i = 5i - 4$, i = 1, 2, …, 15 ( $x_i = 2{,}5i - 1{,}5$ , i = 1, 2, …, 29). Thus, we obtain values $y_i = \exp(\beta_0 + \beta_1 x_i)$. To these accurate values a random effects

$\varepsilon_i \approx N(0, \sigma^2)$ are added, while the standard deviation $\sigma$ is gradually selected, i.e. 25, 50, 75, …, 400. Thus, for each value $x_i$, i = 1, …, 15 (29) we will obtain the only value $Y_i = \exp(\beta_0 + \beta_1 x_i) + \varepsilon_i$. For each of the 16 models we then have 15 (29) values of the dependent-variable, altogether 240 (464) simulated values.

The unknown regression parameters $\beta_0, \beta_1$ of the linearized model $\ln y = \beta_0 + \beta_1 x$ will be estimated for each simulated random sample in three ways, i.e. least-squares method, iterated weighted least-squares method, and Gauss iteration method. As the initial approximation for the Gauss method we will select the estimate provided by the least-squares method.

Because we know the parameters of the model, we will explore the properties of the estimates not with the help of the residual sum of the squares but by the distance of the estimate $b_i$ from the $\beta_i$ value of the parameter. The estimate $b_i$ obtained by the least-squares method or iterated weighted least-squares method will be considered "reliable", if it is satisfied the inequality $|b_i - \beta_i| < \Delta$, where $\Delta$ is one half of the width of the 95% confidence interval for the $\beta_i$ parameter, namely confidence interval for the Gauss iteration method (see [12]). The relative frequency of these reliable estimates will be marked as the relative reliability of the estimate. This relative reliability is then related to the estimate obtained by the Gauss iteration method.

### 4.2 Results of simulation

The Fig. 5 and 6 show the dependency of the relative reliability of the estimate on the standard deviation for various numbers of observations (15 and 29).

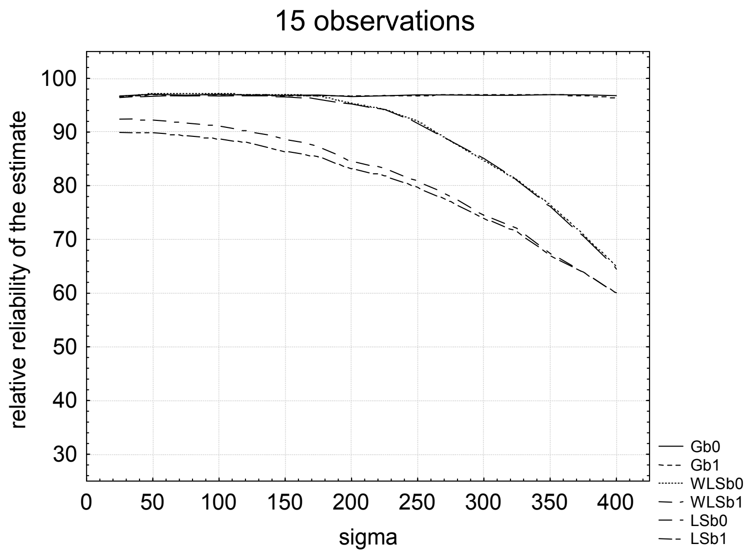


Fig. 5

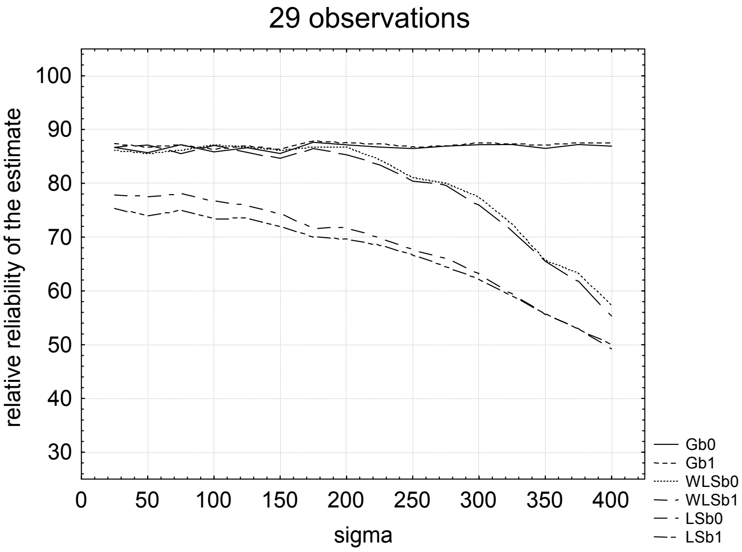


Fig. 6

*Clue:*
*Gbi … estimate provided by Gauss iteration method*
*WLSbi … estimate provided by the iterated weighted least-square method*
*LSbi … estimate provided by least-squares method*

We can see that for both regression parameters the inaccuracy of estimates grows with the growing variability of observations. Also it is evident that the most reliable estimates are provided by Gauss iteration method, then the iterated weighted least-squares method, and then the least-square method. We receive much more accurate estimates for the $\beta_1$ parameter than the $\beta_0$ parameter, i.e. there is a difference of two orders.

The Gauss method provides the best results, but it does not necessarily always converge (it very much depends on the initial approximations). This is why the use of it is not very widespread. Also, the iterated weighted least-squares method is not very known (not even common statistic sources mention it). least squares method, which is most used, gives the worst results.

## 5 Conclusion

The computer simulations have been used for teaching probabilities and statistics for a long time, for example to illustrate the force of the law of large numbers or central limit theorem. In this paper, we attempted to introduce other options to study the properties of various statistical tests and methods with the help of simulated random samples.

## References

[1] ANDĚL, J. (1978) *Matematická statistika.* STNL/Alfa, Praha.
[2] BOX, G. E. P., JENKINS, G. M. and REINSEL G. C. (1994) *Time Series Analysis: Forecasting and Control*. 3rd ed. Upper Saddle River, Prentice-Hall, New York.
[3] BROWN, M. B. and FORSYTHE, A. B. (1974) *Robust Tests for the Equality of Variance*. Journal of the American Statistical Association, 69, 364-367.

[4] DURBIN, J., and WATSON, G. S. (1950) *Testing for Serial Correlation in Least Squares Regression, I*. Biometrika **37**, 409 - 428.

[5] DURBIN, J., and WATSON, G. S. (1951) *Testing for Serial Correlation in Least Squares Regression, II*. Biometrika **38**, 159 - 179.

[6] HAIČMAN, M. (2011) *Simulace a testy normality*. MU Brno. On the website: http://is.muni.cz/th/150689/prif_b_a2/simultestynormality.pdf

[7] Critical Values of Durbin – Watson test. On the website: www.nd.edu/~wevans1/econ30331/Durbin_Watson_tables.pdf

[8] LEVENE, H. (1960) *Contributions to Probability and Statistics*. Stanford University Press.

[9] MAROŠ, B. (2001) *Empirické modely I.* CERM, Brno.

[10] Mathcad software. On the website: www.dtn.mathsoft.cz

[11] SAVIN, N.E. and White, K.J. (1977) *The Durbin-Watson Test for Serial Correlation with Extreme Sample Sizes or Many Regressors*. Econometrica **45**, 1989 - 1996.

[12] STATISTICA software. On the website: www.statsoft.com/products/analytics.htm and www.statsoft.com/textbook/stnonlin.html.

[13] ZVÁRA, K. (1989) *Regresní analýza*. Academia, Praha.


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# MOTIVATION IN MATHEMATICS AND PHYSICS – MPEMBA´S EFFECT

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**Abstract.** Nowadays, in times characterized by decreasing interest of young generation in studying of physics and chemistry plays reading role motivation. It is well known that interest of pupils and students can be awaked by their own research of interesting phenomena. The typical example of such phenomenon is so called Mpemba´s effect, i.e. special phenomenon where "hot water supposedly freezes faster than cold water". The aim of our contribution is experimental research of Mpemba´s effect. The results of our work did not confirm this phenomenon and showed that Mpemba´s effect is most probably mere artifact.

**Key words.** Water, freezing, evaporation, convection, time dependent temperature, Mpemba´s effect, artifact.

*Mathematics Subject Classification:* Primary 80-05 ; Secondary 97D40.

## 1 Introduction

The phenomenon that hot water may freeze faster than cold water is often called the Mpemba effect. Erasto Mpemba, Tanzanian high school student was usually making together with his colleagues ice cream by mixing boiling milk with lot of sugar. He was supposed to wait for the milk to cool before placing it to the refrigerator. But sometimes he put his milk in without cooling it and was very surprised that hot milk froze into ice cream earlier than cold milk of other students.
Mpemba discussed it with his teacher, but he did not believe him and in addition made fun of him ("That is Mpemba´s physics and mathematics"). Later, Dr. Osborne, a professor of physics at Dar es Salam University repeated Mpemba´s experiment and his repeated tests gave the same results [1] Later Dr. Kell confirmed Mpemba´s effect and explained it by more intensive evaporation of hot water. But he was unaware of Osborn´s experiments [2] which invalidated Kell´s explanation. Also many others scientists repeated Mpemba´s and Osborne´s experiments with similar results and explained them by following basic hypothesis [3 - 6]:

1. Evaporation. (Initially warmer water loses significant amounts of water to evaporation). Theoretical calculations have shown that evaporation cannot only explain experiments that were done in closed containers.

2. Dissolved gasses. Hot water holds less dissolved gas than cold water. This difference can influence convection currents cooling water. But there are some experiments that disprove such hypothesis.
3. Convection. Initially hot water develops more intensive currents, cooling the water i.e. cooling is also more intensive (that at initially cool water). But there is a basic objection: hot water all the time made up for temperature of cold water, which is in every time cooler than initially hot water.
4. Surroundings. An explanation of Mpemba´s effect based on assumption of melting in ice layer on the bottom of the refrigerator (which is much higher for hot container than for cold one) has not scientific justification: the conditions of experiments should be the same for both containers with hot and cold water [6].

Also other theories and explanations as overcooling of water or change of structure between cold and hot water [3 - 6]cannot reliably explain Mpemba´s effect. It is clear that many other experiments at exact physical conditions should be made: the same amount of hot and cold water, the same conditions of heat exchange and very accurate thermometers in both containers (making possible measuring of time dependence of temperature).

## 2. Experimental procedure and results

Experimental verification of so called Mpemba´s effect has been investigated by means of very precise thermometer Giessinger GMH 3250, enabled measurement of time dependent temperature in small volumes (accuracy ±0.1 $^0$C). We obtained many graphs $t = f(\tau)$ ($t$ is temperature in $^0$C and $\tau$ is time in s). The liquids measured were distilled water at various initial conditions, some acid and salt solutions and other homogenous liquids (milk, ice creams, cider and so on).
Two identical amounts of hot and cold liquids were placed into two small containers made from plastic and placed into freezer on a polystyrene mat. At such arrangement we were sure that initial conditions were equal without any parasitic phenomena (as could be for example melting hot container into ice on the bottom of freezer).
We have measured very wide set of liquids: distilled water one times or many times boiled, various temperature intervals between initial temperatures of hot and cold water (10 $^0$C – 50 $^0$C), drinking water, rainwater, NaCl solutions of various concentrations, ascorbic acid of various concentrations, milk with sugar (according to Mpemba), cider and two sorts of ice creams. At all measurements we obtained typical curves with more or less explained part of phase transition (Fig. 1 -10). ***In none of our measurements we have not observe Mpemba´s effect.***
Apparent "Mpemba´s effect we have observed at experiments where the amount of hot water was less than cold water (Fig. 8 – 10). From that figures we can see that it was for the difference cca 10 ml (12 wt %).

## 3. Conclusion

The results of our experiments confirm that if all parasitic phenomena are removed, so called Mpemba´s effect is not observed (the same conclusion was presented also in [7]). Evidently the simple logical deliberation is valid: hot liquid reach the temperature of cold water later and so phase transition finish also later. In our opinion Mpemba´s effect is artefact, probably caused by the fact that Mpemba (and maybe also some other authors) placed into freezer less amount of hot liquid than cold one (a part cold be evaporated). Other explanations are pointless if we take the view that Mpemba´s effect is artefact.

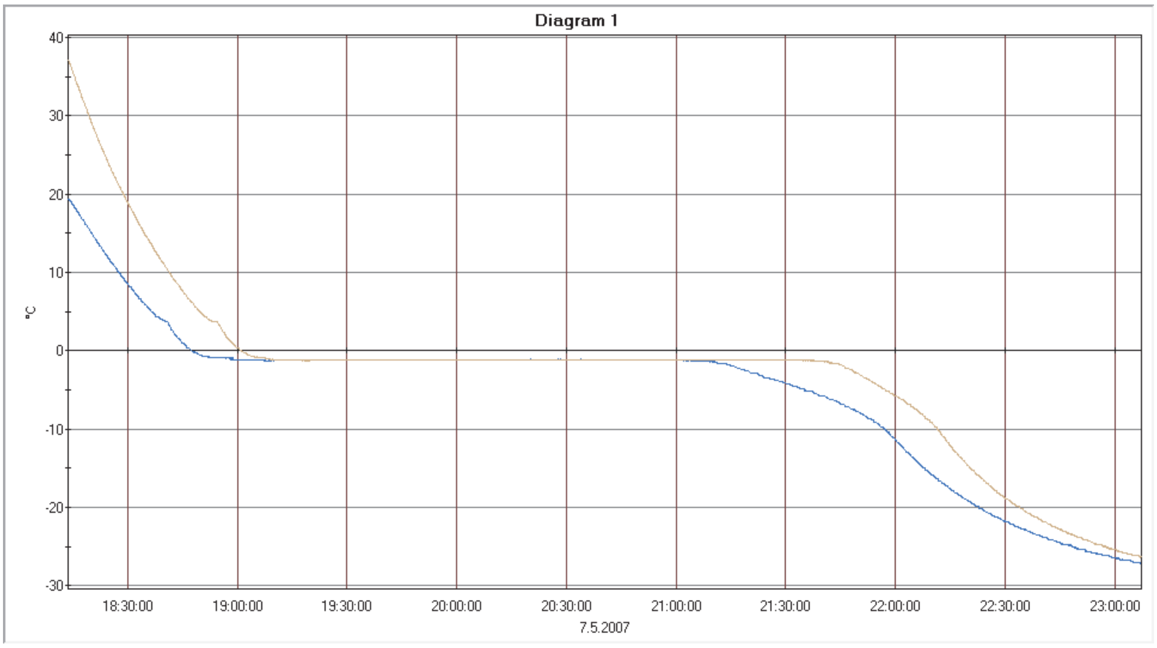


Fig.1. Distilled water

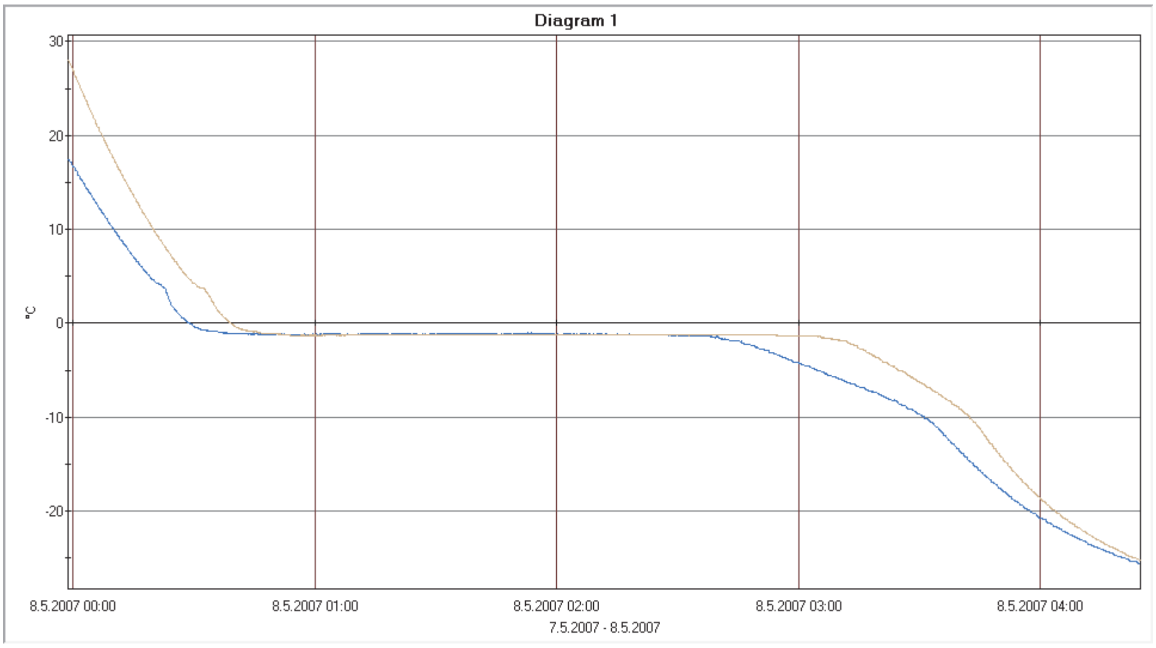


Fig.2. Drinking water

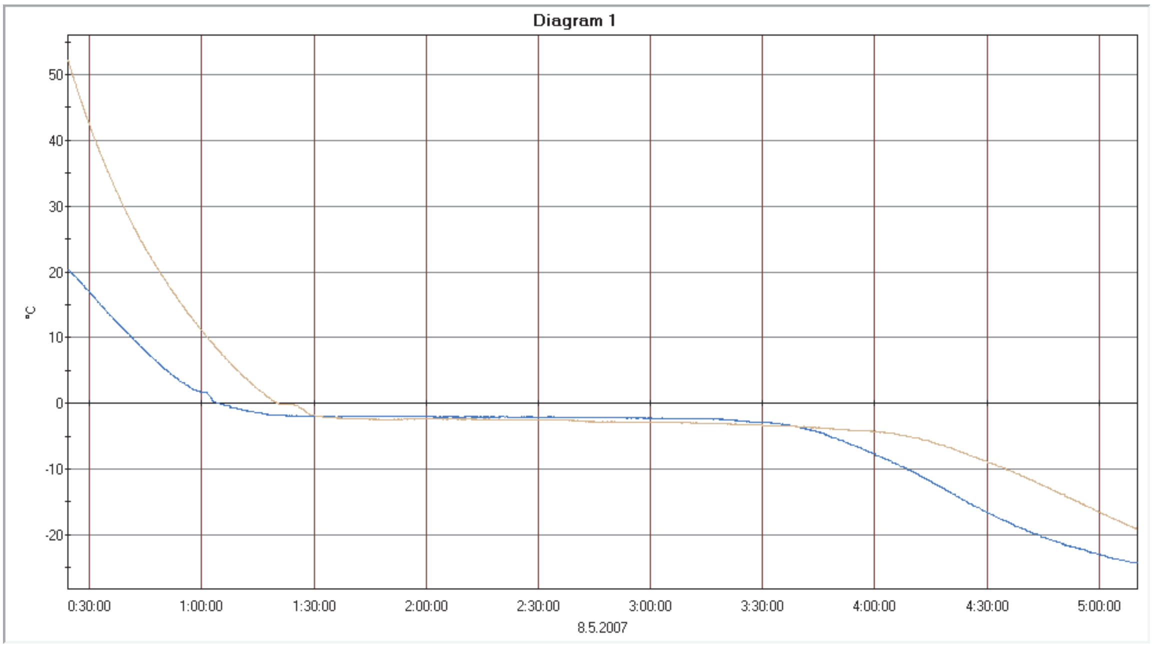


Fig.3. Water solution of NaCl (1.0 wt% NaCl)

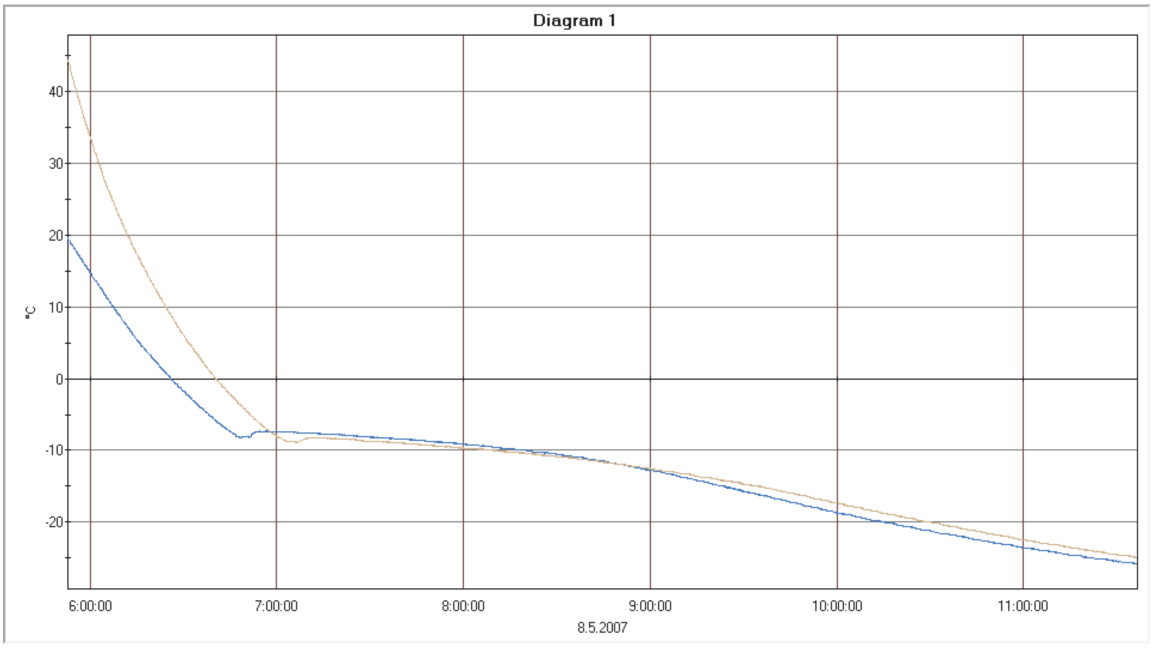


Fig.4. Water solution of NaCl (10.0 wt% NaCl)

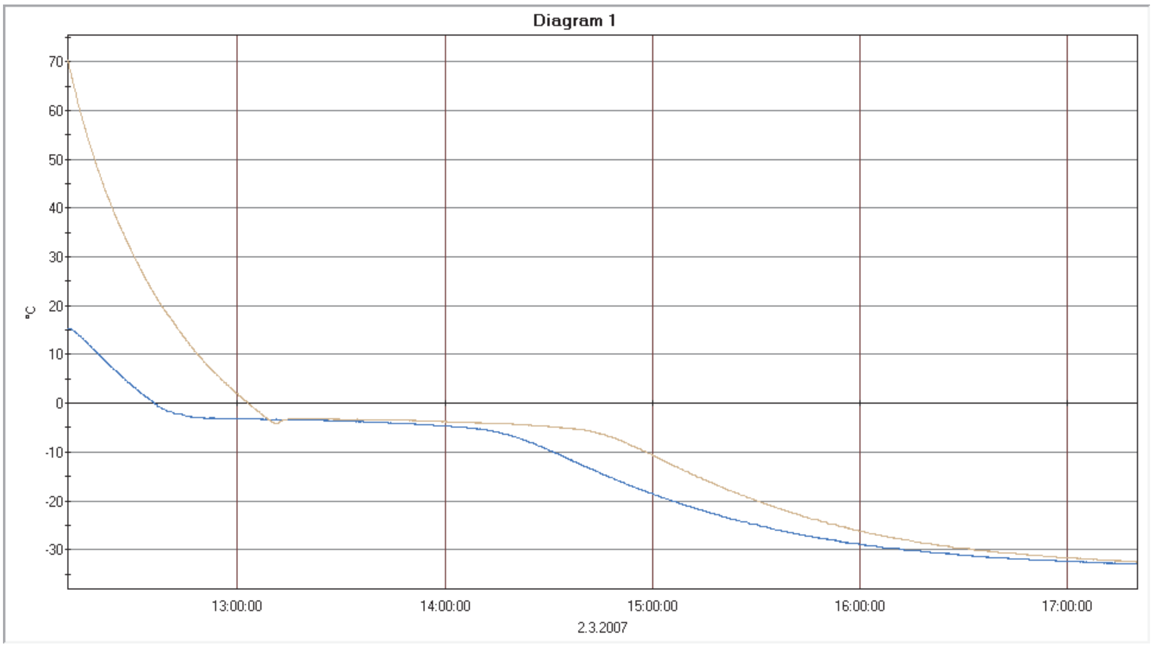


Fig.5. Milk and sugar (according to Mpemba)

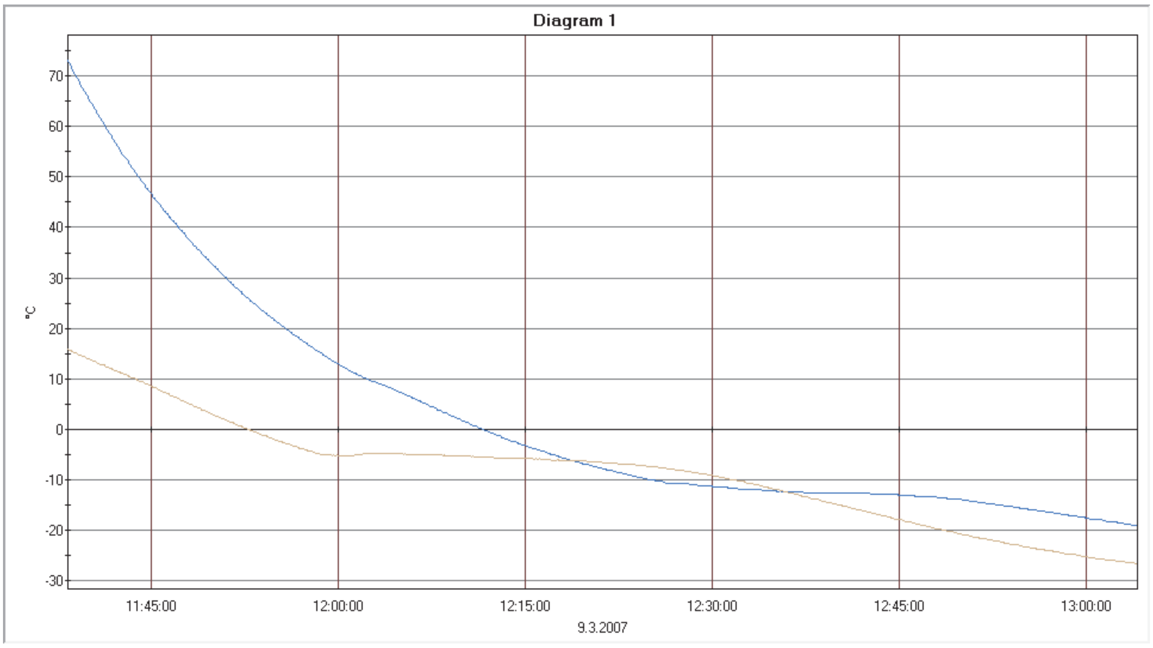


Fig.6. Russian icecream

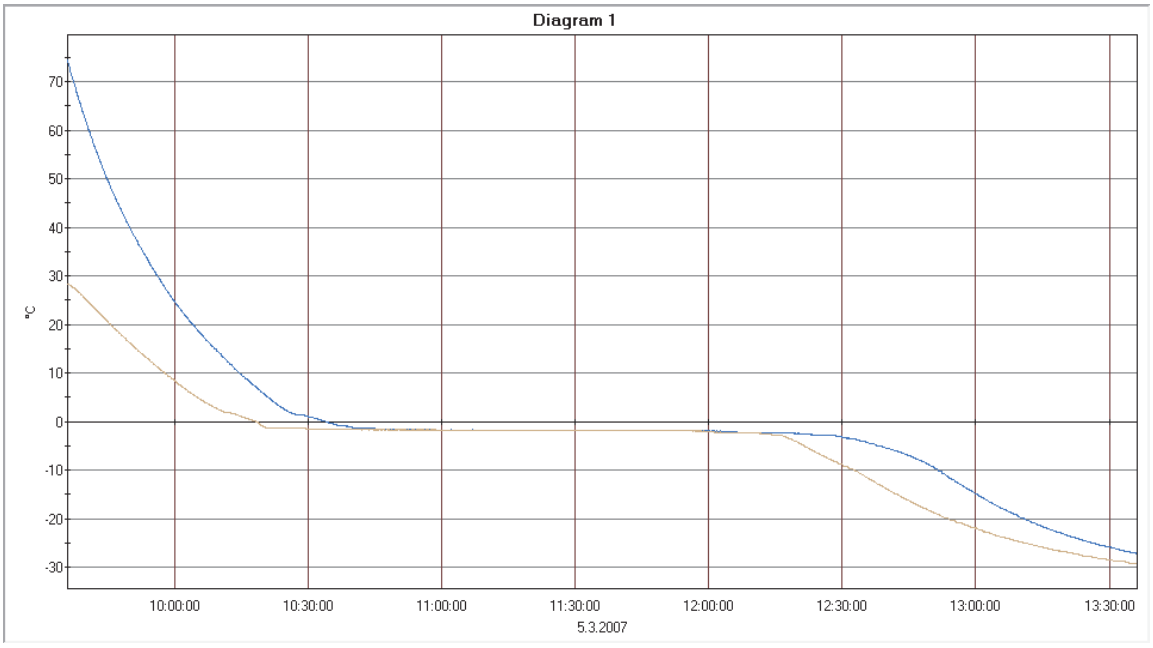


Fig.7. Water + ascorbic acid (1 wt.%)

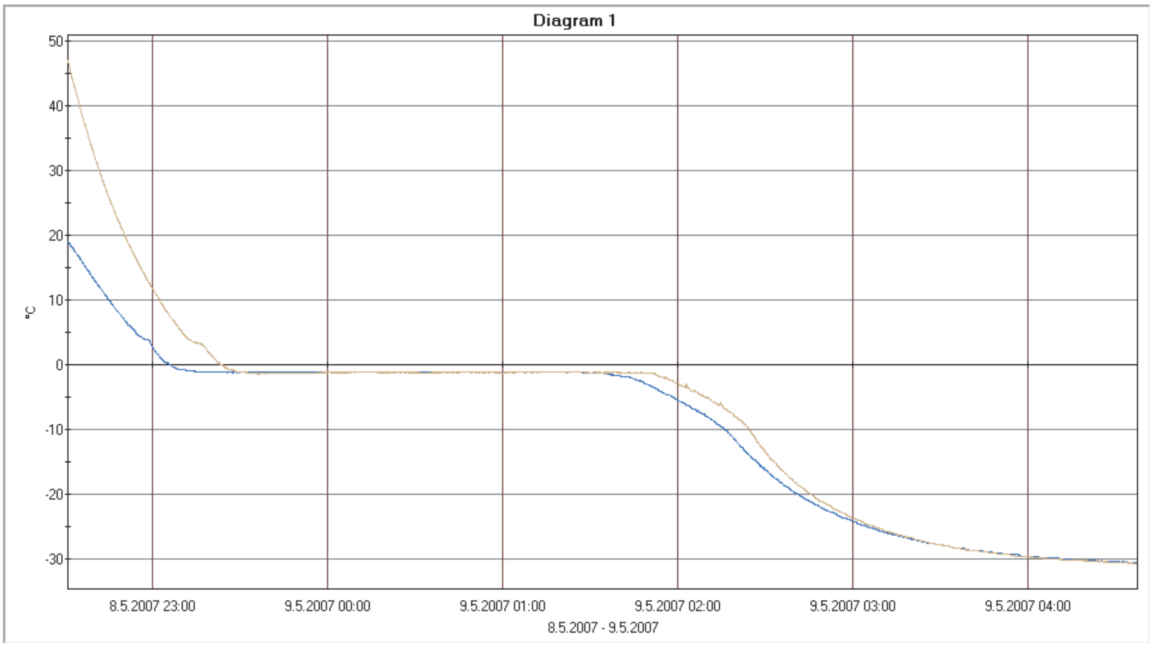


Fig.8. Amount of hot water 5 wt% lower

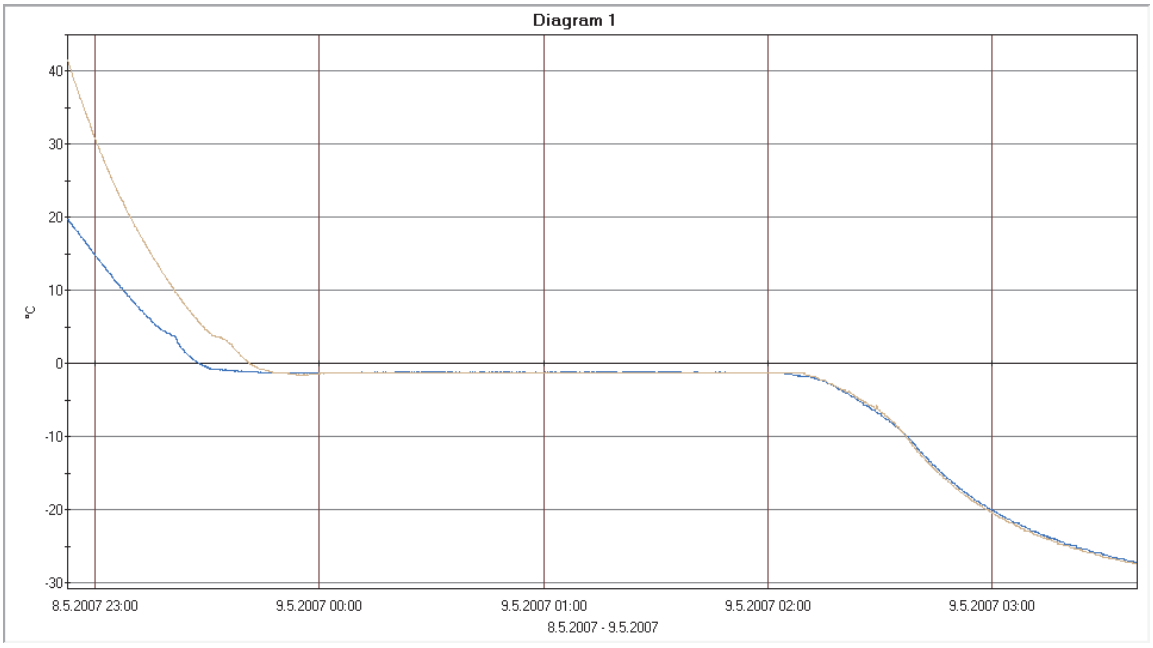


Fig.9. Amount of hot water 12 wt% lower

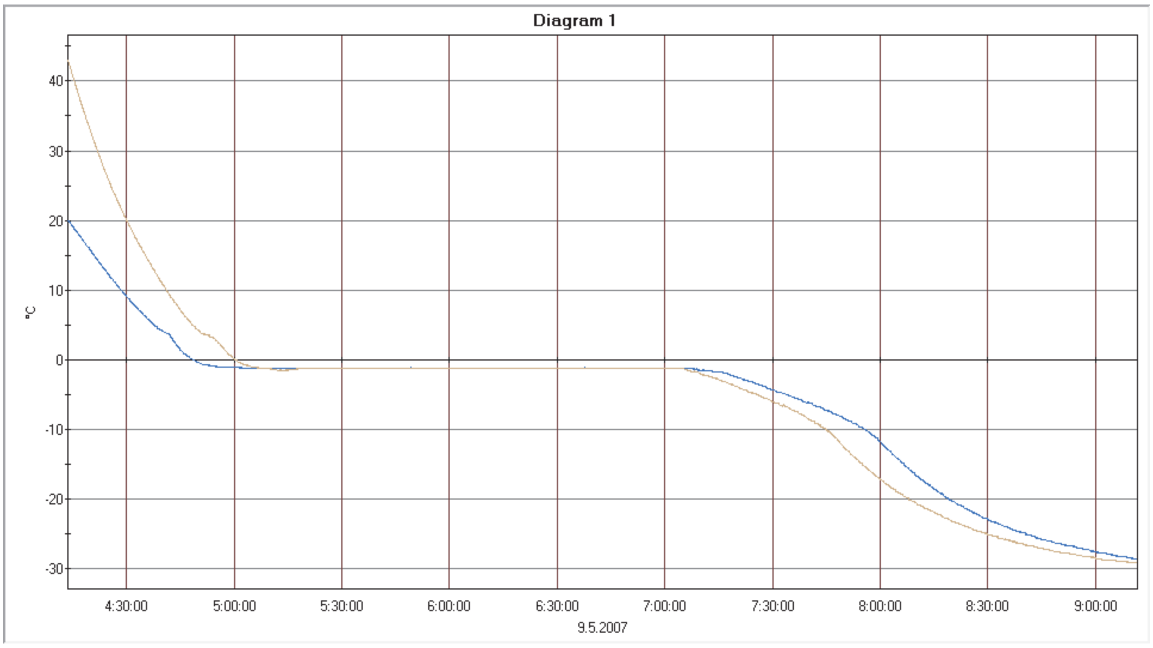


Fig.10. Amount of hot water 20 wt% lower

## References


[1] OSBORNE, J.: *Mind on Ice.* Phys. Educ. Vol.14, pp. 414 – 417, 1979.

[2] KELL, G.S.: *The Freezing of Hot and Cold Water*. Amer. Journ. of Phys. Vol 37, 564 – 565, 1969.

[3] AUERBACH, J.: *Supercooling and the Mpemba effect* . Am. J. Phys., Vol.63, 882 –

885,1995.

[4] DEMO, P.: *Ještě jednou o zamrzání horké vody*. Vesmír, Vol. 82, p. 375, 2003.

[5] JENG, M.: *The Mpemba Effect: When can hot water freeze faster than cold water?* Am.J.Phys., Vol. 74, pp. 514-522, 2006.

[6] WALKER, J.: *The Amateur Scientist: Hot water freezes faster than cold water. Why d*oes *it do so?* Sci. Am., Vol.37, n.3, pp. 246-257, 1977.

[7] BÖHM, P.: *Mpembův jev – skutečnost nebo fikce?* Diplomová práce, Kat. did.fyz. MFFUK Praha, 2006.

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# FROM CHANGING STUDENTS' "CULTURE OF PROBLEMS" TOWARDS TEACHER CHANGE

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**Abstract.** The article presents the changes in the collaborative research group resulting from a long-term collaboration of teachers and researchers. We observed the changes in the approaches of all the participants in the research (teachers, researchers and students). However, the shift could be observed to the highest degree in the teachers' participation in the research and change in their approaches to their own teaching strategies and practices.

**Key words.** Teacher change, collaborative research, culture of problems

*Mathematics Subject Classification:* 9700.

## 1 Introduction

What do students say about the problems that they solved? What relationships to the mathematical subject does their knowledge have? Are their opinions specific? Similar? Unanimous or diverse? How do they apprehend them? Do they share their opinions about the topic? Is there a difference between individual classes? In school mathematics, problems are more often seen as instruments of checking if and what students have learned than as an instructive experience. How can then students themselves assess problems? The objective of the studies that are treated in this article is how to modify students' "culture of problems" in an inherent and positive way. For more details see (Brousseau & Novotná, 2008; Novotná, 2009).

The nature of these researches requested such form of cooperation of teachers, researchers and students that is original with regards to deontological and technical principles of separation of the proper functions that Brousseau applied in his previous researches, namely in COREM (Brousseau, 1997; Brousseau, Davis & Werner, 1986). Teachers, students and researchers to a certain extent share the same school culture of problems that is imposed on them by their environment. To break or modify it without knowing it would probably result in great difficulties. If these difficulties are to be overcome, they must be observed and understood. The main goal of this article is to report on this conception, the evolution and results of such cooperation.

## 2 Theoretical background

### 2.1 Cooperation between teachers and researchers

Discussion about conditions of cooperation between researchers (in mathematics, psychology, education etc.) and teachers has been in the centre of interest since the 1960s. Its aim was to offer teachers the scientific background by providing information about research results which is stimulating and which influences their work. Subsequently, the movement reversed and teams of researchers and teachers worked together, either in order to create and disseminate tools for improving education (curriculum, materials, recommendations) or to answer the ongoing needs of certain researchers.
This enabled the merging of the roles into one single status of a teacher-researcher in mathematics education. Teacher trainers were keen to simultaneously support knowledge exchanges, their scientific validation, proof of their usefulness and their dissemination. In 1980, Bent Christiansen instigated the formation of an international group seeking "systematic co-operation between theory and practice in mathematics education".
A cooperation respecting simultaneously ethical, professional and scientific conditions of teachers and researchers was developed (1968) by G. Brousseau and carried out in COREM (1973-1998). In COREM, all the participants if they wanted and were able to could perform any function (teacher, researcher, observer, student) but never more than one at the same time or in the same research, always respecting the rules of each of the functions. (Salin & Greslard Nédélec, 1999)
TSG 24 "Research on classroom practice" appeared on ICME 11 programme.
The cooperation of teachers and researchers-university educators (shortly researchers) in mathematics education represents a broad, relevant topic. The focus is mostly on the improvement of the quality of mathematics teaching and learning (Brown & Coles, 2000). Many discussions have been carried out within the last decade about the impact of this type of projects on mathematics education – see (Goos, 2008). The different experiences and knowledge proved to be an influencing factor on the findings (see e.g., Novotná et al., 2006; Tichá & Hošpesová 2006).

### 2.2 Background: research on school culture of mathematical problems

Problems are not merely tasks where an individual applies the taught knowledge. They are navigation to an individual mathematical activity accompanying this knowledge, an instrument of students' acculturation in mathematical practice difficult to be manifested explicitly and therefore rather difficult for being transferred directly.
When correcting the solution of a problem, the teacher and students are supposed to use the institutionalized knowledge only, that is the accepted as correct and having been already taught in the class. But the ability to solve a problem depends also on the non-institutionalized knowledge. Foremost, problems develop students' ability to reconstitute and pursue mathematical texts; much less the ability to pose questions, reflect conjectures, follow intuition etc. (This resulted in extending the notion of a problem to the notion of situation in order to better model real mathematical activity.)
Observations of mathematics classes show that in the course of students' mathematical activities, namely when solving a problem, the teacher uses the knowledge that the students have in order to let them produce or accept the correct proposals. He/she uses them by means of a corpus of precepts and epistemological and heuristic practices that are not the real knowledge but are indispensable for him/her (Sarrazy, 1997). All this knowledge creates a culture specific but necessary for each class. The study of this culture allows perceiving problem solving in a completely different perspective which is much closer to a spontaneous activity of mathematicians and much more suitable for learning. This change of approach towards solving problems in mathematics requires an essential change in the teachers' approaches to their practices.

Knowing and developing this culture of problems was the object of observations, testing specific situation. Like for example the competition of problem assignments on various interpretations of division (Brousseau & Brousseau, F1987, E2009) and of one method for evoking explicit knowledge.

## 3 Our research

In this paper, we focus on studying cooperation in the scope of our ongoing research. The research consists of three parts:
The goal of the first part is to find out if there exists (if it may be proved by our statistical method) such a culture, even embryonic; in other words, if students from the same class have or do not have converging judgements. If yes, the method will then allow enriching the description of students' experiences with problems independently on the teacher's pedagogical strategy.
The second part focuses on modifying this experience by placing students in the position of posing problems, of choosing them for exposing topics, of judging and criticizing them etc. This project of collective reversing of relationships is supposed to prioritize understanding and problem solving but mainly to improve students' dispositions and motivation.
The third part requires a more complex project design. Its goal is to combine activities in mathematical situations that focus on producing knowledge (most brought into by the teacher) and are furnished with properties that do not provoke too many questions with those that provoke students to pose questions.

### 3.1 Conditions of the research: Accepted or investigated difficulties

The team was aware of the difficulties related to the subject: Teachers organize interesting activities with regards to mathematics, aiming at invocation of diverse interest and knowledge, however, they cannot set them as the aim, nor can they acknowledge them as acquisition of knowledge.
Moreover, it deals with developing students' new intellectual practices and behaviours, unforeseen, often implicit and sometimes unconscious on the teachers' side and unknown to researchers. Therefore all participants are equally involved in the research.
In addition, communicational difficulties among the partners were come across resulting from the language difference and translations. Class events could never be observed by all and the previous French experiences were only referred to and summarized.
This arrangement was also advantageous by obliging partners to express explicitly what they wanted and subsequent analysis of what was ignored and impossible to be corrected.

### 3.2 Our observations, questions and aims about the cooperation in this research

The realisation, analysis and evaluation of the experiment are done in cooperation with secondary school teachers from Prague. At least three classes participate in the experiments every year.
In this article, we point out issues related to teacher change as referred to above. Moreover, coming out from our experience from the common research, we address the following more general questions:

- What is the relation between the teacher's participation in a research and the change of his/her approaches in their own teaching strategies and practices?
- Which situations or circumstances may serve as motivation for teacher change?

We asked teachers to carry out the intended activities; this could, but not necessarily, make all of us change our knowledge. Our intention was not to change teachers' practices before our research

results were known. We tried to find whether the teachers took advice directly regardless of our conclusions.
We do not look for generally true answers because we believe such answers do not exist. Based on our experience, we illustrate one successful way of teacher and teacher educator change, developed during this longitudinal common research. Our research still continues and we believe the changes observed will grow richer.
The following methods were used to determine the influence of collaboration of the teachers in the research: interviews, teachers' written self-reflection, questionnaires and observation in their mathematics classes.

### 3.3 Teachers' contributions to change design

After three years of cooperation, the experiment runs as follows with the described modifications. For details see (Novotná, 2009):

1st stage
The central question of this stage is to find out whether there is or is not a certain common accord among students from the same class about several non-mathematical criteria presented to them. The criteria are distributed in (subjective, objective) x (personal, collective) ones: *Length of the text, Difficulty, Attraction, Usefulness, Comprehensibility,* and *Length of the resolution*.
Changes in the original organisation: This stage proved to be well designed from the beginning. Seeing the accord in certain classes related with certain conditions – homogeneity (see the Attachment), teachers were assured about the scientific nature of our observations for the future. They proposed only to add the *length of the resolution* criterion. It was included in order to recognize if students apply solving strategies independently and creatively, or rely on algorithms taught by the teacher.
Requirements on teachers: There is no correct or incorrect answer. Teachers are used to assigning tasks for which they are able to uniquely determine whether the answer is correct or not. Dealing with tasks where this is not the case asks for change of their approaches to teaching.

2nd stage
The aim of this stage is to draw attention of students to *mathematical models* of problems without teaching it through its formal definition. Simultaneously, students should gradually realize that it is possible (but not necessary) to solve some problems "in the same way". The teacher prepares a problem with a simple mathematical model which is solved by the class. Students then work in groups. They are asked to pose problems that can be solved "in the same way". Each group presents the posed problems together with their solutions. This is then collectively discussed. The goal is to attract students' attention to the utility of the equivalence of mathematical models to the success in solving the problems they posed.
Addition to the original organisation: This stage was not included in the original experimental setting.
Requirements on teachers: The teacher must present the whole activity in a clear way but without defining the mathematical model explicitly. This needs a thorough preparation of the whole activity, especially of what the teacher can and cannot tell his/her students. It is important that the teacher's participation in the discussion among students should be minimal.

3rd stage
This stage should prepare such an environment where the notion of mathematical model of a problem comes out in a natural and clear way. The teacher prepares "type-problems" (of various

mathematical models) and presents word problems to students, asking them to assign them to the type-problems according to their "mathematical similarity". In the set of word problems, there are also some that cannot be meaningfully assigned to any of the presented type-problems. This is followed by a collective discussion where groups present their allocation of word problems to type-problems and justify their solution. In the discussion, the similarity based on non-mathematical items (such as the "environment" of the story in the assignment, presence of some signal words etc.) should be rejected.
<u>Changes in the original organisation</u>: This stage was a part of the original settings, organized as a competition; see for example (Bureš, Hrabáková, 2008). The notion of mathematical model was explicitly used by students and the teacher. The students' task was to determine "original problems". An original problem is to be understood as a problem whose mathematical model is different from the models of all other problems dealt with. The explicit use of the notion of "original problem" (relative to a group of problems) resulted in the teacher's teaching of this notion instead of students focusing on the similarities/differences in the solving processes. The collective discussion of the allocations was included from the beginning. In all cases, it proved to be an important part of the stage.
<u>Requirements on teachers</u>: The algorithm for establishing the relationship between two problems is quite complex. The teacher can be easily led to explicit indication of the belonging or not of one or more word problems to one problem-type. However, the teacher should not teach a method in this activity. His/her task is to ensure that all students, including the weaker ones, participate by presenting their ideas.

4th stage
The goal of this stage is to foster an environment for reflection on the problems from the perspective of their mathematical solving procedure. In other words, to help students to understand that it is possible "to produce the solution of one problem by solving another one". The teacher prepares six word problems from a mathematical domain known to the students. The students solve them using the solving procedure that they choose. In groups, they look for similarities among the six problems. The criteria of similarity are not given. The groups formulate in writing the criteria for each grouping. In the following collective activity, they record their groupings in the table on the blackboard and for each grouping they explain the used criteria. The activity continues in groups. The teacher selects one of the problems and asks the groups to choose from the remaining five problems such a problem that they would offer as an aid to somebody who does not know how to solve the problem selected by the teacher. The choices made by the groups are then collectively discussed. This second part of the stage helped to reenter students' focus on the similarity according to mathematical criteria.
<u>Addition to the original organisation</u>: This stage was not included in the original experimental setting. It was introduced in order to strengthen students' focus on the mathematical model without teaching what it is.
<u>Requirements on teachers</u>: In the discussions it appeared necessary to use a precise terminology. There exist many kinds of equivalences between problems (N. and G. Brousseau, 1987, p. 269-273): Two problems are a-equivalent if it is sufficient to substitute the numbers from one assignment in the solving algorithm of the other and we obtain the correct result; or f-equivalent, if there exist a common formula linking the data in the same way; etc. Two problems are called *from the same family* if there are parts of their solving algorithms that are equivalent.
To teach this terminology to students would change the activity to a "classical school lesson" instead of offering students the space for reflection on problem solving. In order to avoid this risk, a change of the teacher's way of classroom management is necessary.

Explanation of changes in stages 2 to 4: The notion of *mathematical model* of problem proved to be rather difficult for its introduction to students. The danger of metadidactical shift (Brousseau, 1997) became evident. As already mentioned, our goal is not to teach problem solving but we want to develop students' debate about problems, a change in their approaches to problems and their solving. It was necessary to modify organisation of the original stages based on mathematical model in such a way to enable the students to make sense of this notion gradually, without being it explicitly taught by the teacher.

5th stage
The aim of this stage is to summarize and precise the knowledge acquired by students during the previous three stages. It has the form of students' problem posing. Students pose word problems with a given non-mathematical topic (for example problems from a supermarket). Each group contributes one problem to a set of problem that will be used in the activity in the lesson. During the first lesson of this stage, students in groups compare the presented problems, look for similarities of their solving procedures and decide if their problem is useful for the solution of one or more other presented problems. If this is the case they have to pose a new problem that is used in the second lesson of this stage.
During the second lesson of this stage, each group presents their problem to the whole class (the original one or the newly posed in case that they created a new one), the solving procedure and gives evidence that their problem cannot help to solve another problem originally submitted. The other students may discuss and react to it.
Changes in the original organisation: This stage was substantially modified. Originally, teachers did not set any limits for the assignment; it resulted in a great variety of posed problems based on the variety of other criteria than this of mathematical model (for example the number of posed questions, their "wittiness"). The number of problems treated in the collective discussion was 15. The discussion finally resulted in the students' decision that "all the posed problems were from the same family because in all of them we have to use the four arithmetical operations".
Requirements on teachers: This activity was well evaluated by students. They appreciated the possibility to discuss, the necessity to justify their decisions, the opportunity to revise their solution and to think about different solving procedures of the same problem instead of only solving the problem as usual. If students are to be offered all this, there are many requirements on the teacher, mainly in classroom management: What to say and what not to say in order to open space for independent efficient work of students? When to intervene and in which form in order not to stop the student activities? Of course, the activity requires the ability to react to unexpected situations in the classroom in both – mathematics and classroom management. The teacher needs to have a good command of mathematical knowledge for opening all the above mentioned possibilities to the students.

### 3.4 Does experimentation change teachers?

The success in changing students' relationship to solving problems requires not only deep teachers' involvement in realisation of the designed activities but also their active involvement in the project design. This change of their role goes hand in hand with the change of their pedagogical approaches and beliefs.
In our collaborative group, there are two different teacher approaches towards research: Some participating teachers are not actively involved in the organisation and planning of the research activities. They mostly appreciate the stages of the research as non-traditional classroom activities which can be used to identify students' difficulties and assess their understanding of mathematics. They like to realize the proposed activities in their classes. The experience from our research

indicates the change in attitudes and behaviours of teachers of this type. Gradually, they become active in the experiment design and other activities, originally perceived as the researchers' domain. This change is not sudden but has happened in our group in all cases.
The other group consists of teachers who have been deeply involved in the preparation and organisation of the whole experiment from the beginning, who have actively influenced the experiment design and are open to the use of non-traditional classroom activities. Having gained more experience with the activities, they propose modifications and improvements in the experiment design.
We detected changes in all aspects: their ability to design and organize efficient a-didactical situations in the classes, their ability to analyse situations, evaluate their course and results and distinguish between the rules of the situation and contingency, their active involvement in designing, realisation and analysis of the research in collaboration with researchers, and their ability to function successfully in two different roles, the teacher and the researcher. The findings are based on teachers' self-reflections and researchers' observations.
We illustrate the changes detected in our group on two teachers, who are referred to as Teacher A and Teacher B. Both of them are full-time teachers and doctoral students at Charles University, Faculty of Education. In their case, the participation in the common research significantly contributes to their development as researchers in their research for PhD studies. In our collaborative group, the teacher has the last say in the design of the teaching unit that he/she is teaching following the rule used in COREM.
We observed a significant increase in the teachers' autonomy. During the preparation and realisation of the changes in our experimental settings, the teachers gradually took the roles of those who actively influence the stage design and the problems used. It was not only significantly influenced by their realisation of the experiment stages, but to a great extent also by their participation at the team meetings where the experiences and preparation of the consequential steps were discussed.

***Extracts from the teachers' self-reflections illustrating the above stated observations***
Teacher A

- Experience with teacher's "new" role during the a-didactical situation: "The teacher should rather become an observer, moderator of discussions and of the work in the classroom. This role is demanding and from the perspective of traditional teaching unusual. When you listen to students during group work and see that they are close to the solution, it is not easy to answer their question and not to intervene in their work."
- Experience with moderating students' discussion: "I learned to 'listen and intervene only when it is a must'; if not, there is a danger that I divulge students something that they could find themselves if I have given them more space."
- Experience with group work: "Before the project, I used it rarely. I was afraid that I would not succeed in involving all students in the activity so that each was active. The experiment showed that with an appropriate choice of activities, this is possible."
- Experience with the students' peer control: "The teacher is not the only who can tell students what is correct and what not. It proved to be more efficient when this evaluation was formulated by their schoolmates."
- Teacher B
- Experience with underestimating students' abilities: "This experiment showed me the conflicts between my expectations and that what the students really do. At the beginning, I was embarrassed that I did not manage to get from them what I wanted; it motivated me to a

deeper reflection on the ways of presenting the stages to students. At present, I find that it is not a shame if students do something differently because we all can learn from it."

- Experience with feedback from students: "The experiment evoked my need of the feedback from children. It should become an integral part of school work. Before the project, I could not imagine that mathematics in school could be a subject where more discussions take place than in other lessons."
- Experience with organizing research projects: "I see the shift; at the beginning I expressed myself only to organisational items, such as the number of problems, dividing into groups. After gaining experience I find that I am solving more fundamental issues, such as definition of the mathematical model of a problem, links in school mathematics."

We could observe a change in the teachers' perception of organising student problem posing. Before the participation in the project, teachers assigned problems to students themselves; they saw it mostly as the only appropriate way for managing the teaching/learning process.

The usefulness of students' problem posing emerged gradually during the experiment stages. There was a visible shift in the teachers' approaches to students' problem posing. At the beginning of implementation, teachers perceived it as an activity prepared by students outside school; it was only during the project development that they became ready to use it also inside mathematics lessons in school. They compared both ways and this comparison did not discriminate one way to the other. The advantage of the "out of school" way was the time that students could devote to this activity and the (possibly) resulting greater care devoted to it. The experience showed that some problems were thoroughly elaborated, but some of them were copied from various sources and some of them were slapdash.

The main concern of teachers about students' problem posing during the lessons in school concerned the difficulties with its organisation. Their experience proved that this concern was unnecessary. Preparation and organisation themselves are not too difficult. More demanding are the analyses of the activity. It is dependent on the contingency (composition of groups, instantaneous disposition etc.). It does not say much about the individual knowledge of the students but the discussion in the group facilitates the devolution and the shift of responsibility for the quality of the posed problems to the students.

Teacher A

- "Before the project I never suspected how funny and beneficial for students their own problem posing could be. Since, I use this activity regularly in my teaching, at all ages and cognitive levels of students. I receive the feedback from students. I see if they really understand. It offers me the opportunity to discover insufficiencies.
- Teacher B
- "Before the experiment, the idea that students could independently pose problems, that it could be useful, never crossed my mind. I was afraid that its use would not be positively accepted by students, they would not identify with it. I was surprised by the heterogeneity in the approaches to problem posing and differences in problems posed. ... The influence of contingency on the process and result of problem posing was visible."
- "I think that this activity is meaningful for students of all levels of abilities. I see the minimal benefit in the discussion on the problem assignment structure and data analyses. I see the maximum benefit in looking for relationships between problems, in students' increased sensitivity to details in data in the problem assignment."
- The teacher must take into account the students he/she works with in order not to have too high expectations. For example in my case, I expected that the posed problems would be much more sophisticated than they were in reality. I underestimated that the students will prefer problems from the domains where they have already a good command of their

solution. This activity serves also as a diagnostic tool for students' confidence in mastering a certain domain of school mathematics."

## 4 Concluding remarks

Our described experience from the collaborative research is similar to the examples of cooperation between researchers and teachers presented in (Novotná et al., 2006). In our collaborative group, we perceived a significant teacher change towards the development of teachers' sensitivity to the students' ability to use knowledge in a variety of contexts, towards a significant increase of their identification of why difficulties appear in teaching and how to overcome them, and towards understanding of the social processes happening in their classrooms. The collaboration did not end in proposing didactical situations designed for changing the students' culture of solving problems and their piloting, and analysis of their effects but also in the change of the relations in our team: The division of roles and responsibilities moved from a significantly unbalanced one with most responsibility put in the hands of the researchers towards real cooperation with a clear division of responsibilities where everybody brings his/her expertise into play.
The project considerably influenced all members of the collaborative group, the teachers as well as the researchers. If the work of the team is to be successful, all the participating persons must collaborate. And the change can be observed not only on the teachers' side. Also the researchers gained much from the collaboration. The teachers' input helps to precise the experimental settings as well as to analyze the project results.

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## References

[1.] BROUSSEAU, G.: *Theory of Didactical situations in mathematics 1970-1990.*: Kluwer Academic Publishers, Dordrecht, 1997.
[2.] BROUSSEAU, G., DAVIS, R.B., & WERNER, T.: *Observing students at work*. In B. Christiansen, A.G. Howson, & M. Otte (Eds.), Perspective on mathematics education, pp. 205-241, Reidel, 1986.
[3.] BROUSSEAU, G., NOVOTNA, J.: *La culture scolaire des problèmes de mathématiques*. In B. Sarrazy (Ed.), Les didactiques et leurs rapports à l' enseignement et à la formation. Quel statut épistémologique de leurs modèles et de leurs résultats? Université Bordeaux IV, Université Bordeaux 2, 2008. [CD ROM]
[4.] BROUSSEAU, N., BROUSSEAU, G.: *Rationnels and décimaux dans la scolarité obligatoire*. Université Bordeaux I, I.R.E.M. Bordeaux, 1987. English version 2009.
[5.] BROWN, L., COLES, A.: *Same/different: a 'natural' way of learning mathematics*. In T. Nakahara & M. Koyama (Eds.), Proceedings PME 24, Hiroshima: PME, Vol. 2, pp. 113-120, 2000.
[6.] BUREŠ, J., HRABÁKOVÁ, H. (2008). *Création d'énoncés de problèmes par les élèves*. In Actes du XXXVe colloque COPIRELEM, Bordeaux, 2008.

[7.] GOOS, M.: *Critique and transformation in researcher-teacher relationships in mathematics education*. Symposium on the Occasion of the 100$^{th}$ Anniversary of ICMI, 2008. Retrieved December 13, 2008, from http://www.unige.ch/math/EnsMath/Rome2008/partWG3.html.

[8.] NOVOTNÁ, J.: *Contribution à l'étude de la culture scolaire. Cas de la résolution de problèmes dans l'enseignement des mathématiques*. In L. Poirier (Ed.), Proceedings CIEAEM 61. In print.

[9.] NOVOTNÁ, J., LEBETHE, A., ROSEN, G., ZACK, V.: *Teachers who Navigate between their Research and their Practice*. In N.A. Pateman, B.J. Dougherty, & J. Zilliox (Eds.), Proceedings PME 27/PME NA 25, Honolulu: PME, Vol. 1, pp. 69-99, 2003.

[10.] NOVOTNÁ, J. et al.: RF01: *Teachers researching with university academics*. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), Proceedings PME 30, Praha: Atelier Guimaec, Vol. 1, pp. 95-124, 2006.

[11.] SALIN, M.-H., GRESLARD NÉDÉLEC, D.: *Relations between classroom practice and research in mathematics education.,* In F. Jaquet (Ed.), Proceedings CIEAEM 50. Neuchâtel: IRDP, 1999.

[12.] SARRAZY, B.: *Sens et situations: une mise en question de l'enseignement des stratégies méta-cognitives en mathématiques*. Recherches en didactique des mathématiques, 17/2, pp. 135-166, 1999.

[13.] TICHÁ, M., HOŠPESOVÁ, A.: *Qualified Pedagogical Reflection as a Way to Improve Mathematics Education*. Journal for Mathematics Teachers Education. Special Issue: Inter-Relating Theory and Practice in Mathematics Teacher Education, 9(2), 129-156, 2006.

**Appendix**

Kendall rank correlation coefficient

| | **Difficulty** | **Attraction** | **Length of text** | **Length of resolution** | **Comprehensibility** | **Usefulness** |
|---|---|---|---|---|---|---|
| **Whole class** | NS | S 0,10 | S 0,01 | NS | S 0,10 | S 0,01 |
| **Good** | S 0,10 | S 0,05 | S 0,01 | NS | S 0,05 | S 0,05 |
| **Weak** | NS | NS | S 0,01 | NS | NS | S 0,05 |

Note: The Kendall rank coefficient is used as a test statistic in a statistical hypothesis test to establish whether two variables may be regarded as statistically dependent. This test is non-parametric, as it does not rely on any assumptions on the distributions of $X$ or $Y$.

S $p$ … the significance level or critical $p$-value; NS stands for non-significant.

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# MATHEMATICS FOR ECONOMISTS

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**Abstract.** An attempt to improve teaching mathematics at the University of Economics in Prague is introduced. On the basis of an inquiry, which was held in 2008 and addressed all the teachers of the university, some mathematical models and instruments, which are used in economical applications, are pointed out and are illustrated in easy economic examples.

**Key words and phrases.** matrix, Markov process, Taylor polynomial, inquiry.

*Mathematics Subject Classification.* 97M40 .

## 1 Introduction

In 2008 Jiří Henzler, a member of the Department of Mathematics, asked all the teachers of the University of Economics in Prague by email about their opinion on the content of lectures on mathematics at the University. They were asked about the topics they consider important for the further study of economic subjects.

The basic course of mathematics 4MM101 content: Arithmetical vectors and algebraic operations with them. Matrices, rank of matrix. Systems of linear equations. Multiplication of matrices, regular matrix, inverse matrix, matrix equations, determinants, Cramer's rule. Limits of sequences and limits and continuity of real functions of one real variable. Derivatives of real functions of one real variable, higher order derivatives, L'Hospital's rule. Increasing and decreasing functions, increasing-decreasing test, critical points test, the first derivative test and the second derivative test for graphing and maximum-minimum problems, the second derivative and concavity, the second derivative test for concavity, inflection points, drawing graphs. Real functions of two real variables, partial derivatives, partial derivatives of second order, smooth functions of first and second order, local maxima and local minima, constraint maxima and minima, method of Jacobi's determinant, method of Lagrange multipliers. Indefinite integral, integration by parts and the substitution method, integral of rational functions. Definite integral, improper integral. Differential equations (first and second order differential equations with constant coefficients and special right side).

Some of the answers to the inquiry contained the same issues. For instance the matrix calculus, especially eigenvectors and eigenvalues, the derivative for functions of one variable and its applications, especially convexity, concavity, optimization and the Taylor polynomial, homogeneous functions of n variables, differential and difference equations are all needed for study microeconomics, macroeconomics, statistics, demography, statistics, etc. Some of these topics are not taught in the basic course of mathematics at the University at present.

Not many members of the academic staff took part in the inquiry. On $15^{th}$ November 2011 we asked the heads of all the departments at the university which parts of mathematics their students need for mastering subjects taught by their departments. The answers differed according to the faculties.

The Faculty of Finance and Accounting and the Faculty of Business Administration would need deeper study of derivatives, elasticity, sensitivity, constraint maxima and minima, the method of Lagrange multipliers, eigenvalues, eigenvectors of matrices and Taylor polynomial.

Students need to know polynomials, power functions, exponential and logarithm functions, which are widely used in economic models, less gonimetric and cyclometric functions.

At the Faculty of Informatics and Statistics some students attend one more course of mathematics Mathematics for Informatics a Statistics 4MM103, which spreads the basic course and in which eigenvalues and eigenvectors, Taylor polynomial, difference equations, etc. are studied. This could be the way for other faculties. The department of mathematics could prepare special courses of mathematics for particular faculties.

Within the project FRVŠ 517/2011 "Innovation of Teaching Mathematics for Economists " we collected many application examples, which will be published in a course book for students of economics. Examples will be completed with the theory that is so far beyond the content of the basic courses of mathematics in our university.

The goal of the prepared course book is to show student easy examples from the economic use in which mathematical models are applied. The examples are not numerically difficult (like e.g. in numerical mathematics, where problems are simulated by mathematical software). They are easy enough not to be necessary to explain difficult economic theories. They just illustrate the use of mathematical tools.

## 2 Linear models, eigenvalues, eigenvectors, limits of sequences

In the model of multimarket equilibrium it is set supply equal to demand in each market. Then the equilibrium prices are determined by solving a system of linear equations.

**Example 2.1** *Let in markets for two goods be supply and demand given by the following equations*

$$\begin{aligned} q_{1s} &= -1 + p_1 \quad , \quad q_{1d} = 31 - 2p_1 - p_2 \quad , \\ q_{2s} &= \phantom{-1+{}} p_2 \quad , \quad q_{2d} = 40 - 2p_2 - p_1 \quad , \end{aligned}$$

*where $p_1$, resp. $p_2$, is the price per unit of good $g_1$, resp. $g_2$, and $q_{1s}$, resp. $q_{2s}$, and $q_{1d}$, resp $q_{2d}$, are the quantities supplied and demanded of those goods.*

*Derive the equilibrium prices of the goods.*

*We solve the system of linear equations*

$$\begin{array}{rclclcl} q_{1s} & = & q_{1d} & \Leftrightarrow & -1+p_1 = 31-2p_1-p_2 & \Leftrightarrow & 3p_1 + p_2 = 32 \ , \\ q_{2s} & = & q_{2d} & \Leftrightarrow & p_2 = 40-2p_2-p_1 & \Leftrightarrow & p_1 + 3p_2 = 40 \ . \end{array}$$

*We find that the system has a unique solution* $p_1 = 7$, $p_2 = 11$.

Another models using systems of linear equations, and, in some cases, eigenvalues of matrices, are linear models of productions. Suppose that a hypothetical economy produce $n$ goods, each of the goods is produced by a production process. (We may consider labor as a commodity that is not produced, but we do not.) A production process is a list of amounts of goods needed to produce a unit of output, an input-output coefficient $a_{ij}$ denotes the input of good $i$ needed to output a unit of good $j$, $i, j = 1, 2, ..., n$. The coefficients $a_{ij}$ form an input-output matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

Nonnegative coefficients $c_i$ denote the consumer demand for good $i$, $i = 1, 2, ..., n$. They form a matrix

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{bmatrix}.$$

The law of supply and demand requires that produced output must be used in production or in consumption. Let us denote $\boldsymbol{x} = (x_1, x_2, ..., x_n)$, where $x_i$ denotes the amount of output $i$, than we have a system of linear equtions

$$x_i = a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n + c_i, i = 1, 2, ..., n,$$

which is equivalent to

$$\begin{array}{rcccccccl} (1-a_{11})x_1 & - & a_{i2}x_2 & - & \dots & - & a_{in}x_n & = & c_1, \\ -a_{11}x_1 & + & (1-a_{i2})x_2 & - & \dots & - & a_{in}x_n & = & c_2, \\ . & & & & & & & & \\ . & & & & & & & & \\ -a_{11}x_1 & - & a_{i2}x_2 & - & \dots & + & (1-a_{in})x_n & = & c_n. \end{array}$$

This system of linear equations can also be considered as a matrix equation

$$X = AX + C, \text{ where } X = \boldsymbol{x}^T = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}.$$

This system is equivalent to

$$(J - A)X = C, \text{ where } J \text{ is an identity matrix.}$$

If $J - A$ is an invertible matrix we have a unique solution

$$X = (J - A)^{-1}C.$$

**Example 2.2** *Consider a hypothetical farm that produces wheat and fertilizer. Suppose that the production of* 1 *ton of wheat requires* 0.2 *ton of wheat and* 0.2 *ton of fertilizer. The production of* 1 *ton of fertilizer requires* 0.4 *ton of wheat. The farm needs* 10.4 *ton of wheat and* 1 *ton of fertilizer for consumers. How much wheat and fertilizer has the farm to produce?*

*Denote $x_1$ the amount of produced wheat and $x_2$ the amount of fertilizer. The production of $x_1$ of wheat needs $0.2x_1$ of wheat and $0.2x_1$ of fertilizer. The production of $x_2$ of fertilizer requires $0.4x_2$ of wheat. The farm needs $x_1 = 10.4 + 0.2x_1 + 0.4x_2$ of wheat and $x_2 = 1 + 0.2x_1$ of fertilizer. We solve the system of linear equations*

$$\begin{array}{rcrcl} 0.8x_1 & - & 0.4x_2 & = & 10.4 \quad , \\ -0.2x_1 & + & x_2 & = & 1 \quad , \end{array}$$

*which can be considered as a matrix equation*

$$(J - A)X = C, \ \text{where } J - A = \begin{bmatrix} 0.8 & -0.4 \\ -0.2 & 1 \end{bmatrix}, \ A \text{ is the input-output matrix} \begin{bmatrix} 0.2 & 0.4 \\ 0.2 & 0 \end{bmatrix}.$$

*We find*

$$(J - A)^{-1} = \frac{5}{18} \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}.$$

*Then*

$$X = \frac{5}{18} \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 10.4 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 4 \end{bmatrix}.$$

*The unique solution of the system is $x_1 = 15, x_2 = 4$. The farm will produce* 15 *tons of wheat and* 4 *tons of fertilizer.*

In the previous model each good can have different natural units, it is convenient express them in millions of dollars, for instance. Now the coefficient $a_{ij}$ denote the sum of millions of dollars of good $i$ that is needed to produce a million of dollars of good $j$. The sum of coefficients in each column represents the total cost of producing a million dollars of the product $j$. If we assume that the industry should make a positive profit the sum of the coefficients in each column should be less than 1. If there is no profit it should be equal to 1.

It can be proved, see [1], Theorem 8.13, that if all entries in the matrix $A$ are nonnegative and the sum in each column is less then 1, then $(J - A)^{-1}$ exists and all its entries are nonnegative.

If the sum in each column is equal to 1 (the industries make the zero profit), then $(J - A)^{-1}$ does not exists. If, moreover, there were no consumers, we would solve the matrix equation

$$X = AX \text{ or } (J - A)X = O, \text{ where } O \text{ is a zero matrix.}$$

As $J - A$ is not an invertible matrix we get infinitely many solutions, which can be considered as eigenvectors for the matrix $A$ and its eigenvalue 1.

**Example 2.3** *Let* $A = \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}$ *be an input-output matrix of the liner model of production. Let no goods be consumed, which means that* $C$ *is zero matrix. Determine the outputs to maintain the market equilibrium.*

*We find an eigenvector of* $A$ *corresponding to the eigenvalue* $1$. *We solve the equation* $(J - A)X = O$. *As* $(J - A)$ *is not invertible we solve a homogeneous system of linear equations*

$$\left[\begin{array}{rr|r} 0.1 & -0.3 & 0 \\ -0.1 & 0.3 & 0 \end{array}\right].$$

*The solution is* $X^T = \boldsymbol{x} = t(3, 1)$, $t \geq 0$.

*That means that the ratio of outputs should be* $3 : 1$.

Let us consider a process that at any time $n$ is in one and only one of $k$ states $S_1$, $S_2, ..., S_k$. Let us have a matrix $M$ of transition probabilities, which means that each $m_{ij} \geq 0$, $i$, $j = 1, 2, ..., k$, denotes the probability that the process will be in state $i$ at time $n + 1$ provided it is in the state $j$ at time $n$. We suppose that the system moves into one and only one state. Thus, the sum of the elements of each column must be 1. Such a matrix $M$ is called a Markov matrix. Let us denote by $x_{i,n}$ the probability that the state $i$ occures at the period $n$, and denote by $\boldsymbol{x}_n$ the vector of the probabilities $(x_{1,n},\ x_{2,n},\ \dots,\ x_{k,n})$ . A Markov system (process) is a system of difference equations

$$x_{i,n+1} = m_{i1}x_{1,n} + m_{i2}x_{2,n} + ... + m_{ik}x_{k,n}, i = 1, 2, ..., k,$$

which can be written as

$$\boldsymbol{x}_{n+1}^T = M\boldsymbol{x}_n^T.$$

If the matrix $M$ has distinct real eigenvalues $r_1$, $r_2, ..., r_k$ and corresponding eigenvectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2, ..., \boldsymbol{v}_k$ then the general solution of the previous system of difference equations, see [1], Theorem 23.6, can be written as

$$\boldsymbol{x}_n = c_1 r_1^n \boldsymbol{v}_1 + c_2 r_2^n \boldsymbol{v}_2 + ... + c_k r_1^n \boldsymbol{v}_1,\ c_1,\ c_2,\ \dots c_k \in \mathcal{R}.$$

**Example 2.4** *Let in a state probably* $10$ % *of inhabitants of cities move from the cities to the country (and, thus,* $90$ % *o them stay in the cities), and* $8$ % *of inhabitants of the country move from the country to the cities (and, thus,* $92$ % *of them stay in the country) every year. The initial number of people living in the cities is* $30$ *mil., in the country* $15$ *mil. How many people will be probably living in the cities and in the country after two years?*

*We use the equation* $\boldsymbol{x}_{n+1}^T = M\boldsymbol{x}_n^T$.

$$\boldsymbol{x}_0 = (30, 15), \quad M = \begin{bmatrix} 0.9 & 0.08 \\ 0.1 & 0.92 \end{bmatrix}, \quad \boldsymbol{x}_1^T = M\boldsymbol{x}_0^T, \quad \boldsymbol{x}_2^T = M\boldsymbol{x}_1^T.$$

$$\boldsymbol{x}_1^T = \begin{bmatrix} 0.9 & 0.08 \\ 0.1 & 0.92 \end{bmatrix} \cdot \begin{bmatrix} 30 \\ 15 \end{bmatrix} = \begin{bmatrix} 28.2 \\ 16.8 \end{bmatrix}, \qquad \boldsymbol{x}_2^T = \begin{bmatrix} 0.9 & 0.08 \\ 0.1 & 0.92 \end{bmatrix} \cdot \begin{bmatrix} 28.2 \\ 16.8 \end{bmatrix} = \begin{bmatrix} 26.724 \\ 18.276 \end{bmatrix}.$$

*There will be probably* $26.724$ *mil. people living in the cities and* $18.276$ *mil. people living in the country.*

**Example 2.5** *Let us consider the same situation as in the previous example and decide how many people will be living in the cities and in the country generally after n years.*

*We find the eigenvalues and the eigenvectors of the matrix* $M = \begin{bmatrix} 0.9 & 0.08 \\ 0.1 & 0.92 \end{bmatrix}$.

*First we find the eigenvectors:*

$$|M - \lambda J| = \begin{vmatrix} 0.9-\lambda & 0.08 \\ 0.1 & 0.92-\lambda \end{vmatrix} = \lambda^2 - 1.82\lambda + 0.82 = 0.$$

*The roots are* $\lambda_1 = 1$, $\lambda_2 = 0.82$.

*We find the eigenvectors for* $\lambda_1 = 1$. *It is sufficient to find one of them. We substitute* $\lambda = 1$ *into the equation*

$$(M - \lambda J)\boldsymbol{x}^T = \boldsymbol{o}^T$$

*and get*

$$\begin{bmatrix} -0.1 & 0.08 \\ 0.1 & -0.08 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{aligned} -0.1x + 0.08y &= 0, \\ 0.1x - 0.08y &= 0. \end{aligned}$$

*This homogeneous system has a general solution* $t(4,5)$. *We take* $t = 1$, *and, thus,* $\boldsymbol{x}_1 = (4,5)$.

*We find the eigenvector for* $\lambda_2 = 0.82$. *We substitute* $\lambda = 0.82$ *into the equation*

$$(M - \lambda J)\boldsymbol{x}^T = \boldsymbol{o}^T$$

*and get*

$$\begin{bmatrix} 0.08 & 0.08 \\ 0.1 & 0.1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{aligned} 0.08x + 0.08y &= 0, \\ 0.1\ x + 0.1\ y &= 0. \end{aligned}$$

*This homogeneous system has a general solution* $t(1,-1)$. *We take* $t = 1$, *and, thus,* $\boldsymbol{x}_2 = (1,-1)$.

*The general solution of the difference equations is* $\boldsymbol{x}_n = c_1 \cdot 1^n \cdot (4,5) + c_2 \cdot 0.82^n \cdot (1,-1) = c_1(4,5) + c_2 \cdot 0.82^n \cdot (1,-1)$.

*The particular solution will be found by substituting the initial values:*

$$\boldsymbol{x}_0 = c_1(4,5) + c_2 \cdot 0.82^0 \cdot (1,-1) = c_1(4,5) + c_2(1,-1) = (30,15).$$

*We solve the linear system*

$$\begin{aligned} 4c_1 + c_2 &= 30, \\ 5c_1 - c_2 &= 15. \end{aligned}$$

*The solution is* $c_1 = 5$, $c_2 = 10$, *we get the particular solution*

$$\boldsymbol{x}_n = 5 \cdot (4,5) + 10 \cdot 0.82^n \cdot (1,-1) = (20,25) + 0.82^n \cdot (10,-10).$$

*After n years there will be probably* $20 + 10 \cdot 0.82^n$ *people living in the cities, and* $25 - 10 \cdot 0.82^n$ *people living in the country.*

*For example,* $\boldsymbol{x}_2 = (20,25) + 0.82^2 \cdot (10,-10) = (26.724; 18.276)$, *which is the solution of the previous example.*

The steady-state or stationary values in difference equations are values $x_{i,n}$, $i = 1, 2, ..., k$, at which the system comes to rest. That means that $x_{i,n+1} = x_{i,n}$, $i = 1, 2, ..., k$, see[2].

**Example 2.6** *The vector* $\boldsymbol{x} = (20, 25) = 4 \cdot (4, 5)$ *is one of the eigenvectors of the eigenvalue* 1 *for the matrix* $M$*, see the previous example. This is a steady-state of the equations.*
*Let us denote by* $\boldsymbol{x}_n = (x_n, y_n)$ *the solution of the previous example, and compute*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (20 + 10 \cdot 0.82^n) = 20,$$
$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} (25 - 10 \cdot 0.82^n) = 25.$$

*The solution converges to the steady-state. That means that after many years there will be probably almost no migration. There will be probably about* 20 *mil. people living in the cities and about* 25 *mil. people living in the country.*

If the solution converges to the steady-state, we say that the solution is asymptotically stable, see [1].

## 3 Taylor polynomial, duration and convexity of bonds

The present value of a bond is the amount repaid to the investor by the time the bond matures. It can be considered as a real function of a real variable $r$:

$$P(r) = \sum_{i=1}^{n} \frac{C_i}{(1+r)^{t_i}}, \qquad (1)$$

where $C_i$ is a coupon payment (periodic interest payment) at the time $t_i$, $i = 1, 2, .., n-1$, $n$ is the number of payments, $C_n$ is the sum of a coupon payment at the time $t_n$ and the face value of the bond, which is also paid at the time $t_n$, and $r$ is yield to maturity.

**Example 3.1** *Let us consider a two year bond with a face value 1000 crowns, a coupon payment* $C = 80$ *crowns and a present value* $P(r) = 960$ *crowns. Compute r (yield to maturity).*
*We solve the equation* (1)*:*
$960 = \frac{80}{1+r} + \frac{80}{(1+r)^2} + \frac{1000}{(1+r)^2} \Rightarrow 24r^2 + 46r - 5 = 0 \Rightarrow r_1 = 0.10,\ r_2 = -2.02.$
*The solution is* $r = 0.10$, *i.e.* 10%.

Investors are interested in the change of the present value of a bond $P(r)$ provided the yield to maturity $r$ is changed due to moving interest rates. A small change of the value of a bond can be expressed by the Taylor polynomial $T_1(r)$, or, more accurately, $T_2(r)$ in the point $r_0$.
If we use $T_1(r)$, we get

$$P(r) \approx T_1(r) = P(r_0) + P'(r_0)\frac{(r-r_0)}{1!} = P(r_0) + P'(r_0)(r - r_0),$$

and the change of the bond can be approximated by

$$P(r) - P(r_0) \approx P'(r_0)(r - r_0).$$

If we use $T_2(r)$, we get

$$P(r) \approx T_2(r) = P(r_0)+P'(r_0)\frac{(r-r_0)}{1!}+P''(r_0)\frac{(r-r_0)^2}{2!} = P(r_0)+P'(r_0)(r-r_0)+P''(r_0)\frac{(r-r_0)^2}{2},$$

and the change of the bond can be approximated by

$$P(r)-P(r_0) \approx P'(r_0)(r-r_0)+\frac{1}{2}P''(r_0)(r-r_0)^2.$$

**Example 3.2** *Let us consider a two year bond with a face value 1000 crowns, a coupon payment $C = 80$ crowns and a yield to mature 9 % , i.e. $r_0 = 0.09$. Approximate the change of the present value of the bond using Taylor polynomials of $1^{st}$ and $2^{nd}$ order, if the yield to mature suddenly decreases from 9 % to 8 %, i.e. $r = 0.08$.*

*First we approximate the change of the bond by $T_1(r)$, then by $T_2(r)$. Let us denote by $P_{T_1}, P_{T_2}$ the estimations of the values. The function $P$ is given by*

$$P(r) = \sum_{i=1}^{2}\frac{80}{(1+r)^i}+\frac{1000}{(1+r)^2} = \frac{80}{1+r}+\frac{80}{(1+r)^2}+\frac{1000}{(1+r)^2} = \frac{80}{1+r}+\frac{1080}{(1+r)^2}.$$

*We compute the present value of the bond*

$$P(r_0) = P(0.09) = \sum_{i=1}^{2}\frac{80}{(1+0.09)^i}+\frac{1000}{(1+0.09)^2} = \frac{80}{1.09}+\frac{1080}{(1.09)^2} = 982.409$$

*and the first derivative in $r_0$*

$$P'(r) = 80\cdot(-1)\cdot(1+r)^{-2}+1080\cdot(-2)\cdot(1+r)^{-3} = -\frac{80}{(1+r)^2}-\frac{2160}{(1+r)^3},$$

$$P'(r_0) = P'(0.09) = -\frac{80}{(1.09)^2}-\frac{2160}{(1.09)^3} = -1735.251.$$

*Thus*

$$P(r)-P(r_0) \approx P'(r_0)(r-r_0),$$

$$P(0.08)-P(0.09) \approx 1735.251\cdot(0.08-0.09) = 17.352.$$

*The estimation is $P_{T_1}(0.08) \approx 982.409+17.352 = 999.761$ crowns.*
*We compute the second derivative in $r_0$*

$$P''(r) = -80\cdot(-2)\cdot(1+r)^{-3}-2160\cdot(-3)\cdot(1+r)^{-4} = \frac{160}{(1+r)^3}+\frac{6480}{(1+r)^4},$$

$$P''(r_0) = P''(0.09) = \frac{160}{(1.09)^3}+\frac{6480}{(1.09)^4} = 4714.145.$$

*Thus*

$$P(r)-P(r_0) \approx P'(r_0)(r-r_0)+\frac{1}{2}P''(r_0)(r-r_0)^2,$$

$$P(0.08)-P(0.09) \approx -1735.251\cdot(0.08-0.09)+\frac{1}{2}\cdot 4714.145\cdot(0.08-0.09)^2 = 17.352+0.236 = 17.588.$$

*The estimation is* $P_{T_2}(0.08) \approx 982.409 + 17.588 = 999.997$ *crowns.*
*The present value of the bond (1) for the yield to mature* $r = 8$ *% is*

$$P(0.08) = \frac{80}{(1+0.08)} + \frac{80}{(1+0.08)^2} + \frac{1000}{(1+0.08)^2} = 1000 \ \textit{crowns} .$$

*It is clear that the second approximation is more accurate. The higher is the order of the Taylor polynomial the more accurate is the approximation.*

Solution of the previous example can be generalized: The first and the second derivatives of the function $P$ are

$$P'(r) = -\frac{1}{(1+r)}\sum_{i=1}^{n}\frac{t_iC_i}{(1+r)^{t_i}},$$

$$P''(r) = \frac{1}{(1+r)^2}\sum_{i=1}^{n}\frac{t_i(t_i+1)C_i}{(1+r)^{t_i}}.$$

Substituting the derivatives into $T_1(r)$ we get

$$P(r) - P(r_0) \approx -\frac{1}{(1+r_0)}\sum_{i=1}^{n}\frac{t_iC_i}{(1+r_0)^{t_i}}(r-r_0),$$

or, more accurately, into $T_2(r)$ we get

$$P(r) - P(r_0) \approx -\frac{1}{(1+r_0)}\sum_{i=1}^{n}\frac{t_iC_i}{(1+r_0)^{t_i}}(r-r_0) + \frac{1}{2(1+r_0)^2}\sum_{i=1}^{n}\frac{t_i(t_i+1)C_i}{(1+r_0)^{t_i}}(r-r_0)^2.$$

A **duration** of a bond, see e.g. [3], is the weighted average of the times until fixed cash flows are received:

$$D(r) = -\frac{(1+r)\cdot P'(r)}{P(r)}.$$

A **convexity** of a bond is:

$$K(r) = P''(r).$$

Now the approximated change of the present value of the bond by Taylor polynomial can be expressed ( by using duration and convexity):

$$P(r) - P(r_0) \approx -D(r_0)\frac{P(r_0)}{(1+r_0)}(r-r_0), \tag{2}$$

$$P(r) - P(r_0) \approx -D(r_0)\frac{P(r_0)}{(1+r_0)}(r-r_0) + \frac{1}{2}K(r_0)(r-r_0)^2. \tag{3}$$

When we denote $P(r) - P(r_0)$ by $\Delta P$ and $r - r_0$ by $\Delta r$ we get

$$\Delta P \approx -D(r_0)\frac{P(r_0)}{(1+r_0)}\Delta r,$$

$$\Delta P \approx -D(r_0)\frac{P(r_0)}{(1+r_0)}\Delta r + \frac{1}{2}K(r_0)\Delta r^2.$$

**Example 3.3** *Let us consider a two year bond with a face value 1000 crowns, a coupon payment* $C = 80$ *crowns and a yield to mature* $9\ \%$ *, i.e.* $r = 0.09$*. Compute duration and convexity of the bond.*

$$D(r) = \frac{\sum_{i=1}^{n} \frac{t_i C_i}{(1+r)^{t_i}}}{P(r)},$$

$$D(0.09) = \frac{\left(\frac{80}{1.09} + \frac{2\cdot 80}{(1.09)^2} + \frac{2\cdot 1000}{(1.09)^2}\right)}{P(0.09)} = 1.925 \ years \ ,$$

$$K(r) = \frac{1}{(1+r)^2}\sum_{i=1}^{n}\frac{t_i(t_i+1)C_i}{(1+r)^{t_i}},$$

$$K(0.09) = \frac{1}{(1+0.09)^2}\left(\frac{2\cdot 80}{1.09} + \frac{2\cdot 3\cdot 80}{(1.09)^2} + \frac{2\cdot 3\cdot 1000}{(1.09)^2}\right) = 4714.145.$$

We will approximate the change of the value of a bond using duration and convexity.

**Example 3.4** *Let us consider a two year bond with a face value 1000 crowns, a coupon payment* $C = 80$ *crowns and a yield to mature* $9\ \%$ *, i.e.* $r = 0.09$*. Approximate the change of the present value of the bond if the yield to mature suddenly decreases from* $9$ *to* $8\ \%$*, i.e.* $r = 0.08$*.*

*From Examples 3.2 and 3.3 we know that the value of the bond is* $982.409$ *crowns, duration is* $1.925$ *years and convexity is* $4714.145$*. We subtitute the duration and convexity into (2) and (3). First we use* $T_1(r)$*, then* $T_2(r)$*. The approximated prices will be denoted by* $P_{T_1}, P_{T_2}$

*We subtitute the duration into (2):*
$P(0.08) - P(0.09) \approx -1.925 \cdot \frac{982.409}{(1+0.09)} \cdot (0.08 - 0.09) = 17.350$*. The estimation is* $P_{T_1}(0.08) \approx 982.409 + 17.350 = 999.759$ *crowns.*

*We subtitute the duration and convexity into (3):*
$P(0.08) - P(0.09) \approx -1.925 \cdot \frac{982.409}{(1+0.09)}(0.08 - 0.09) + \frac{1}{2}\cdot 4714.145 \cdot (0.08 - 0.09)^2 = 17.350 + 0.236 = 17.586$.
*The estimation is* $P_{T_2}(0.08) \approx 982.409 + 17.586 = 999.995$ *crowns.*

*The difference between these results and the estimation by Taylor polynomials in Example 3.2 are caused by rounding during computation.*

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## References


[1] SIMON, C.P., BLUME, L. : *Mathematics for Economists*, W.W.Norton & Co, New York, Mass. - London, 1994 (2001)

[2] HOY., M., LIVERNOIS, J., at al. *Mathematics for Economics*, MIT Press, Cambridge, Mass. - London, 2001

[3] CIPRA, T.: *Matematika cenných papírů*, HZ - Praha, 1995


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# SIEVE OF ERATOSTHENES TO FIND NEW NUMBERS

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**Abstract**. Why is Math so hard for some children? We want to start from the vision and the teorethical dissertation in [16], Supported by educational and philosophical theories that here are presented, to use the concept of reification. A new element in *contemporary reason* is therefore the *re-emergence of the subject*, the reconsideration of the distinction between objective and subjective, between that which belongs to the subject and that which belongs to the object. We want to submit an experience of study with 10-year old children, at primary school. The aim is to help children in forming the abstract concept of prime numbers and to use this concept for build other concepts.

**Key words.** Objectivization and subjectivization, sieve of Eratosthenes.

*Mathematics Subject Classification:* Primary 97A30, 97C30; Secondary 97F30.

## 1 Mathematical concept development

In [16], the author outlines two main perspectives that influence the formation of mathematical concepts: procedural and structural perspectives. Historically, the formation of a mathematical concept took place according to different stages: the preconceptual, the operational and the structural stage. The preconceptual stage: mathematicians were getting used to certain operations on the already known concepts; the operational stage, during which a new kind of concept begun to emerge out of the familiar processes; the structural phase: the concept in question has eventually been recognized as a fully-fledged mathematical object

To sum up, the history of numbers was presented as a long chain of transitions from operational to structural conceptions: again and again, processes performed on already accepted abstract objects have been converted into compact wholes, or *reified* (from the Latin word *res – a* thing), to become a new kind of self-contained static constructs. Abstract notions can be conceived in two fundamentally different ways: *structurally* - as objects, and *operationally* - as processes.

Sfard individualise three stages in concept development *interiorization, condensation* and *reification*. At the stage of interiorization a learner gets acquainted with the processes which will eventually give rise to a new concept. These processes are operations performed on lower-level mathematical objects. Gradually, the learner becomes skilled at performing these processes.

The phase of condensation is a period of "squeezing" lengthy sequences of operations into more manageable units. At this stage a person becomes more and more capable of thinking about a given process as a whole, without feeling an urge to go into details. This is the point at which a new concept is "officially" born.
The condensation phase lasts as long as a new entity remains tightly connected to a certain process. Only when a person becomes capable of conceiving the notion as a fully-fledged object, we shall say that the concept has been reified. *Reification,* therefore, is defined as an ontological shift, a sudden ability to see something familiar in a totally new light. Reification is an instantaneous quantum leap: a process solidifies into object, into a static structure. Various representations of the concept become semantically unified by this abstract, purely imaginary construct. The stage of reification is the point where an interiorization of higherlevel concepts begins.
It seems that the structural approach should be regarded as the more advanced stage of concept development. I*n the process of concept formation, operational conceptions would precede the structural.* This statement is basically true whether historical development or individual learning is concerned.

## 2 Objectivization and subjectivization

Bacon had already stated that the error made by *ancient science* was to consider *objective knowledge* possible only by excluding the action of man from natural reality. Objective knowledge was interpreted as an *absolute view* of the world and as *reason and indisputable truth* in understanding phenomena which govern not only physical laws, but particularly a perspective view of world knowledge.
With the advent of the so-called *science of complexity* a new *thinking reasoning* originated, which places the dimensions of *logic* and *time* at the centre of *philosophical and speculative* interests, and particularly, has led to a different interpretation of the concept of time. Ilya Prigogine, in his interesting work "*From being to becoming*" deals with this theme, identifying a form of temporality which he defines *creative time*. His thinking represents a focal point in extrication from reflection on the phenomena which lead to the interpretation of the *concept of knowledge*.
Prigogine, in explaining his theory, essentially wanted to demonstrate that life does not simply consist of the execution of a *pre-determined programme*, but is defined within a recursive cycle as *creation of new*, as *invention*. We have before us a *creator time* which requires its own particular logic: the logic which comes closest to this kind of temporality is *the logic of non-linearity,* better defined as *logic of complexity*.
In this view, *contemporary scientific rationality* is a *reasoning* in which universalizing principles do not exist, neither a *globality of basic concepts* exists, nor a *universality of methods* applicable to all fields of knowledge. From this context and on these theoretical bases arises the need to think up new ways of interpreting *didactics* and the dynamics which characterize teaching theory, along with the necessity to re-qualify the cognitive and structural background on which to base a new *teaching professionalism*.
A new element in *contemporary reason* is therefore the *re-emergence of the subject*, the reconsideration of the distinction between objective and subjective, between that which belongs to the subject and that which belongs to the object. Every human being produces his own world as a result of being produced by this world (Maturana and Varela). A "subject reality" and an "object reality" do not exist; autopoiesis creates simultaneously the object and the *particular view* for which that object acquires, at that moment, a specific personal meaning.
From Piaget onwards, there has been the consideration that the *biological basis of the human organism* is the condition from which the *intelligent processes* flower and, at the same time, the

flowering of these processes is the condition for recognizing a biological base: the tree of knowledge is the tree of life, feeding back on itself.
For Piaget, the passage from *psyche* to *logic* is the passage from *irreversible* to *reversible* and is linked to the discovery of reversibility: it is the same type of relationship that Piaget establishes between *biological structure* and *stages of intelligence*. Using the model of Piaget's theory it is possible to affirm that *cognitive strategies* are the capacity for *self-regulation* (intellectual ability) through which the mind faces the environment. The pillar of this conception is found in the passage from *intellectual ability* to *cognitive strategies*: it is possible to solve a problem for which we do not know *objective rules* for solving it, because there is the possibility for the mind to feed back infinitely upon itself, transforming and intersecting the very rules that have formed it. In brief, while intellectual abilities are learned in situations of objectivization of the teaching-learning processes and are translated into objectively verifiable applications, the possibility of learning *cognitive strategies* is linked to a nonlinear logic. These are taught through nonlinear didactics (designed to favour autoreferentiality) in a field of subjectivization of educative processes.

## 3 Our experience and results

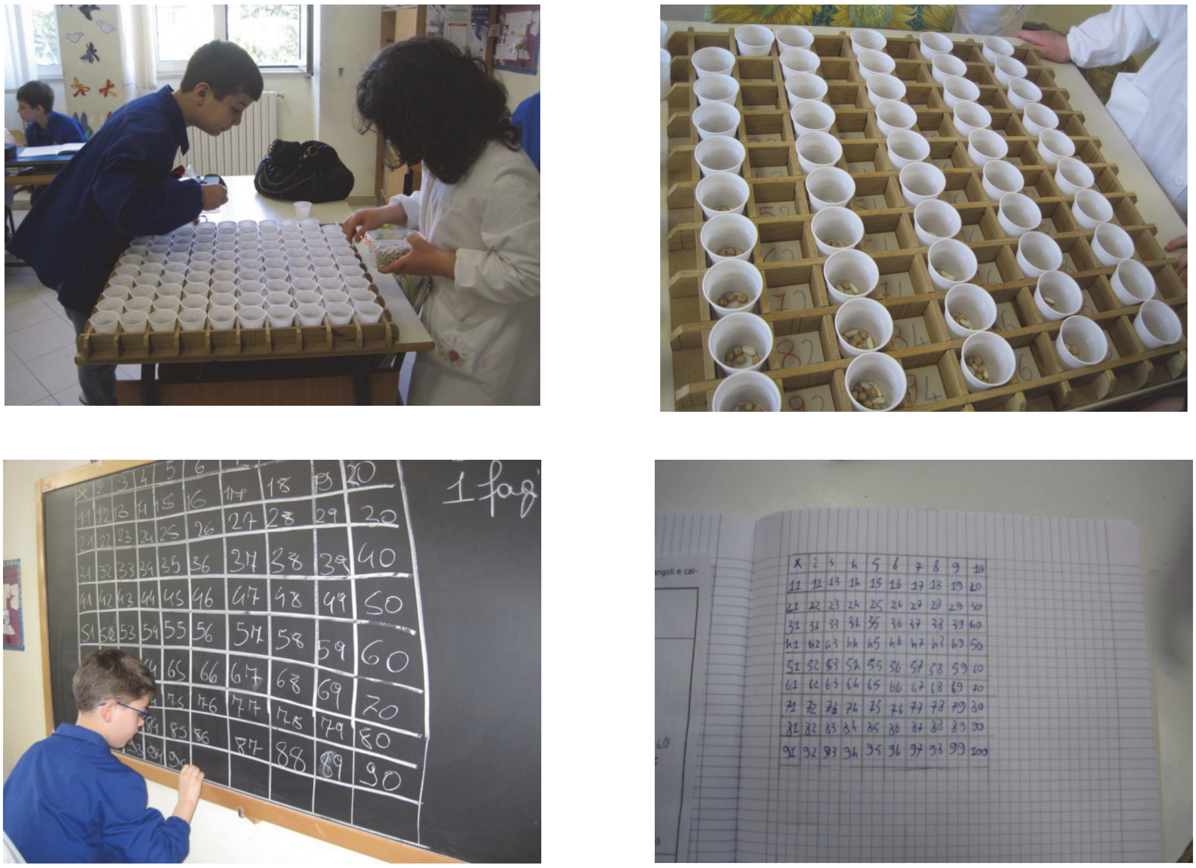
We conducted an experimental study with 49 10-year old children, at primary school. The aim was to help children in forming the abstract concept of prime numbers. The children have also improved their knowledge on concepts already known but not yet well established as: divisor, multiple, divisibility rules, and so on. The teacher turned several questions to the children:

The sieve in class.

- ✓ What is it?
- ✓ What is its utility?
- ✓ In your opinion what are the "natural sieves"?
- ✓ What can we do with a sieve in mathematics?
- ✓ Why do we call Sieve of Eratosthenes a sieve in mathematics?

The teacher asked them if they had already heard of Eratosthenes; All said they did not know who he was, so she lead to reflect on the fact that they had heard of the scientist during the history lesson; they studied the history of Egypt and its important library.

It reads a difficulty by children in considering the same subject in the context of different disciplines.Have all numbers the same properties?
The children built the Sieve of Eratosthenes by using chickpeas, peas and beans.



Some stages of the experience

Then the teacher asked them which means a number is a multiple of another number and she called 10 children with the names „one“, „two“, ..., „ten“ and other two children with the names „unità“ („units“) and „decina“ („tens“).
The teacher said to wait to „one“ child, while the „two“ child had to delete on the sieve all all numbers divisible by two; the “three” child had to delete on the sieve all all numbers divisible by three and so on. Before these operations, they had to remember the respective divisibility rules.
We can say that this first part of our experience corresponds to the *interiorization stage*. A learner gets acquainted with the processes which will eventually give rise to a new concept. These processes are operations performed on lower-level mathematical objects.
Subsequently, the teacher asked the children to control what the numbers were left on the sieve. He said them to write the numbers and to call those “prime numbers”. Gradually, the learner becomes skilled at performing these processes. So, This is the phase of condensation: the period of "squeezing" lengthy sequences of operations into more manageable units. At this stage children become more and more capable of thinking about a given process as a whole, without feeling an urge to go into details. This is the point at which a new concept is "officially" born.

Finally the children calculated primes up to 600. Children become capable of conceiving the notion as a fully-fledged object, so we We reached the stage of *Reification,* and it is possible to use this new concept for new procedures.

## References


[1.] G. ACONE, *L'identità teorica della pedagogia tra complessità e postmodernità,* Pedagogia e vita, La Scuola, Brescia 2005.
[2.] G.T. BAGNI, *La introducción de la historia de las matemáticas en la enseñanza de los números complejos. Una investigación experimental en la educación media superior*, *Revista Latinoamericana de Investigación en Matemática Educativa*, 2001.
[3.] BROUSSEAU, *Fondaments et méthodes de la didactique des Mathématiques - Recherches en didactique des mathématiques*, 1986.
[4.] M. CAPPUCCIO, *Neurofenomenologia,* Mondatori, Milano 2006.
[5.] B. D'AMORE, *Elementi di didattica della matematica*, Pitagora, 1999;
[6.] H. MATURANA and F.J. VARELA, *Autopoiesi e cognizione,* Marsilio, Padova 1985.
[7.] H. MATURANA and F.J. VARELA, *L'albero della conoscenza*, Garzanti, Milano 1987.
[8.] G. MINICHIELLO, Ricerche sul problema della razionalità in pedagogia, Edisud, Salerno 1999.
[9.] G. MINICHIELLO, Elementi di didattica generale, Edisud, Salerno 2003.
[10.] F. PALLADINO, L. LOMBARDI, N. Palladino, *Algoritmi elementari del calcolo aritmetico e algebrico*, Pitagora editrice , Bologna, 2005.
[11.] J. PIAGET, *Biologia e conoscenza,* Einaudi, Torino 1983.
[12.] J. PIAGET and A. SZEMINSKA, *La genesi del numero nel bambino*, La Nuova Italia, Firenze, 1976.
[13.] K.R. POPPER, Conoscenza oggettiva. Un punto di vista evoluzionistico, Armando, Roma 1975.
[14.] I. PRICOGINE, Dall'essere al divenire, Einaudi, Torino 1986.
[15.] RADFORD, *On Psychology, Historical Epistemology and the Teaching on Mathematics: Towards a Socio-Cultural History of Mathematics*, For the Learning of Mathematics, 1997.
[16.] A. SFARD, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*, Cambridge University Press, 2008
[17.] A. SFARD, *On the duale nature of mathematical conceptions:: reflections on processes and objects as different sides of the same coin*, Educational Studies in Mathematics, Volume 22, Number 1, 1991.
[18.] VYGOTSKIJ, *Pensiero e linguaggio*, Giunti&Barbera, Firenze, 1966.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# ANIMATION OF ESSENTIAL CALCULUS CONCEPTS IN MAPLE

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**Abstract.** The notion of the limit of a real function belongs to essential concepts of calculus. A part of higher schools students has learned the notion in the secondary school mathematics, but, as a rule, the concept of the limit is one of the crucial introductory topics in calculus lectured at universities. Some students understand the concept only intuitively, mainly due to ideas based on the movement of a body. There are different ways how a teacher could demonstrate the correspondence between the intuitive insight and the exact definition of the limit. The paper shows how the concept of limit of a function can be illustrated with Maple animation. The paper also deals with the principle of some other animations, e.g. continuity of function, the derivative of a function at a point, and the Riemann's definite integral. More general problems on teaching and learning are discussed, especially those connected with the special role of learning outcomes in calculus.

**Key words.** Limit, Continuity, Derivative, Definite Integral, Animation, Computer Algebra Systems, Maple, Learning Outcomes.

*Mathematics Subject Classification:* Primary 97I40, 97I50; Secondary 97U50.

## 1 Introduction

The article deals with problems and corresponding procedures how to communicate visually the basic concepts of mathematics in teaching, especially those in calculus (limits of functions and derived notions as continuity, derivative, definite integrals etc.). Here a key role of a teacher is to force learners to touch on analysis of real numbers, even in a dynamic, not static way. On the other hand, it is known that our recipient – a student prefers the information supplied in a visual form. We share the opinion that the access to technological tools affords the ways how to motivate them, how to provide the insight into the core principles, build up the knowledge, and, after acquiring skills, to incorporate them into problems solving. A part of ideas concerning the topic can be found in [4].

## 2 The limit of a function

The notion of the limit of a real function belongs to essential concepts of calculus. Construction of subsequent calculus concepts uses just the limit concept: the derivative of the function at the point, Riemann's definite integral, multivariable calculus notions, etc. The concept of the limit is also the key notion in iteration processes, especially when searching for equation solutions, or investigation of equilibrium state, of optimal values, for area evaluation, etc. Several mathematical theories include applications of procedures based on limit constructions, and for this reason, they are used fruitfully in other sciences, not only in mathematics.

The intuitive understanding of the notion of limit could be found in considerations of calculus inventors in 17th century. The attempt was connected with the effort of capturing the rules valid for mechanical motion. Issac Newton and Wilhelm Leibniz, both treated now as such inventors, used in argumentations and calculations the notion of the flux, resp. of the infinitely small (infinitesimal) quantity (see [2]). The form of the limit definition has been accepted after 200 years later. It was introduced in lectures of Karl Weierstrass, see, e.g. [5]; he reformulated it, eliminating the notion of infinitesimals. Hence, the key calculus notions required the strong intellectual effort at its very involving – this effort repeats nowaday when a student is forced to move from his/her intuition insight, or from the pre-concept, into the exact interpretation of the concept required.

As it is well-known, searching for the limit of a given real function $f$ of one variable means to ask on values of the function $f$ at those $x$ in its domain that a variable $x$ approaches to the given number $a$, not being equal to, independently on fact whether $a$ belongs to the domain of the function or not. In case all the values $f(x)$ can be made as close as desired to the number $A$, a unique one, the number $A$ is to be called the limit of the function $f$ at the point $a$. Augustin-Louis Cauchy in Preliminaries to *Cours d'Analyse de l'École Royale Polytechnique: Première Partie: Analyse Algébrique (Analysis Course of the Royal École Polytechnique: First Part: Algebraic Analysis) in* 1821 wrote:

*When the successively attributed values of one variable indefinitely approach a fixed value in such a way that they finally differ from it by as little as desired, then that fixed value is called the limit of all the others.*

Such description has been not fully accepted, as the word „approach“ leads to interpretation as a motion or movement is involved and, as we know, mathematics works only with a static concepts. Instead of the intuitive, and dynamic approach, with the support of geometrical interpretation, it turned out that the more appropriate definition of the limit has to be based on the arithmetical approach. It was provided by Bernard Bolzano in 1817, and in a less precise form by Augustin-Louis Cauchy. Karl Weierstrass formalized it in 1851; in his formulation he uses wording „as close as possible“- the variable has been treated as a static one, not depending on time, with any values close to some other given value.

“ The limit of the function $f$ at the point a is the real number $A$, if for each real number $\varepsilon$, $\varepsilon > 0$, there exists a real number $\delta$ , $\delta > 0$, such that for all numbers $x$ with $0 < |x - a| < \delta$, it follows $|f(x) - A| < \varepsilon$.”

In other words: „The limit of the function $f$ at the point $a$ equals to $A$, if for any given real number $\varepsilon$, $\varepsilon > 0$, the distance between values $f(x)$ and $A$ is smaller than $\varepsilon$, $\varepsilon > 0$, provided the distance between points $x$ and $a$ is smaller than the given real number $\delta$, $\delta > 0$, and $\delta$ depending on the choice of $\varepsilon$.“

The work with the preceding definition, having the concrete choice for one pair of numbers $\varepsilon, \delta$ enables to present one static picture expressing the distances in function values and corresponding arguments of that function. In the classical study, the learner then is forced „to play a game“ with those $\varepsilon, \delta$, where $\delta$ depends on $\varepsilon$. The more intelligent version, supporting the learning (not only learning the limit) in a substantial way, consists in the use of MAPLE tools; then single pictures will move into the animation sequence, moreover, controlled by teacher/learner.

## 3 Animation in Maple

MAPLE, the Computer Algebra System (CAS, see e. g. [3]) enables to provide numerical computations, and symbolic ones as well. The principle how to generate animation of relations which concern limits with the real number $\varepsilon$, $\varepsilon > 0$, $\varepsilon$ as the parameter, consists in the use of procedures. Then the animation itself means to invent the construction of the very fast sequence of appropriate graphs. The MAPLE command recommended for to animate the sequence is animate, see [2]. Working with the task defined above, it turned out that it would be more reasonable to use the command seq: in result of its application, one gets the required sequence of pictures/graphs. Those pictures could be mapped fastly in an ordered sequence using the command display, which is available after calling plots. The command is to be used with the parameter insequence=true.

## 4 Presentation of essential concepts of calculus

Introducing the notion of the limit of a function and animated it in teaching, based on the previous remarks, one can use therefore also animations derived from that one prepared for the limit as its core.

### 4.1 The limit of a function at a point

How to interpret geometrically the relation „to be as closed as possible to some given value“, the crucial in definition of the limit: drawing, it means that for arbitrary given $\varepsilon$, $\varepsilon > 0$ we construct two paralel lines $y = A + \varepsilon$ and $y = A - \varepsilon$, and we shall find the corresponding $\delta$, $\delta > 0$ such that for any $x$, $x \neq a$, $x$ between $a - \delta$ and $a + \delta$, it holds $A - \varepsilon < f(x) < A + \varepsilon$, i.e. the graph of the function $f$ is located between horizontal lines $y = A - \varepsilon$, $y = A + \varepsilon$, see [7].

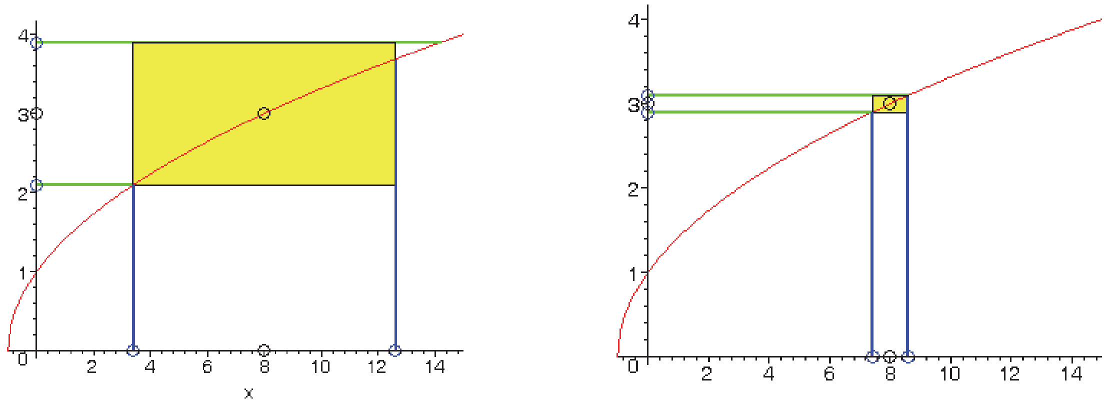



*Fig. 1 Geometrical interpretation of the notion of limit of the function* $f : y = \sqrt{x - 1}$ *at the point* $a = 8$ *with the value* $\varepsilon = 9/10$ *or* $\varepsilon = 1/10$.

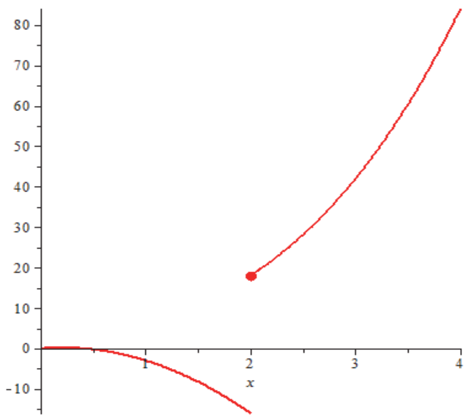
For the animation sequence, consider the set of new and new, decreasing values of ε, $\varepsilon > 0$ and repeat the construction of the picture of finding the interval between $a-\delta$ and $a+\delta$ as described above. It means, the experiment will help and is worthy to go on. Use the commands available in MAPLE for the experiment – for the construction of such sequence, and in this way animate the notion of the limit of a function. For learning purposes, it is useful to provide details in protocol on parameters values.

### 4.2 The continuity of a function at a point

MAPLE tools enable also to manipulate with graphs of function in cases when we, in view of the learning purpose, require to join its different parts, fitting some possible parameters, with the aim to get its graph as the graph of the continuous function on a given set. As an example, take the function defined on the set of all real numbers as

$$f : y = \begin{cases} cx^2 + 2x, x < 2, \\ x^3 - cx, x \geq 2, \end{cases}$$

where $c, c \in R,$ is the parametr. The problem is formulated as the search for which value of the parameter $c$ the function $f$ should be continuous at the point $x = 2$.



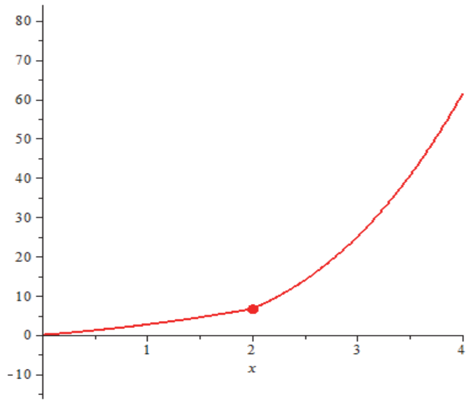




*Fig. 2 The function fails to be continuous at* $x = 2$ *when the parameter value is* $c = -5$.
*The function is continuous at* $x = 2$ *when the parameter value is* $c \doteq 0{,}67$.

### 4.3 The derivative of a function at a point

The next calculus concept which requires to be acquainted with the concept of the limit is the derivative of the function at a given point. As it is well-known, the geometrical interpretation of this concept means the slope of the tangent line to the graph of the function at the given point. As a support of learning, it would be worthy to follow on the classical approach: to produce the sequence of section lines, each at the given point, and show how section lines are closer and closer to the tangent line at this given point, depending on the existence of the limit for values of section lines slopes. Some phases in running the sequence are shown on Fig. 3. Let us remark that it is valuable to provide also an animation in which we chose as the point in question just the inflection point of the given function.

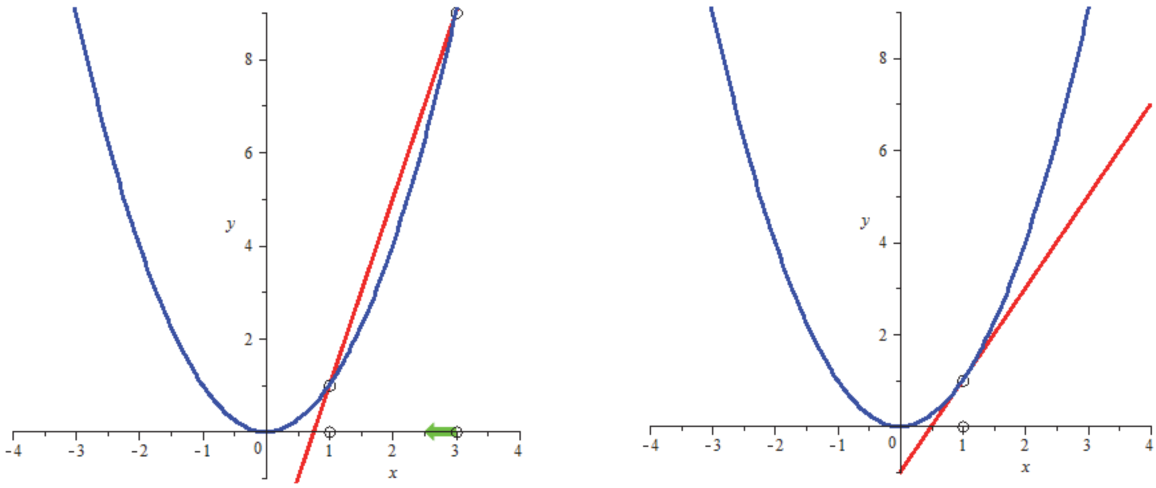


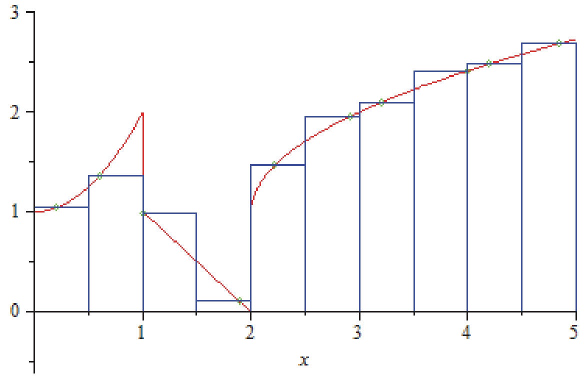

*Fig. 3 The geometrical interpretation of the section line – the section line, passing through given points, is approaching to the tangent line at this point. The geometrical interpretation of the derivative of the function at a point: the tangent line, as „the limit location of section lines“, passing through the given point on the graph.*

### 4.4 Riemann's sums

Lecturing on Riemann's definite integral, the animation could be used where for the given bounded function defined on the closed interval and the given division of that interval, one constructs the sequence of Riemann's sums. One item of that sequence is generated as a result of a refinement of preceding interval division. The sequence shows the method how to estimate the area of the plane region under the graph of the function, as the sum of areas of rectangles. The series of single pictures connected into the sequence provide the animation of Riemann's definite integral defined as the limit of the (special) sequence of Riemann's sums. MAPLE suggests to apply the CAS predefined procedure named RiemannSum. Figure 4 shows some sequence items. In teaching, the geometrical interpretation enables to introduce immediately the formal definition of the concept in question, see, e.g. [6]. The definition has been shown, and its principal steps requiring the limit attempt are demonstrated – MAPLE worked as the experiment tool.

## 5 The value of animation in learning: a wider frame

Why to discuss here the basic, usual mathematical approach in details? Implementation of LMS in learning of mathematical concepts/procedures is included among key activities of the project „REFIMAT" solved at Faculty of informatics and Management, UHK (for its description, see „Acknowledgement" notice added). Faculty students are supplied with the rich base of learning



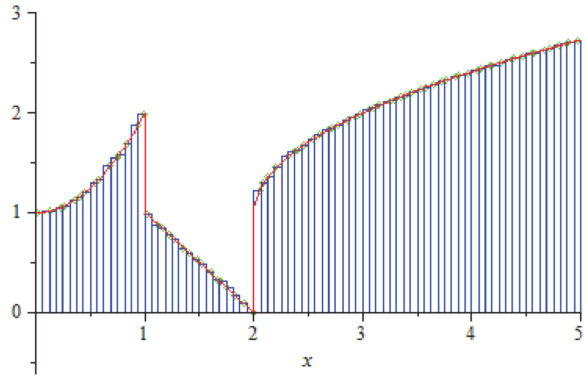




*Fig. 4 The construction of Riemann's sums; 10 subintervals of the given interval.*
*The geometrical interpretation of the definite integral – area of the plane region under the graph of the non-negative function.*

material, study supports of different types and styles, including their e-learning versions. Still, research based on the questionnaire in the project, proved the existence of the strong demand of learners on providing them materials of the type „show us how to manage, evaluate, construct, compute, solve – provide us working instructions". Here two streams meet together:

A. Faculty teachers involve learning outcomes into the study of subjects with the mathematical content, as the defined targets for understanding, knowledge and skills. Learning outcomes are defined for programmes or single courses; in the contribution context, learning outcomes of the course mean:

*Upon completion of the course, the student will acquire the knowledge, understanding and skills, based on calculus tools, on real functions of one variable, on their basic properties and use in practice, and their roles as principal tools applied for formulating, modeling and solving practice problems.*

Concerning the topic limit of a function at the point, learning outcomes for this course module are stated as follows:

*Upon completing this section, the student acquires*

- *knowledge: using the relevant knowledge on subsets of the set of all real numbers, the students is able to define the limit of the function at a point, he/she provides basic properties of the limit, and he/she learned how to provide the geometrical interpretation of the limit;*
- *he/she understand the notion of the limit at the point, and the local behaviour of the function close to the point, on base of the existence/non-existence of the limit; he understand how to check the existence/non-existence of the limit;*
- *skills: he applies the definition and rules followed from the definition in the numerical evaluation of the limit*

Animations support the learning outcomes in a significant way: they show

- the effective method how to organize a way to an insight, or some kind of experiment on the concepts/procedures,

- they enable, after the starting position, to repeate a great number of necessary steps, which cannot be done by pencil-and-paper method,
- the experiment could be taylored on a learner, on his/her cognitive abilities,
- animations provide steps convergent to the vision, and enrich the vision itself, which cannot be done effectively using only the pencil-and-paper method,
- a learner could be provided by an inspiration how to read/organize an animation, or the experiment, for related topics concepts.

It could be stated therefore that an animation, using CAS included, develops

- understanding assistance
- acquiring of knowledge
- training the skills

Why just MAPLE: instructive system, relatively easily understandable and applicable; also necessarily for the uniform teaching management of large groups of faculty learners, their access to the system is guaranteed and inputs/outputs could be controlled by instructors.

B. The second stream is a valuable declaration of students, showing their active attitude to the study, not to be neglected: „provide us with instructions preferably"; naturally, this does not mean to reduce the study onto the work with solved/unsolved exercises only, or to overestimate the technological aid of the study.

This voice coincides with the common knowledge in the society on how the characteristics of new generations of students is changed; the group of learners born between 1975 - 1981 called X Generation, after 1981 Y Generation (Millennial Generation, Gamer Generation, Net Generation), and some derived notation is used for the next ones etc. show different behaviour patterns, described by Howe and Strauss as follows (see [9]):

- Ability to read visual images—they are intuitive visual communicators
- Visual-spatial skills—perhaps because of their expertise with games they can integrate the virtual and physical
- Inductive discovery—they learn better through discovery than by being told
- Attentional deployment—they are able to shift their attention rapidly from one task to another, and may choose not to pay attention to things that don't interest them
- Fast response time—they are able to respond quickly and expect rapid responses in return

In [9], the characteristics are recollected into a table:

| ***Birth Dates*** | ***1900–1946***<br>***Matures*** | ***1946–1964***<br>***Baby Boomers*** | ***1965–1982***<br>***X Generation*** | ***1982–1991***<br>***Net Generation*** |
|---|---|---|---|---|
| Description | Greatest generation | Me generation | Latchkey generation | Millennials |

| | | | | |
|---|---|---|---|---|
| Attributes | Command and control<br>Self-sacrifice | Optimistic<br>Workaholic | Independent<br>Skeptical | Hopeful<br>Determined |
| Likes | Respect for authority<br>Family<br>Community involvement | Responsibility<br>Work ethic<br>Can-do attitude | Freedom<br>Multitasking<br>Work-life balance | Public activism<br>Latest technology<br>Parents |
| Dislikes | Waste<br>Technology | Laziness<br>Turning 50 | Red tape<br>Hype | Anything slow<br>Negativity |

In the next table, Net Gen Characteristics, Learning Principles, Learning Space, and IT Applications are aligned – as the challenge for the necesary management of learning space for individuals raised in the direct contact with ICT:

| ***Net Gen Trait*** | ***Learning Theory Principles*** | ***Learning Space Application*** | ***IT Application*** |
|---|---|---|---|
| Group activity | Collaborative, cooperative, supportive | Small group work spaces | IM chat; virtual whiteboards; screen sharing |
| Goal and achievement orientation | Metacognition; formative assessment | Access to tutors, consultants, and faculty in the learning space | Online formative quizzes; e-portfolios |
| Multitasking | Active | Table space for a variety of tools | Wireless |
| Experimental; trial and error | Multiple learning paths | Integrated lab facilities | Applications for analysis and research |
| Heavy reliance on network access | Multiple learning resources | IT highly integrated into all aspects of learning spaces | IT infrastructure that fully supports learning space functions |
| Pragmatic and inductive | Encourage discovery | Availability of labs, equipment, and access to primary resources | Availability of analysis and presentation applications |
| Ethnically diverse | Engagement of preconceptions | Accessible facilities | Accessible online resources |
| Visual | Environmental factors; importance of culture and group aspects of learners | Shared screens (either projector or LCD); availability of printing | Image databases; media editing programs |
| Interactive | Compelling and challenging material | Workgroup facilitation; access to experts | Variety of resources; no “one size fits all” |

## Conclusion

Considerations generate also questions of another type, namely on learning space implications (see [9]). The role of a teacher as a facilitator requires to let enough space (in a classroom, or as informal, or a virtual one) for study experiments, searching for learner's own way under own priorities and on the own sequencede steps for a learner. It seems that the intelligent access to technologies together with course management, and with the aim of the effective learning, leads to build up the space as an „Intelligent Tutor"; then, it could be supported by an appropriate institutional LMS, where multiple educational resources are available. Learning science indicates that successful learning is often active, social, and learner-centered ([9]). We share also ideas that with the appropriate use of technology, learning can be made more active, social, and learner centered - but the uses of IT are driven by pedagogy, not technology ([9]).

Animations described in the text are of initial value when approving that Intelligent Tutor as the learning space could be an adequate and functional institutional response to learner's expectations. We hope that this gives the more general meaning to the phenomenon indicated and denoted as *„two streams are meeting together"*.

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## References

[1] DUNHAM W. *The Calculus Galery*, Princeton University Press, Princeton, 2005.

[2] EDWARDS C.H. *The Historical Development of the Calculus*, Springer-Verlag, New-York, Berlin, Heidelberg, 1979.

[3] GARVAN F. *The Maple Book*, Chapman & Hall/CRC, 2002.

[4] PRAŽÁK P., GAVALCOVÁ T., PRAŽÁKOVÁ B. Animace pojmu limita funkce v systému počítačové algebry Maple, *Sborník příspěvků ze semináře a soutěže eLearning 2007*, Gaudeamus, Hradec Králové, 2007, pp. 400 – 405. (in Czech)

[5] SCHWABIK Š. Druhá krize matematiky aneb potíže růstu diferenciálního a integrálního počtu, *Sborník Matematika v proměnách věků*, Prometheus, Praha, 1998, pp. 7-60. (in Czech)

[6] SCHWABIK Š. *Integrace v R (Kurzweilova teorie)*, Nakladatelství Karolinum, Praha, 1999. (in Czech)

[7] ŠKRÁŠEK J., TICHÝ Z. *Základy aplikované matematiky I*, SNTL, Praha 1983. (in Czech)

[8] *Augustin-Louis Cauchy and the Rigorisation of Analysis*, [cit. 20.11.2011]. Available on www:
   a. http://everything2.com/title/Augustin-Louis+Cauchy+and+the+Rigorisation+of+Analysis

[9] DIANA G. OBLINGER and JAMES L. Oblinger (Editors): *Educating the Net Generation, Educause,* 2005, [cit. 20.11.2011]. Available on www:
   a. www.educause.edu/educatingthenetgen/

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